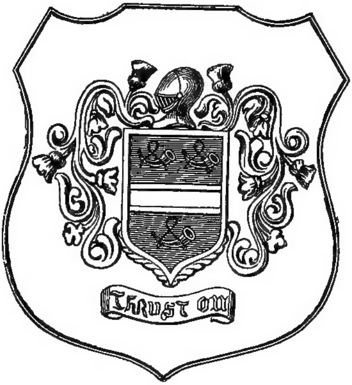

Robert Henry Thurston

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A TREATISE OF MECHANICS, THEORETICAL, PRACTICAL, AND DESCRIPTIVE.

BY

OLINTHUS Gilbert GREGORY,

OF THE ROYAL MILITARY ACADEMY, WOOLWICH.

VOL. I.

CONTAINING

The Theory of Statics, Dynamics, Hydrostatics, Hydrodynamics, and Pneumatics.

—— Philosophia mater omnium bonarum artium est. Cicero.

—— Meta autem Scientiarum vera et legitima, non alia est, quam ut dotetur vita humana novis inventis, et copiis. Bacon. *Nov. Org.*

LONDON:
PRINTED FOR GEORGE KEARSLEY, FLEET-STREET.
1806.

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TO THE RIGHT HONOURABLE

JOHN JOSHUA PROBY,

EARL OF CARYSFORT;

KNIGHT OF THE MOST ILLUSTRIOUS ORDER OF ST. PATRICK,
&c. &c. &c.

MY LORD,

ALTHOUGH the promotion of the arts and sciences has been ever deemed the proper province of the great and noble, yet it has not always been the good fortune of those who employ their time and exertions in the dissemination of knowledge, to meet with a patron who is at once distinguished by his rank, his talents, and his zeal in the encouragement of useful performances. I cannot, therefore, but esteem it a high honour, and a source of considerable gratification, that I am permitted to present a Treatise of Mechanics to the public, under your Lordship's protection: and I have been doubly solicitous that the work which you have thus been pleased to encourage, may not be found altogether unworthy the approbation of such a patron.

When I reflect upon your lordship's eminent and accurate acquaintance with many of the subjects discussed in the following sheets, my diffidence might be justly alarmed; but I throw myself upon the candour and kindness I have so often experienced: and, though I am not vain enough to imagine that my undertaking will be found free from faults and errors, or that these can escape your notice; yet I have the satisfaction to know, notwithstanding, that whatever will stand the test of a correct and solid judgment, or may be found calculated to enlarge the stock of human knowledge, or more widely to diffuse an acquaintance with the principles of a branch of science of acknowledged utility, will not fail to receive your lordship's commendation and support.

Allow me, my lord, publicly to thank you for the numerous favours you have conferred upon me in different periods of my life, and to subscribe myself, with unfeigned respect, and the sincerest gratitude,

My lord,

Your lordship's most obliged

and most obedient Servant,

OLINTHUS G. GREGORY.

PREFACE.

IT has of late been a too prevailing opinion in this country, that a man may become celebrated as a natural philosopher, and be very successful in the application of his knowledge to practical purposes; particularly in the construction of machinery; while he is completely unacquainted with the principles of mathematics. Among a variety of unpleasant consequences which have resulted from this erroneous opinion, may be reckoned, the rapid decline of the mathematical sciences in Britain, the dissemination of superficial and vague notions on physical topics, and the absolute necessity of having recourse to foreign publications for profound and extensive information on those subjects, which we should have supposed, had not experience convinced us of the contrary, Englishmen would have been proud to cultivate; since they were first placed upon an unshaken basis in the Principia of our own countryman, the illustrious Newton. It has been asserted by a competent judge, Dr. Robison, and it is mortifying to reflect that the assertion is true, "that while the continent has supplied us with most elaborate and useful treatises on various articles in physical astronomy, practical mechanics, hydraulics, and optics, there has not appeared in Britain half a dozen treatises worth consulting for the last forty years." We have, it is readily admitted, many popular works on mechanics, hydrostatics, &c. in which the principles are exhibited, and their application to the useful purposes in life illustrated by examples; but since their authors have scarcely ever aimed at demonstration, those who rest satisfied with the information they can gather from these treatises, can at best acquire but a kind of second-hand knowledge of unproved principles loosely floating in the mind; and must generally hesitate in every case presenting a novel aspect, for want of that confidence in principles and results which the irresistible evidence accompanying the deductions of pure science alone can inspire. I would not be understood as intending to cast any censure upon the authors of familiar treatises on scientific subjects: I only wish it to be recollected that these are much fitter for children than for men; that they may be useful as introductory books for pupils, but of little or no immediate service in the active concerns of a more advanced state; and that, because they merely excite a thirst for knowledge which they cannot gratify. Nor would I wish to insinuate that there are not in our language some excellent works on the *theory* of mechanics; but I cannot help being apprehensive that most of these (chiefly on account of the small encouragement likely to be given them) are too confined in their object, and for that reason fail in being so beneficial to the public, as the talents of their authors might otherwise lead us naturally to expect.

For some years I have seen (or *thought* I have seen), and often regretted, that a forbidding distance and awkward jealousy seem to subsist between the theorists and the practical men engaged in the cultivation of mechanics in this country: and it is a desire to shorten this distance, and to eradicate this jealousy, that has been a principal stimulant in the execution of the following performance. I have by long habit, combined perhaps with early acquired prejudices, been much delighted with the investigations of theorists: but while I prize the deductions of sound theory as highly as any person, and rest as firmly upon them; yet am I desirous not to forget that, as all general principles imply the exercise of abstraction, it would be highly injudicious not to regard them in their *practical* applications as approximations, the defects of which must be supplied, as indeed the principles themselves are deduced, from experience. Habits of abstraction and theorizing may be carried to excess; and crude experience without reflection will never be productive of essential good. But, as an eminent philosopher * (for whose talents and virtues I entertain great respect) remarks, "Care "should be taken to guard against both these extremes, and to "unite habits of abstraction with habits of business, in such a

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* Professor Dugald Stewart; Elements of the Philosophy of the Human Mind, p. 251, &c.

" manner as to enable men to consider things either in general or " in detail, as the occasion may require. Whichever of these " habits may happen to gain an undue ascendant over the mind, " it will necessarily produce a character limited in its powers, and " fitted only for particular exertions.—When theoretical knowledge " and practical skill are happily combined in the same person, the " intellectual power of man appears in its full perfection, and fits " him equally to conduct with a masterly hand the details of ordinary business, and to contend successfully with the untried difficulties of new and hazardous situations. In conducting the " former, mere experience may frequently be a sufficient guide: " but experience and speculation must be combined together to " prepare us for the latter." 'Expert men,' says Lord Bacon, ' can execute and judge of particulars, *one by one;* but the *general* ' counsels, and the plots, and the marshalling of affairs, come best ' from those that are learned.'

Admitting the truth of these observations—and their truth I think cannot well be denied—it will thence follow, that theoretical and practical men will most effectually promote their mutual interests, not by affecting to despise each other, but by blending their efforts: and further, that an essential service will be done to mechanical science, by endeavouring to make all the scattered rays of light they have separately thrown upon this region of human knowledge converge to one point. In conformity with these ideas, I have undertaken a task, attended, I am aware, with so many difficulties, that even failure will not be disgraceful; while, on the other hand, success, if my labours should fortunately be crowned with it, will be doubly gratifying; as I shall then flatter myself with having contributed in some measure to the union of accurate theory and judicious practice, and thus, ultimately, to the promotion of arts and manufactures.

The Treatise of Mechanics I now presume to lay before the public is comprised in two volumes, besides a volume of plates. Of these, the first volume is devoted chiefly to the theory: the second is practical and descriptive. The theoretical part is divided into five books, and these are subdivided into chapters, as the nature of the several subjects seemed to require.

Book I. is appropriated to the subject of STATICS, and is subdivided into six chapters. It commences, as indeed do all the books, with definitions and preliminary remarks; which are succeeded by a statement and illustration of the Newtonian laws of motion and rest, which are assumed as fundamental principles to guide our subsequent enquiries. The nature of statical equilibrium, with the composition and resolution of forces, are then discussed; forces being considered, 1. as disposed in one plane, and concurring in the same point: 2. as directed to one point, but not confined to one plane: 3. as situated in one plane, but applied to different points of a body: 4. as not confined to one plane, and directed to various points of a body. This portion of the work occupies rather more space than has been commonly assigned to it in treatises of mechanics; but this circumstance will, I hope, be found to facilitate the application of the parallelogram of forces to the investigations which follow. The remaining chapters in this book treat of the centre of gravity, and the centrobaryc method; the simple machines, or, as they are usually called, the mechanical powers; the strength and stress of timber and other materials; and the equilibrium, tension, and pressure of cords, arches, and domes. The minuter topics connected with these general points of enquiry cannot well be specified here.

The second book relates to DYNAMICS. It comprehends six chapters; treating of, motion, uniform and variable; the descent and ascent of heavy bodies in vertical lines; the motion of projectiles, with observations on ricochet-firing; descents along inclined planes and curves; the vibrations of pendulums, and the curve of swiftest descent; central forces; the rotation of bodies about fixed axes, and in free space, with theorems relative to the centres of oscillation, gyration, percussion, spontaneous rotation, &c.; the physico-mathematical theory of percussion, nearly as first delivered in Don George Juan's Examen Maritimo; and the motion of machines, and their maximum effects; closing with some remarks shewing in what points of view machines ought to be considered by those who would labour beneficially for their improvement.

Book III. is devoted to the subject of HYDROSTATICS. It is divided into four chapters—on the pressure of non-elastic fluids; the determination of the specific gravities of solid and fluid bodies; the construction of hydrometers or areometers; the equilibrium, stability, and oscillations of floating bodies, with a particular reference to ships and canal boats; and, on the phenomena of attraction in capillary tubes. This book contains an extensive, and, I trust, correct table of specific gravities.

In the fourth book I have given some of the most useful propositions and theorems relative to HYDRODYNAMICS. But as this is a very complex and intricate subject, in which the most elaborate theory can do but little, independent of numerous experiments, it seemed prudent, rather than to delude the student with a display of mathematical accuracy where it is so far from being attained, to curtail the theoretical part with a view to the introduction of some valuable experimental results. This book comprises, then, only four chapters. 1. On the discharge of fluids through apertures in the bottom and sides of vessels, and on spouting fluids: 2. an account of experiments made by different philosophers (as Bossut, Venturi, Eytelwein, Young, Vince, &c.) on the discharge of water through apertures and tubes; and the practical deductions from those experiments: 3. on the effect of water upon the motion of water-wheels: 4. an account of Mr. Smeaton's experiments on water-wheels.

The fifth book, which terminates the first volume, is on PNEUMATICS, and contains six chapters. The subjects stated and discussed in this book are, the equilibrium of elastic fluids; the measurement of altitudes with the barometer and thermometer; the motion of air when the equilibrium of pressure is removed; the theory of air-pumps, and pumps for raising water; the resistance of fluids to bodies moving in them; and results of experiments on the resistance of fluids.

The second volume of this work, which, as before mentioned, is chiefly practical, commences with an introduction occupying about 80 pages, and comprising general remarks, rules, and directions,—on the construction and simplification of machinery; on rotatory, rectilinear, and reciprocating motions; on bevel-geer, and proportioning the number of teeth; on friction and the rigidity of cords, with the experiments of Vince, Coulumb, &c. and an example of the power of the capstan, allowing for friction and the stiffness of cords; on water and wind as movers of machinery, with Smeaton's rules relative to windmills; on the strength of fired gun-powder; on steam as a mover of machinery, with the theorems and results of Bettancount and Dalton; and, on the strength of men and horses, according to the best and most accurate observations. The remaining part of this volume is appropriated to the description of a variety of mechanical contrivances, in number exceeding an hundred and fifty. These are arranged alphabetically, that they may be consulted with most convenience; and several of them relate to machines and instruments which have never before been publicly described—and for accounts of which I am indebted to some of our most celebrated civil engineers, whose names will appear with the respective articles. A general enumeration in this preface would be tiresome, especially as the articles are separately mentioned in the Table of Contents to the second volume. It may, however, be proper just to observe, that the greater part of the contrivances there described are such as are highly useful in various arts and manufactures, and the most important purposes of human life: among them will be found descriptions of air-pumps, balances, bark-mills, barometers, bellows, clocks, cranes in great variety, file-cutters, fire-engines, flax-mills, flour-mills, foot-mills, hand-mills, hydraulic engines of various kinds, hygrometers, kneading-mills, lathes, and the most curious turning apparatus, locks, oil-mills, ordnance boring machines, contrivances for what are called parallel motions, pendulums, pile-engines, planing machinery, presses, and pressure-engines, several pumps, pyrometers, Ramsden's dividing machines, saw-mills, scapements, steam-engines, curious and accurate steel-yards, stream-measurers, telegraphs, thermometers, thrashing-mills, tide-mills, watches, water-mills, weighing apparatus, wind-mills, yarn-mills, &c. In some instances where the subject seemed very important, and not admitting of sufficiently copious discussion, among so many other particulars, I have inserted catalogues, arranged chronologically, of the chief writings on that respective subject, whether smaller dissertations, or full and extensive treatises: these catalogues may be seen under the words *clock*, *fire-engines*, *hydraulic-engines*, and *mills*. In a few cases too, where the theory of a machine had not been correctly exhibited in any English work with which I am acquainted, I have given such theory in the second volume, for the sake of more readily deducing the most useful practical results: this has been done with respect to *Barker's mill*, the *Persian-wheel*, and *Archimedes's screw*.

In the composition of the first volume of this treatise, I have derived material assistance from the labours of several of my predecessors in this department of science; though I have not, perhaps, so frequently cited my authorities as some readers may be apt to expect: but this will not, I trust, on consideration, be thought a culpable omission; for, although I have not, for example, ascribed to Prony what I found in succession in the writings of Varignon, Belidor, Bezout, and D'Alembert, nor to Parkinson, or Atwood, what had previously appeared in the writings of Galileo, Wilkins, Wallis, Desaguliers, or Emerson, esteeming whatever I found, in such circumstances, as common property, to be adopted without hesitation; yet, in all cases where I could speak confidently of the *original* author, and particularly where the matter quoted had been but seldom published, I have not failed to make the corresponding reference. As to the second volume, it is professedly a compilation; and I have no other merit to claim respecting it, than that of having employed much labour and pains in consulting a great many volumes of journals, transactions, acts, encyclopedias, theatres of machines, &c. published in England, France, and Germany, and having selected from these numerous and often voluminous works, such particulars as were most likely to be serviceable to my countrymen, when presented to them (separate from every thing extraneous) in a moderate sized single volume.

In the descriptive part I have but seldom given accounts of machinery for which the inventors have taken out *patents*. In this measure I have been guided solely by views of utility: when a patent machine appeared to exhibit some peculiar ingenuity in its construction, or when the specification developed some general methods which are not merely serviceable with respect to the inventions to which they are now applied, but may after the expiration of the patent be advantageously appropriated to other machinery, I have thought it right to insert a description of such contrivance; taking care at the same time to mention the exclusive right secured for the term of the patent. An indiscriminate account of what are called *inventions* in this fruitful age for the revival of old contrivances, would have added to the bulk of the work; but it must necessarily have operated as a dead weight to sink its respectability.

It may, probably, be imagined by some, that considering the advantages for that purpose, with which my situation at Woolwich furnishes me, I should insert descriptions of the various kinds of machinery used in the *artillery* service. This, however, I have not done: chiefly because such descriptions, by Muller and other respectable authors, are already easily procured in works appropriated exclusively to that important object, and where the accounts are given with far greater copiousness than they could be in a performance where so many other subjects are handled.

In the exposition of the theory I have generally proceeded by a series of connected propositions and dependent corollaries; those in each chapter having a manifest relation to each other, and flowing naturally from the same source: and I have at the same time endeavoured to attain a just medium between that fatiguing prolixity of detail which leaves nothing to be struck out by the ingenuity of the pupil, and that obscurity which commonly results from the suppression of intermediate ideas. I have, therefore, occasionally introduced such general scholia and more limited remarks, as seemed calculated to clear up a point in the history, or a doubt in the principle, or to suggest an application: and it was my first intention to have these printed in a different type from the propositions and corollaries; but I relinquished that design, on finding that it would much enhance the price of these volumes, a circumstance which I have been all along solicitous to prevent.

I fear many persons who may take up this work will be dissatisfied with it on account of the frequent introduction of the fluxional analysis into the theoretical part: this dissatisfaction is engendered in some by a prejudice to which I have already adverted in the beginning of my preface; and in others, by an undue attachment to the geometry of the ancients. I would beg the attention of the latter class of objectors to a few observations. I am very ready to allow that in the study of geometry, the diagrams to which our attention is directed serve as a continual check upon our reasoning powers; they, besides, exhibit to the senses a variety of relations which the language of algebra by no means suggests, and which it is perhaps too general to express: still, as every geometrical investigation *may* be expressed algebraically, and as, moreover, there have been excellent geometricians who were blind and never *saw* a geometrical figure; the circumstance of the diagrams does not constitute so essential a difference between geometry and algebra as some may be apt to suppose. In many physical investigations, it is true, where a geometrical process can be applied, it carries with it a degree of luminousness certainly not to be exceeded by a process conducted according to the modern analysis; and it must further be acknowledged, that this analysis, without a cautious exercise of the judgment, and a frequent recurrence to first principles, may terminate in an absurd conclusion: yet, on the other hand, it must be recollected, that physical enquiries conducted geometrically by the most eminent men, have sometimes led to very erroneous results; and, that in numerous cases, the geometrical method can be applied with difficulty, if at all, in which an investigation carried on by the modern analysis, has issued in some very important discoveries. If, with regard to the fluxional calculus in particular, it be recollected that it was invented by *an Englishman* who will be the immortal honour of his country, and that this is neither the least brilliant nor the least valuable of his discoveries, it cannot but be considered as singular, that many among his countrymen, who profess themselves the friends of science, should discountenance the introduction of his analysis into our scientific works. Philosophical men, when contemplating the progress of knowledge, have been delighted to remark that "the discoveries which in one age were confined to the studious and enlightened few, become in the next the established creed of the learned; and, in the third, form part of the elementary principles of education:" how completely different from this the progress must be, when the discoveries of a century back are disregarded, not because of their being useless or obsolete, but because of their *difficulty*, is too palpable to need more than an allusion, and cannot be dwelt upon without regret. If what has been here advanced should not be thought sufficient to justify the occasional adoption of the fluxional analysis (I do not contend for the exclusive use of it), let it be recollected that, in our best treatises on mensuration, guaging, &c. it is allowed, and for this sufficient reason, because such treatises would be very deficient, imperfect, and confined in their utility without it: unless, therefore, a similar occasion will not justify a similar mode of proceeding in a treatise of mechanics, I shall hope to be excused, even by those I am now addressing, when it is considered that some of the most important branches of mechanical enquiry, and especially great part of what relates to the *maximum* effects of machines, must either be entirely omitted, or treated by the fluxional calculus, or some modern calculus analogous to it. In consistency with the preceding observations, I have followed a kind of mixed course, and have deduced the chief truths commonly developed in our elementary books of mechanics, either after the geometrical manner, or by introducing the algebraical notation in aid of the deductions from geometry; and in the more curious and abstruse enquiries which occur in various parts of the work, I have had recourse to the fluxional analysis, not for the sake of displaying my mathematical attainments (being well aware how scanty those attainments are), but in order to render the treatise in some degree complete: thus giving it such latitude and comprehensiveness, that those who consult it may draw from it some such notions of the science of mechanics in general as may be learned respecting an existing structure, by examining a model shewing its several parts in due proportion.

It may happen, after all, that the arguments adduced in favour of the method I have chosen may fail in their impression, on account of their not being stated with sufficient force: let me then fortify my opinion, and shelter this part of my plan from censure by high mathematical authority. Mr. *Simpson*, in the preface to his TRACTS, speaking of the problems in mechanics and physical astronomy, says "In treating of this subject I have chiefly adhered to "the *analytic method* of investigation, as being the most direct and "extensive, and best adapted to these abstruse kinds of speculations. "Where a *geometrical* demonstration could be introduced, and "seemed preferable, I have given one: but, though a problem "*sometimes*, by this last method, acquires a degree of perspicuity "and elegance not easily to be arrived at any other way; yet I can"not be of the opinion of those who affect to shew a dislike to "every thing performed by means of symbols and an algebraical "process; since, so far is the synthetic method from having the "advantage in all cases, that there are innumerable enquiries into

" nature, as well as in abstracted science, where it cannot be at " all applied to any purpose. Sir Isaac Newton himself (who per- " haps extended it as far as any *man* could) has, even in the most " simple case of the lunar orbit (*Princip.* b. 3. prop. 28.), been " obliged to call in the assistance of algebra; which he has also " done in treating of the motion of bodies in resisting mediums; " and in various other places. And it appears clear to me, that " it is by a diligent cultivation of the *modern analysis* that *foreign* " *mathematicians* have, of late, been able to push their researches " further in many particulars than Sir Isaac Newton and his fol- " lowers here have done: though it must be allowed, on the other " hand, that the same neatness and accuracy of demonstration is " not every where to be found in these authors; owing, in some " measure, perhaps, to too great a disregard for the geometry of " the ancients."

When treating of the theory of mechanics, I have not attempted to explain the *nature* of gravity, impulse, or the other sources of the motion of bodies: but this, I hope, will not be considered as a defect, when it is recollected that the general definition of the term *force* in a mechanical sense, supersedes the necessity of enquiring into the essence of the various kinds of forces which may operate upon matter. Those who carefully contemplate the process of that gradual refinement of language, which results from the necessary demands occasioned by the progress of civilisation, will see how requisite it is to appropriate terms originally of a laxer or of a grosser signification, to some peculiar modification of thought; and hence, that such words as *power* and *force*, primarily used to denote animal energy, are now, by a natural extension, grounded upon an obvious analogy, employed to express *efficiency* in general. In the philosophical acceptation, then, we define *force or power to be that,* whatever it be, *which causes a change in the state of a body, whether that state be rest or motion:* and this definition does not require entering into any metaphysical disquisitions relative to the nature of causes, or the connection of cause and effect: that every effect is brought about by some cause, or something which precedes in the order of occurrence, is a truth which I think none will be disposed to deny; but what is the agency, or where it actually resides, we can seldom know, except perhaps in the case of our own voluntary actions. It is not, then, the business of the mechanist, strictly speaking, to enquire into the *modus operandi:* we learn from universal experience, that the muscular energy of animals, the operation of gravity, electricity, pressure, impact, &c. are sources of motion, or of modifications of motion; and hence, without pretending to know the essence of either of these, we do not hesitate to call them mechanical forces; because it is incontrovertible that bodies exposed to the free action of either, are put into motion, or have the state of their motion changed. Forces, therefore, being known to us only by their effects, can only be measured by the effects they produce in like circumstances, whether those effects be creating, accelerating, retarding, deflecting, or preventing motions: and it is by comparing these effects, or by referring them to some common measure of ready appreciation, not by ascertaining the essential nature of any forces, that mechanics is made one of the mathematical sciences.

Besides, what is meant by the *nature* of any thing? As we are ignorant of its essence, or what makes it that thing and no other thing, we must content ourselves with the discovery of its qualities or properties; and it is the assemblage of these which is commonly called its *nature:* yet this is very inaccurate, since these are only the *consequences* of the essence. Hence, we can give no definition of even the *simplest* of things which comprehends its real essence; and it is justly observed by Locke *, that *nominal essences* only, which are the creatures of our own minds, can be properly comprehended by us, or can be properly defined; and even of these there are many too simple in their nature to admit of definition. The various properties of a body which fall under our observation, offer themselves to our notice simultaneously; while we have the power of an intellectual separation, by which we can form different branches, each constituting a particular object of research: thus, the extension, the materiality, the impenetrability, the colour, &c. are coexistent qualities of a body; but our minds being too limited to examine these all at once, we divide and assort them, and consider them singly, or in kindred groups; hence arise the sciences of geometry, mechanics, chromatics, &c. It would seem, then, that to the weakness of our reason is to be ascribed the origin of the sciences: and this, if it is not very gratifying to our pride, may teach us at least to moderate our expectations when we attempt to investigate those ultimate questions, which, sooner or later, occur in all objects of human enquiry. I trust it will not be supposed that I mean to discourage an examination of physical and metaphysical problems; on the contrary, I am of opinion, that by looking attentively into such subjects, our minds are opened and enlarged: an enquiry, properly conducted, will lead to at least one of two good effects; though it should not make us more knowing, it may make us modest; though it may not preserve us from error, it may from the *spirit* of error; and may teach us to be aware of dogmatism, when so much labour may terminate in so much uncertainty.

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* This great philosopher, so far from conceiving, as many of the moderns have done, that the nature of a mechanical operation or event was sufficiently explained by reducing it to some supposed case of impulsion, considered even the connexion between impulse and motion as inexplicable, as a *conjunction* gathered solely from experience, and not as a consequence deducible from any reasoning *à priori.* The following passages on this subject are highly deserving of the attention of the student. " Another idea we have of body, is the power of " communicating motion by impulse; and of our souls, the power of exciting " motion by thought. These ideas, the one of body, the other of our minds, " every day's experience clearly furnishes us with: but, if here again we enquire " how this is done, we are equally in the dark. For, in the communication of " motion by impulse, wherein as much motion is lost to one body as is got to the " other, which is the ordinariest case, we can have no other conception but of " the passing of motion out of the one into another; which, I think, is as ob- " scure and inconceivable as how our minds move or stop our bodies by thought, " which we every moment find they do." Again: " Sensation convinces us " that there are solid extended substances; and reflection, that there are think-

The preceding observations may serve to suggest the advantage of a mathematical process, in which we have a clear perception, not of the things themselves, but of the agreement or disagreement of our ideas at every step of the enquiry, over other processes in which there is much that is conjectural, or gratuitously assumed, and in which a mass of error may be produced in the result, propor-

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" ing ones: experience assures us of the existence of such beings; and that the " one hath a power to move body by impulse, and the other by thought. If " we would enquire further into their nature, causes, and manner, we perceive " not the nature of extension clearer than we do of thinking. If we would ex- " plain them any further, one is as easy as the other; and there is no more dif- " ficulty to conceive how a substance we know not should by thought set body " in motion, than how a substance we know not should by impulse set body into " motion." Locke's Essay, book ii. chap. 23.

tioned to the length to which the chain of reasoning is extended: although in the mathematics the relations of quantity are so various, yet, as they are all along susceptible of exact mensuration, these, if any, are the sciences in which reason may triumph, while in others its trophies are comparatively inconsiderable. On these accounts I have been careful in the composition of the following sheets, to associate nothing with the theory of mechanics but the results of important experiments, and the descriptions of ingenious and useful machinery; omitting all those discussions which, in my judgment, appeared to be merely conjectural and speculative.

Hitherto I have been using the language of justification, with which I have, unawares, detained the reader much longer than I at first intended: it is time to adopt the tone of apology. In the execution of this work, I have experienced the wide difference between forming a project and accomplishing the object I had in view: although I have sometimes pleased myself with imagining I have struck out a more satisfactory illustration, or a more convincing demonstration, than has been usually given, I have too frequently found it necessary to rest below that perfection which I had previously fancied within my reach. I am conscious of some deficiencies and of some mistakes in arrangement, especially in the second volume, which were discovered too late to be now rectified. I hope, however, none of them will be found of much consequence. Other mistakes, it may be apprehended, will be detected by the acute reader; I shall rejoice if these are but few and of slight importance. Every exertion has been made on my part to render the performance correct, perspicuous, and useful; and if the *candid* examiner should find, that in the main it possesses these properties, I shall hope to escape severe censure for those imperfections which my want of abilities, not want of care, may have permitted to meet the eye of the public.

Royal Military Academy, Woolwich.
December, 1805.

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MECHANICS.

BOOK I.

STATICS.

Introductory Definitions and Remarks.

1. MECHANICS, the ſubject of this work, is a mixed mathematical ſcience, which treats of forces or powers, and their effects upon bodies, either with or without the intervention of machines.

The *theory* of Mechanics, next to that of Geometry, is juſtly reckoned the moſt certain; for next to that it propoſes the feweſt properties of matter as objects of contemplation. Generally ſpeaking, this ſcience propoſes for conſideration, *time*, *force*, or *power*, and the following properties of bodies: viz. *extenſion*, *figure*, *impenetrability*, *maſs*, *mobility*, and *inertia*. The application of this ſcience to the various practical purpoſes in human life, leads to many other branches of enquiry; ſuch as, the nature of machinery, the advantages, and diſadvantages, of different materials, the effects of friction, &c. Each of which will be brought under diſcuſſion, after the ſeveral parts of the theory are laid down.

2. *Matter* is a term by which we denote that ſubſtance of which every thing our ſenſes perceive is imagined to be compoſed. So far as relates to Mechanics, its eſſence may be regarded as conſiſting in *extenſion*, *impenetrability*, and *inertneſs*.

3. *Body* is ſuch a collected quantity of matter as is palpable, or obvious to ſome of the ſenſes. We ſay that a body is *ſolid*, when it is compoſed of particles, or moleculæ, ſo adhering the one to the other that they cannot be ſeparated without effort: ſuch are metals, ſtone, wood, &c. The term *fluid* we apply to ſuch ſubſtances as are compoſed of particles adhering very ſlightly, and which, yielding to any ſmall effort, are eaſily moved among each other: ſuch are water, beer, air, &c.

All bodies, it is manifeſt, are extended, and therefore are found exiſting under figure, or ſhape, which is the boundary of extenſion. It appears alſo eſſential to matter, that it prevents all other ſubſtances *of the ſame kind* from occupying its place; and that it requires the exertion of ſomething *ab extra* to remove it from its ſituation, or to change its ſtate. The former of theſe properties is called *Solidity* or *Impenetrability;* the latter, *inertneſs* or *inertia;* of which we ſhall have occaſion to ſay more. (18.)

4. All bodies are *diviſible* either in reality or in imagination. By art they may be divided into parts of ſurpriſing minuteneſs; as by the operations of grinding, hammering, wiredrawing, &c. In ſome chemical ſolutions this may be carried farther ſtill. And the conſtituent particles of the ſolids and fluids of the myriads of animalculæ which are diſcovered by the microſcope, muſt be ſmall beyond conception: ſo that we can ſcarcely help imagining the capacity of diviſibility to be without limit. What, however, are called mathematical demonſtrations of the *infinite* diviſibility of matter, fail in their object: they merely prove the infinite diviſibility of extenſion, in a geometrical ſenſe. It is probable that, with reſpect to matter, the actual diviſion may admit of being carried to a different extent in different bodies: but the inquiry is of little, if any, conſequence to the mechanician. Thoſe who wiſh to peruſe a copious diſſertation on this topic are referred to *Keil's Introduct. ad Ver. Phyſic. Lect.* 3, 4, *and* 5.

5. *Space*, is commonly defined as, the order of things which coexiſt: in this ſenſe it is a mere abſtract idea, ariſing from our notion of the actual or poſſible ſituation of things amongſt themſelves. We rather choose to call *ſpace* an extenſion conſidered as without bounds, immoveable but penetrable by matter: this may be regarded as *abſolute ſpace.*

6. *Relative Space* is that variable dimenſion or meaſure of abſolute ſpace which our ſenſes define by its relation to bodies within it.

7. *Place*, or *abſolute place*, is that part of infinite ſpace which a body poſſeſſes. *Relative place* is the ſpace a body occupies conſidered with relation to other objects.

8. *Mobility* is a property of body, by which it is capable of being transferred from one place to another, or of exiſting in different parts of ſpace.

9. All bodies are porous: whence, together with the extreme minuteneſs of their particles, it happens, that fluids will inſinuate themſelves into all bodies; that ſometimes a mixture of two fluids will be leſs in bulk than when they are ſeparate; and that the ſame bulk may contain different quantities of matter, or *maſſes.*

10. *Denſity*, ſtrictly ſpeaking, denotes vicinity or cloſeneſs of particles. But in mechanical ſcience it is uſed as a term of compariſon expreſſing the proportion of the number of equal moleculæ, or the quantity of matter in one body, to the number of

equal moleculæ in the ſame bulk of another body: denſity, therefore, is directly as the quantity of matter, and inverſely as the magnitude of the body.

Since it may be ſhewn experimentally that the quantity of matter in different bodies is proportional to their weight, of conſequence, the denſity of any body is directly as its weight, and inverſely as its magnitude: or, the inverſe ratio of the magnitude of two bodies, having experimentally equal weights (in the ſame place) conſtitutes the ratio of their denſities.

11. *Motion* is a ſimple idea, and therefore admits not of definition: when we ſay that it is a continual and ſucceſſive change of place, we deſcribe it in a periphraſis, by its ſenſible effects. Or, by another circumlocution, motion may be deſcribed as that ſtate of a body which is not conſiſtent with its continuance in the ſame place; or in which it is not, in two ſucceſſive inſtants of duration, at the ſame diſtance from divers fixed points in ſpace: this ſtate is oppoſed to that of *reſt*.

Thus, conceiving in ſpace three planes in known and fixed poſitions, not parallel to each other; if from any material point perpendiculars are demitted on theſe three planes, we ſay the point is in motion when it does not conſtantly retain theſe diſtances, but when in *any* two ſucceſſive inſtants, one of theſe perpendiculars changes its magnitude.

12. The motion of bodies is conſidered either as *abſolute* or *relative*. A body is ſaid to be in *abſolute motion* while it is actually paſſing from one point in fixed ſpace to another; and to be in *relative motion* while its poſition is varying with reſpect to other bodies.

It is obvious that theſe two kinds of motion can only coincide when the bodies, to which the reference is made, are fixed: in other caſes a body in relative motion may or may not be in abſolute motion. The determination of the abſolute motions, by means of obſervations on the relative motions, is always a matter of great difficulty, nay, is generally abſolutely impoſſible. Thus, when a ball is diſcharged from a piece of ordnance, it is poſſible, by means of the balliſtic pendulum, and other contrivances of ingenious men, to aſcertain its relative motion, that is, its motion with reſpect to that place on the earth's ſurface from which it is projected; but in order to determine its abſolute motion, the diurnal and annual motions of the earth about the ſun, and probably the motion of that luminary about the centre of ſome more extenſive ſyſtem, muſt be taken into the account; ſo that on the whole this apparently ſimple enquiry becomes ſufficiently complex to baffle the proudeſt efforts of human intelligence.

13. The conſideration of motion neceſſarily involves that of *time;* for no motion can be inſtantaneous.

14. *Abſolute time* is a portion of duration whoſe quantity is only known by a compariſon with another portion; and conſequently the relation between any two parts of abſolute time is not to be diſcovered. *Relative time* is a part of duration which elapſes during any motion of body, or any ſucceſſion of external appearances.

There is a ſtriking analogy between the affections of ſpace and time: hence it is, that time may be repreſented by lines and meaſured by motions. Hence alſo, we ſay that an *inſtant* is the boundary between any two contiguous portions of time, as a point is the boundary of any contiguous lines. A *moment* is any ſmall portion of time. To render time ſuſceptible of mathematical diſcuſſion, it muſt be conceived as meaſureable; and to this end it is neceſſary to recur to ſome event which we imagine uniformly requires equal times for its accompliſhment. We are furniſhed with ſuch an event in the complete rotation of the earth upon its axis, which marks out a natural day as an apt and obvious *unit* of time. This is divided into 24 equal parts called hours, each of theſe into 60 equal parts called minutes, and each of theſe again into 60 equal parts called ſeconds. A *ſecond* is the unit of time generally employed in mechanical diſquiſitions.

15. *Velocity,* or *celerity,* is that affection of motion which determines its quantity: it is the name expreſſing the relation between the ſpace deſcribed by a moving body and the time which elapſes during its deſcription; and it is meaſured by the ſpace *uniformly deſcribed* in a given time.

A body is ſaid to move with a *uniform, accelerated,* or *retarded* velocity, according as its rate of motion continues the ſame, increaſes or decreaſes: when the increaſe or decreaſe of velocity is the ſame in any equal times, the acceleration or retardation is ſaid to be uniform: and when this increaſe or decreaſe of velocity itſelf increaſes or decreaſes in any equal times, the acceleration or retardation increaſes or decreaſes in the ſame ratio. Theſe circumſtances will be brought more fully into conſideration as we proceed.

16. The *direction of a motion* is the poſition of the line along which it is performed: thus if a body move from a point A to another point B along the ſtraight line which joins theſe points, AB is called the direction of the body: if the body move from B to A along the ſame right line, BA is its direction. If a body move along a curve line, its *direction* is continually changing; it may, however, in any given point be regarded as coinciding with the tangent to the curve at that point.

17. *Force* or *power*, in a mechanical ſenſe, is that which cauſes a change in the ſtate of a body, whether that ſtate be reſt or motion.

We ſpeak here of proximate cauſes, for it is not the buſineſs of mechanics to ſearch into the eſſential and hidden cauſes of motion. The enquiry whether they are material or ſpiritual may exerciſe the talents of ingenious ſpeculatiſts, and may, perhaps, be of ſome importance in a moral point of view; but certainly forms no part of the principles of mechanical ſcience. The muſcular power of animals, as likewiſe preſſure, impact, gravity, electricity, &c., are by us looked upon as forces, or ſources of motion; for it is an incontrovertible fact that bodies expoſed to the free action of either of theſe are put into motion, or have the ſtate of their motion changed. All forces, however various, are meaſured by the effects they produce in like circumſtances; whether the effects be creating, accelerating, retarding, or deflecting motions: the effect of ſome general and commonly obſerved force is taken for unity; and with this any others may be compared, and their proportions repreſented by numbers or by lines: in this point of view they are conſidered by the mathematician; all elſe falls within the province of the univerſal philoſopher or the metaphyſician. When we ſay that a force is repreſented by a right line AB it is to be underſtood that it would cauſe a material point ſituated at reſt in A, to run over the line AB (which we name the *direction* of the power) ſo as to arrive at B, at the end of a given time; while another power ſhould cauſe the ſame point to have moved a greater or leſs diſtance from A in the ſame time.

18. Among other forces it has been cuſtomary to ſpeak of the *vis inertiæ*, or *inert force* of matter; applying the term to that property of bodies by which they tend to retain their preſent ſtate (3.), or are indifferent to motion or reſt. But while we admit that much of the language relating to powers, forces, actions, &c. is metaphorical, we muſt object to ſuch uſe of it in the preſent caſe; this property being improperly called a force: 1ſt. Becauſe were it actually ſuch it muſt be of ſome definite quantity in a given body, and therefore an impreſſed force leſs than that would not move the body; whereas any impreſſed force, however ſmall, will move any body however great. 2dly. Becauſe it ſeems to indicate an active power reſident in matter; or rather, it implies an abſolute contradiction, namely, that a body ſhould be both active and inactive at the ſame time. It is deſirable, therefore, that only the term inertia, or inertneſs, ſhould be retained: for this term will imply, as it ought to do, that matter is a merely paſſive thing. A fact which needs no laboured proof; for this inertia preſents itſelf immediately in all

our obſervations and experiments upon matter, and is inſeparable from it, even in idea. When we conſider any of the active powers of nature, as they are called, ſuch as gravitation, magnetiſm, electricity, or the attractions and repulſions which take place in the coheſions and ſeparations of the ſmall particles of natural bodies, and endeavour to reſolve them into ſome higher and ſimpler principles, the *inertia* is always the common baſis upon which we endeavour to erect our ſolutions. For the active party which is ſuppoſed to generate the gravitation, magnetiſm, &c. in the paſſive one, muſt have a motion, and inertia, whereby it continues in that motion, elſe it could have no power; and by parity of reaſon, the paſſive party muſt be inert alſo, elſe it could not re-act againſt the active party, nor impreſs motion on a foreign body. And this by the way, if the reader will pardon *one* ſlight digreſſion, ſuggeſts a brief but cogent argument for the immateriality of the Supreme Being. For, as the acute *Hartley* obſerves, "let us proceed as far as "we pleaſe in a ſeries of ſucceſſive ſolutions, we ſhall always "find an inertia inherent in matter, and a motion derived to it "from ſome foreign cauſe. If this cauſe be ſuppoſed matter "always, we ſhall be carried on to an infinite ſeries of ſolu-"tions, in each of which the ſame preciſe difficulty will recur, "without our at all approaching to the removal of it. "Whence, according to the mathematical doctrine of ultimate "ratios, not even an infinite ſeries, were that poſſible in this "caſe, could remove it. We muſt, therefore, ſtop ſomewhere, "and ſuppoſe the requiſite motion to be imparted to the ſup-"poſed ſubtle matter, by ſomething which is not matter; *i. e.* "ſince God is the ultimate author of all motion, we muſt ſup-"poſe him to be immaterial."

19. *Equilibrium* may be defined generally, as an equality of weights, powers, or forces, of any ſort. Bodies at *reſt* are in a ſtate of equilibrium, when they are ſolicited by various forces in different directions in ſuch a manner as to be completely balanced, and have no tendency to move in any direction. Bodies in *motion* are in a ſtate of equilibrium, when the reſiſtance to motion and the power producing it are ſo adjuſted, that the reſult ſhall be *uniform motion*. An accurate knowledge of both kinds of equilibrium is indiſpenſably neceſſary, in order that the theory may be applied to good practical purpoſes.

The equilibrium of bodies at reſt is ſometimes characteriſed in two different ways; equilibrium of *rotation*, and equilibrium of *tranſlation*: the former denoting that the tendency to a rotatory motion is counteracted; the latter, that the tendency to a rectilinear motion is prevented; and *reſt* in both caſes enſured.

20. We may now define ſomewhat more diſtinctly the

Science of Mechanics: it compriſes the doctrine of the reſt, the equilibrium, and the motion of bodies. It may be divided into *Mechanics*, properly ſo called, and *Hydraulics:* the firſt compriſes *Statics*, which has for its object the balanced reſt of ſolid bodies; and *Dynamics*, which conſiders the motion of ſolid bodies: The latter compriſes *Hydroſtatics*, which relates to the reſting equilibrium of nonelaſtic fluid bodies; and *Hydrodynamics*, which treats of ſuch bodies in motion. To theſe muſt likewiſe be added *Pneumatics*, which comprehends the doctrine of the weight, preſſure, and effects of elaſtic fluids, as *air*, &c. According to this diviſion we ſhall arrange the ſubjects of our firſt five books.

CHAPTER I.

Axioms, or Laws of Motion and Reſt.

21. IN order that the doctrine of Mechanics may be brought within the boundaries of mathematical inveſtigation, it is neceſſary not only that the quantities it propoſes for diſcuſſion ſhould be meaſurable either in themſelves or in their effects; but alſo that ſome general principles ſhould be exhibited, the truth of which ſhould be incontrovertible, and to which the ſtudent may at all times appeal in the courſe of his reſearches. Such general principles were firſt diſtinctly propoſed by Sir Iſaac Newton in his *Principia*; they have ſince his time been received as Mechanical Axioms, or, as they are commonly called, Laws of Motion; and are as follows.

I. Every body continues in its ſtate of reſt, or of uniform motion in a right line, until a change is effected by the agency of ſome mechanical force.

II. Any change effected in the quieſcence or motion of a body is in the direction of the force impreſſed, and is proportional to it in quantity.

III. Reaction is always equal and contrary to action; or, the mutual actions of two bodies upon each other are always equal, and directed to contrary parts.

Of theſe axioms, the firſt has reſpect to the continuance of bodies in a ſtate of repoſe or of motion, without *any alteration,* except ſo far as ſubſequent cauſes operate; the ſecond aſſigns the quantity and nature of ſuch alterations; and the third has reſpect to the mutual circumſtances of the patient, which ſuffers alteration from any cauſe, and of the agent producing the alteration. So that theſe principles, when eſtabliſhed, have a manifeſt tendency to facilitate the ſtudy of the ſcience in which they are propoſed, not as ſelf-evident truths, but as truths which reſult by legitimate induction from the teſtimony of our ſenſes.

22. The evidences from which our aſſent to theſe axioms is derived, are generally ſtated as threefold. 1ſt. From the conſtant obſervation of our ſenſes, which tend to ſuggeſt the truth of them in the ordinary motion of bodies, as far as the experience of mankind extends. 2. From experiments, properly ſo called. 3. From arguments *à poſteriori*. But beſides theſe it has been

ſuggeſted that they are, in fact, laws of human thought reſulting from the nature of things independent of experience. That the ſtudent may, without heſitation, yield his aſſent to theſe principles, it is thought expedient to ſtate briefly the nature of the different kinds of evidence thus afforded.

23. I. *From conſtant obſervation.* Here with reſpect to the firſt axiom, that a body at reſt cannot put itſelf into motion, is ſuppoſed to be known from univerſal experience: but that a body in motion would continue to move uniformly in a rectilinear path, though equally true, ſeems leſs apparent. Since, however, motion is continued the longer, in proportion as obſtructions become leſs, as a vibrating pendulum moves longer in air than in water, longer yet in an exhauſted receiver, it may be reaſonably inferred that, if all obſtacles could be removed, motion once communicated to any body would never ceaſe. The principle may be farther confirmed by a reference to common occurrences. Thus, when a ſtone is whirled round in a ſling, on being ſet at liberty, it will continue to move with the force it has acquired. Again, when a veſſel, containing a quantity of water, is moved along upon an horizontal plane, the water obeying its tendency to reſt, will at firſt riſe up in a direction contrary to that in which the moving force acts: when the motion of the veſſel is communicated to the water it will perſevere in this ſtate; and if the veſſel be ſuddenly ſtopped, the water by its inertia oppoſing the change from motion to reſt, it will riſe up on the oppoſite ſide of the veſſel. In like manner, if a horſe which was ſtanding ſtill ſuddenly ſtart forward, the rider will be in danger of being thrown backwards: if the horſe when in motion ſtops ſuddenly, the rider will be thrown forwards. In all theſe and numerous other inſtances which might be adduced there is ſtrong evidence that matter has a tendency to continue in its *preſent ſtate;* and this is what the firſt axiom aſſerts.

As to the ſecond, the inferences from experience are equally powerful. Thus a ball moving with a double or triple velocity, generates in another, by impulſe, a double or triple velocity, and the ball loſes the ſame proportions of its own velocity. Two bodies meeting with equal quantities of motion mutually ſtop each other. Two forces, which, by acting ſimilarly during equal times, produce equal velocities in ſome third body, are found, by acting together during the ſame length of time to produce a double velocity. If a new force be impreſſed upon a body in motion, in the direction in which it moves, its motion is increaſed proportionally to the new force impreſſed; if this force acts in a direction contrary to that in which the body moves, it is found to loſe a proportional part of its motion: if the direction of this force be oblique to the direction of the moving body, it gives it a new direction compounded of both. A force which we know to act equably produces equal increments of velocity in equal times. In all theſe examples we may trace a ſtrict conformity to the axiom.

The third principle is likewiſe obſervable in all the motions of nature, obtaining indeed throughout the whole ſolar ſyſtem with the utmoſt preciſion: all which will be obvious enough when it is well underſtood what this axiom affirms. In the communication of preſſure upon any immoveable plane, whether ariſing from the protruſion, gravity, or impact of a body, the meaning of this principle is, that the reſiſtance of the plane, and an oppoſite force equal to that producing the preſſure, have preciſely the ſame effect, as either of them only deſtroys the force of protruſion, gravity, or impact. In the communication of motion by viſible impact the axiom aſſerts that the quantities of motion loſt and gained are equal when eſtimated in oppoſite directions. In the communication of motion by unknown means, as magnetiſm, electricity, &c., it affirms that the body attracting or repelling moves in an oppoſite direction to that of the body attracted or repelled, and with an equal quantity of motion. Thus, to propoſe an inſtance in the caſe of attractions: When a loadſtone and a piece of iron, equal in weight, float in water upon equal and ſimilar pieces of cork, they are found to approach each other with equal velocities; and when they meet, or are kept aſunder by any obſtacle, they ſuſtain each other by equal and oppoſite preſſures. Again, when a force of any kind is counterpoiſed, whether by the exertion of animal ſtrength, the preſſure of a fluid, the force of a ſpring, &c. each of theſe oppoſed forces will be equal to ſome given weight, which if it be ſubſtituted ſo as to act inſtead of the force, will preſerve the equilibrium unaltered. Even an immoveable obſtacle exerts a definite force equal and contrary to that which is applied in any manner or degree to alter its poſition: for if ſuch obſtacle be taken away, a force equal and contrary to that by which it was before urged muſt be ſubſtituted in its place to keep the whole at reſt.

24. II. *Evidence from experiments.* Common experience, however, it muſt be admitted, is not entirely ſufficient to eſtabliſh the truth of theſe general principles: for, as we have already remarked (12.), we know not what is the actual motion of a body, or when it is abſolutely at reſt; ſo that thoſe obſervations which have been adduced as made upon bodies apparently at reſt, or in rectilinear motion, do not ſatisfactorily determine the point they are intended to eſtabliſh. Recourſe muſt, therefore, be had to experiments of a more refined and artificial

complexion, and ſuch may readily be found. Although we cannot appreciate the abſolute motions of bodies, we can obſerve and accurately meaſure their relative motions. For inſtance, if it can be ſhewn experimentally, that bodies ſhew equal tendencies to reſiſt the augmentation and the diminution of their relative motions, they, of conſequence, would exhibit equal tendencies to reſiſt ſuch augmentation or diminution in their abſolute motions: and numerous experiments, it is evident, might be conducted which ſhow this very clearly. And in like manner experiments might be contrived tending to eſtabliſh the other axioms: and it would undoubtedly be found that in proportion as all impediments were removed, the more perfectly would the experiments coincide with what theſe axioms might lead us to expect. Many experiments might eaſily be propoſed; but our buſineſs here is not to exhibit them, but to ſuggeſt the argument which flows from them.

25. III. *From arguments à poſteriori.* The conviction with which we are furniſhed in this manner is very cogent: for, as Mr. Atwood remarks, "Let a propoſition be aſſumed as true "even without evidence of any kind, if by ſtrict and logical "reaſoning, various concluſions are deduced, which upon exa-"mination are found conſiſtent among themſelves, and with "experience, this will be a preſumptive proof in favour of the "principle aſſumed; and our aſſent to it will be the more "ſtrongly enforced in proportion as the concluſions inferred "and the compariſon of them with experience have been more "extenſive. From the Newtonian axioms aſſumed as true, a "ſyſtem has been adduced and compared with phenomena in "numberleſs caſes: it has been applied to the motion of the "planets and comets, to that of bodies on the earth's ſurface; "even to the motion of thoſe minute particles which compoſe "both ſolid and fluid ſubſtances. A *perfect agreement* between "theſe conſequences of the axioms and matter of fact has been "the reſult, *no one inſtance excepted.* Theſe and other ſimilar "arguments, upon the whole amounting to evidence ſcarcely "inferior to mathematical demonſtration, are the grounds on "which the laws of motion are received as axioms, from "which the various theorems concerning the effects of forces "are ſyſtematically demonſtrated." *Treatiſe on Motion.* Page 360.

26. IV. Leſt the combined effects of the preceding arguments ſhould after all be inſufficient, let us enquire whether the truth of theſe general principles cannot be made evident, from the laws of human judgment independent of all experiment? Now with reſpect to the firſt axiom, let it be recollected that we know nothing of forces of any kind but from their effects,

and that we conſtantly infer the agency of a force from its changing the ſtate (as to motion or reſt) of bodies on which it acts. We ſuppoſe both reſt and motion to be *ſtates* or *conditions* of a body, from the one of which it is no more likely to be converted into the other *without cauſe*, than it is to change ſpontaneouſly from a cubical to a pyramidal or globular, or any other form. If this be admitted, as we truſt it will, it follows that a body *muſt continue in its exiſting ſtate of reſt or uniform motion, until that ſtate is changed by ſome mechanical force.* The matter has been ſtated a little differently, thus: "Suppoſe a body "in motion, and that the inſtantaneous action of the moving "force is adequate to make the body deſcribe a certain ſpace; "then, ſince after the firſt inſtant the moving force ceaſes, and "the motion ſtill continues, it muſt be uniform, becauſe the "body cannot of itſelf either accelerate or retard it; the mo-"tion will moreover be rectilinear, for there is no reaſon why "the body ſhould deviate to the right rather than to the left; "and hence, in this caſe, where the body is capable of moving, "during a certain time and independently of the moving cauſe, "the motion will be uniform and rectilinear. A body which "can move itſelf uniformly, and in a ſtraight line during a cer-"tain time will perpetually move after the ſame manner:—for, "ſuppoſe a body capable of deſcribing uniformly a ſtraight line, "of which the two extremities are A and B (Pl. I. fig. 1.); "between A and B take two points C and D, then the body at "D is preciſely in the ſame ſtate as when at C, excepting that "it is in a different place. Therefore the ſame ought to hap-"pen to the body as when at C: but, by hypotheſis, when at "C it can move itſelf uniformly to B; therefore when at D it "will be liable to move itſelf uniformly to a point G, taking "DG=CB; and ſo on for ever." *Monthly Rev.* N. S. vol. xxviii, pa. 316.

Again, with reſpect to the ſecond axiom, it may almoſt be conſidered as an identical propoſition, conſidering force as we do (17.), merely as the cauſe of motion, or of a change in motion: the law is in fact equivalent to ſaying that "we take the "changes of motion as the meaſures of the changing forces, "and the direction of the change for the indication of the "direction of the forces."

The third axiom is indeed a univerſal fact: but we conceive it is likewiſe a neceſſary truth. To aſſert the contrary is, we think, to maintain an abſurdity: for if action and reaction are not equal, the greater either acts againſt nothing and is therefore not action (contrary to hypotheſis), or exiſts without a cauſe, which if once admitted, we know not what can be denied.

27. Theſe axioms relate immediately to the actions of particles of matter upon each other, in free ſpace; or to thoſe caſes in which the whole maſs may be conceived as collected into a point.

When bodies move on fixed axes, the energy of the moving force and reſiſtance of the body moved will depend on the diſtance from the axis: in theſe caſes the inertia of the parts of the ſyſtem, by which they oppoſe motion, being calculated, and a maſs of equivalent inertia being ſubſtituted for the given ſyſtem; the quantity of the moving force when moving with the ſame velocity as the matter moved, is likewiſe computed. The moving force and maſs being thus aſcertained, the reſulting motions may be calculated, and it will thence appear, that in rotatory motions on fixed axes, or on moveable axes in free ſpace, as well as in direct rectilinear motions, action is always equal to reaction. But this is merely hinted by the way; it will be developed more fully when we come to DYNAMICS. (Art. 314, &c.)

CHAPTER II.

On Statical Equilibrium, and the Compoſition and Reſolution of Forces.

28. Equilibrium, according to our definition (19.), is of two kinds, the one relating to bodies at reſt, the other relating to bodies in uniform motion: the former may be properly called *Statical equilibrium*, and may, therefore, whenever referred to in our firſt book (whoſe object is Statics) be characteriſed by the term equilibrium alone; the latter may be called *Dynamical equilibrium*, and will be brought under conſideration in ſubſequent parts of the work.

29. Def. When two or more forces act upon a body, or upon a material point, at the ſame time, the aggregate effect produced by their ſimultaneous action, or a force of a certain magnitude and direction equivalent to their joint energy, is called the *Reſultant*, or the *Equivalent*; and the ſeveral forces of which this is compounded are called *Components* or *Compoſants*.

30. Def. The problem of the *Compoſition of Forces*, is that in which it is propoſed to find the *reſultant* of any given ſyſtem of forces: and the *Reſolution of Forces*, which is the converſe problem, propoſes to trace out the *compoſants*, which by their united energy might produce, or did produce, a given reſultant.

Thus, ſuppoſing two forces C, C′, whoſe magnitudes and directions are repreſented (17.) by CP, C′P, to act at the ſame time upon a material point P (Pl. I. fig. 2.); by the firſt of theſe problems we determine the magnitude and direction of the *equivalent* RP to theſe two forces; and by the latter we enquire the two forces CP, C′P, or various pairs of forces, of which the reſultant is RP. It may be proper to remark here that the Compoſition and Reſolution of *forces*, and the ſimilar Compoſition and Reſolution of *motions*, are completely diſtinct objects of enquiry: the former is entirely a phyſical queſtion, the latter a problem purely mathematical. Some authors have inferred from their demonſtrations of the latter problem, the truth of the former: but this cannot well be admiſſible, becauſe wherever ſtatical equilibrium obtains there can be no motion, and of courſe the principle on which the inference is grounded is foreign to the nature of the thing to be proved. It ſeems moſt conſiſtent with ſcientific preciſion, to eſtabliſh ſubjects ſo very diſtinct, each on its own proper baſis: and though the attempting to accompliſh this may ſometimes give an *appearance* of tautology, it is ſtill thought better than to aim at conciſeneſs by the ſacrifice of accuracy.

31. DEF. *The moment of a force* or power, is a term by which we ſhall denote the product of its magnitude into the neareſt diſtance of its direction (17.) from any fixed point.

Thus, in fig. 2. Pl. I. if the directions of the two powers C, C′, are at the reſpective perpendicular diſtances *d*, *d′* from a point S taken arbitrarily, then will the products C*d*, C′*d′* repreſent what we call the *moments* of thoſe powers. They expreſs the energies of thoſe powers, and differ only in the manner of application from momenta in Dynamics.

32. In every ſyſtem of an invariable form we may take for points of application of powers, any points whatever in the lines of their reſpective directions.

For the diſtances of the ſeveral points of the ſyſtem remaining conſtantly the ſame, no one of the points in the direction of any power can move without drawing into its motion all the points in the ſame line, nor of conſequence without cauſing ſuch motion as would be produced if the power were applied immediately at that individual point.

This propoſition may be illuſtrated by a ſimple and obvious example. If the body AB (fig. 5. Pl. II.) be acted upon by a force *puſhing* againſt it by means of an inflexible bar OQ, the effect upon the body will be juſt the ſame whether the force be exerted on the lever at o, g, or Q: and, if, on the contrary, the body were acted upon by a force *drawing* it by means of an inextenſible cord P E C, the tendency to move the body would be juſt the ſame, whether the cord were pulled at P, E, or C. Here we exclude the conſideration of gravity from the bar and the cord.

33. Hence, if an obſtacle be employed to deſtroy any force, it is ſufficient if ſuch obſtacle be applied at any point whatever in the direction of this force, provided the point be one of thoſe in a ſyſtem of unchanging form: ſo that we may in all ſuch caſes, without danger of error, conceive that the power is immediately applied to the obſtacle.

34. 1. Two forces acting in the ſame right line, and directed the ſame way, are equivalent to one force equal to their ſum.

2. Two equal forces acting in the ſame right line, but in oppoſite directions, deſtroy each other.

3. Several forces acting in oppoſite directions in the ſame right line are equivalent to one only, namely, to the exceſs of the ſum of the forces acting in one direction above the ſum of the forces acting in the contrary direction.

Theſe are obvious truths neceſſarily reſulting from the definition of force (17.), and Newton's third axiom (21.).

35. Since, then, a ſyſtem is in a ſtate of equilibrium when it is ſubject to the joint action of two equal forces exerted in oppoſite directions; a ſyſtem ſubjected to the united energy of various forces is in equilibrio, when any one of the forces is equal and oppoſite to the *reſultant* of all the reſt: and, of conſequence, the general problem of the equilibrium between various powers, is reduced to that of the *Compoſition of Forces* (30.). We ſhall, therefore, now proceed to the ſolution of this problem, and, for the ſake of perſpicuity, ſhall conſider it under four ſeparate heads, according to thoſe diviſions which the order of nature ſeems to preſent, viz.

1. Of forces diſpoſed in one plane, and concurring in the ſame point.
2. Of forces directed to one point, but not confined to one plane.
3. Of forces ſituated in one plane, but applied to different points of a body.
4. Of forces not confined to one plane, directed to various points of a body.

I. OF FORCES DISPOSED IN ONE PLANE, AND CONCURRING IN THE SAME POINT.

36. PROP. *The equivalent of ſeveral forces ſituated in one plane is in the ſame plane.*

For if we ſuppoſe the equivalent to be out of the plane of the forces, on either ſide, we may always find a line on the other ſide of the plane ſituated in a perfectly ſimilar manner; and ſince there can be no reaſon why the reſultant ſhould be in one of theſe directions rather than in the other: it is therefore in neither of them, unleſs we admit the abſurd conſequence that it is in both, that is, unleſs we admit that the ſame forces, acting in like manner, can produce two diſtinct effects.

COR. The reſultant of two equal forces muſt be in their plane; and it muſt be in a line which biſects the angle of their direction, ſince there is no reaſon why it ſhould tend more to one ſide than to another.

37. PROP. *If to a material point already kept in equilibrio by a ſyſtem of forces, another ſyſtem is applied alſo in equilibrio, this will not deſtroy the pre-exiſting equilibrium.* This is manifeſt.

COR. Hence, if the three forces C, C′, O (Pl. I. fig. 3.), are in a ſtate of equilibrium, and if each of the forces were doubled, or tripled, or quadrupled, or if they were halved, quartered, &c. or changed in any proportion, the equilibrium would remain, ſo long as they continued to act in the ſame directions, CP, C′P, OP.

COR. 2. Hence alſo, ſince the reſultant R is always equal and oppoſite to one of the forces (as O), it follows that when the magnitude of equilibrated forces concurring in a point are made to vary in any ratio, the reſultant retains its poſition but changes its magnitude in the ſame ratio.

38. PROP. *If three equal forces are inclined to one another in angles each of* 120 *degrees, any one of them will balance the joint action of the other two.*

This is likewiſe incontrovertible; for neither of the forces can prevail.

39. PROP. *Two equal forces inclined in an angle of* 120 *degrees, have for their equivalent a third which has the direction and proportion of the diagonal of the rhombus conſtructed on the lines which repreſent the forces.*

For, if C, C′, are the forces (fig. 3.) acting on the point P, the force O whoſe meaſure is OP=CP,=C′P, and is ſituated ſo that the angles CPO and C′PO are each equal to CPC′, will (38.) enſure the equilibrium. But RP, the meaſure of the equivalent R, is equal and oppoſite to OP (35.): therefore CP=PR=C′P; and becauſe angle CPR=60°=C′PR, CR=CP and C′R=C′P. Conſequently CPC′R is a rhombus, and RP the repreſentative of the equivalent of the forces C, C′, is its diagonal. Q. E. D.

COR. If half the angle CPC′ be denoted by a, we ſhall have PD=PC cos. a=C cos. a, whence the equivalent RP=2C cos. a.

40. PROP. *Any two equal forces have for their equivalent the diagonal of the rhombus conſtructed on the right lines which repreſent them in magnitude and direction.*

For, 1. If this propoſition be true with regard to any two equal forces C, C′, acting in the directions CP, C′P (Pl. I. fig. 4.), and forming with their reſultant R, the angles CPR, C′PR, each for example equal to a, it is true likewiſe for two other equal forces c, c', acting according to the directions cP, c'P, which biſect thoſe angles. In this caſe c may be conſidered (36 *cor.*) as the reſultant of two equal forces x and y, acting in the directions CP, RP; and in like manner c' may be conſidered as the reſultant of two other equal forces x' and y', acting in the directions C′P, RP: ſo that, in lieu of the two equal forces c, c', we may conſider four equal but unknown forces x, x', y, y', acting in the directions juſt aſſigned them. The two firſt of theſe, x, x', acting in the directions C P, C′P, have, by hypotheſis, the diagonal of the rhombus for their reſultant; that is, they are equivalent to a force expreſſed by $2\,x \cos. a$ acting in R P: therefore the reſultant z of c and c', will be equal to $2\,y + 2\,x \cos. a$. But $x = y$, therefore $z = 2\,x\,(1 + \cos. a)$. Now, the angles C P R, c P c', being each equal to half C P C′, are equal to each other; and z being the reſultant of two equal components acting in c P and c'P, while c is the reſultant of two others acting in C P and R P, we have (37, cor. 2.) $z : c :: c : x = \frac{c^2}{z}$. Subſtituting this value of x for it in the preceding equation, we obtain $z = \frac{2\,c^2}{z}(1 + \cos. a) = c \sqrt{2} \sqrt{1 + \cos. a} = c\sqrt{2 \times (1 + \cos. a)}$. But it is known that $\cos. \frac{1}{2} a = \sqrt{\frac{1 + \cos. a}{2}}$ (See *Hutton's Dictionary*, art. Sine, or *Carnot de Géométrie de Poſition*, p. 154.); whence, by reduction $z = 2\,c \cos. \frac{1}{2} a$. Conſequently the propoſition if true for a, is true for $\frac{1}{2} a$.

2. In exactly the ſame manner may the propoſition be proved true with reſpect to the half of $\frac{1}{2} a$, or $\frac{1}{4} a$, and in ſucceſſion for $\frac{1}{8} a$, $\frac{1}{16} a$, $\frac{1}{32} a$, &c. That is, ſince it is true (38.) when the angle C P C′ is meaſured by $\frac{1}{3}$ of the circumference, it is likewiſe true when the angle between the equal components is meaſured by $\frac{1}{6}$, $\frac{1}{12}$, $\frac{1}{24}$, $\frac{1}{48}$, &c. of the circumference, where the ſeries may be continued, *ſine limite*.

3. If the propoſition be demonſtrated for the three angles a, b, and $a - b$, it will be true for the angle $a + b$; that is, if we take two equal components c and c' making with their reſultant x, angles $= a + b$, we ſhall have $x = 2\,c \cos. (a + b.)$ Thus, if in Pl. I. fig. 5. the angles C P R, C′P R, are each equal to a, and c P C, C P d, c'P C′, C′P d', each equal to b: conceiving two forces d P, d P, each equal to c, their reſultant will, by hyp. be $= 2\,c \cos. (a - b)$, becauſe d P R $= a - b$; and this quantity ſubtracted from the reſultant of c, c', d, d', will give x. But c and d have their reſultant C, acting in C P, and $= 2\,c \cos. b$; the ſame thing holding with reſpect to c' and d', we have two forces equal to C, and equivalent to one which is $2\,C \cos. a$ or $4\,c \cos. a. \cos. b$: whence $x = 4\,c \cos. a. \cos. b - 2\,c \cos. (a - b)$. But $\cos. a. \cos. b = \frac{1}{2} \cos. (a + b) + \frac{1}{2} \cos. (a - b)$. (See *Crakelt's Mauduitt*, p. 50. *Hutton's Dict.* art. Sine, or *Carnot Géom. Pos.* p. 153.) Which value of $\cos. a. \cos. b$, ſubſtituted for it in the preceding equation gives $x = 2\,c. \cos. (a + b)$. So that the propoſition when true for a, b, and $a - b$, is true for $a + b$.

4. Let b be taken as ſmall as we pleaſe in the ſeries $\frac{1}{6}$, $\frac{1}{12}$,

$\frac{1}{24}$, $\frac{1}{48}$, &c. and let a be the preceding term in the ſeries, then a, b, $a-b$, $a+b$, are $2b$, b, b, and $3b$ reſpectively, in each of which the propoſition holds: again, if $a=3b$, $a+b=4b$; if $a=4b$, $a+b=5b$, &c. So that the theorem is demonſtrated for all angles in the ſeries b, $2b$, $3b$, $4b$, $5b$, &c. in which b may be taken of a magnitude leſs than any one which can be aſſigned. Conſequently the theorem is true with reſpect to any rhombus whatever: for let any rhombus be propoſed which it is affirmed is an exception to this proof; we can, it is obvious, by chooſing b lower than any aſſigned number, and taking a ſuitable multiple of it, approach nearer the excepted angle than by any aſſignable difference, that is, we ſhow that our theorem is applicable to the angle itſelf. Q. E. D.

41. PROP. *Any two forces having the ratio of the ſides of a rectangle, and whoſe directions coincide with thoſe ſides, have for their equivalent the diagonal of that rectangle.*

Let the two forces C, C′ (pl. I. fig. 6), act in the directions CP, CP′, which compriſe the right angle P: complete the parallelogram CPC′R, and draw its diagonals; parallel to CC′, draw cc' terminated by Cc, C′c', which are drawn parallel to the reſulting diagonal. Conceive c and c' to be two equal forces acting in the equal lines cP, c'P, oppoſite to each other, and conſequently annihilating each other's effects; then cPDC and c'PDC′ being rhombi, the force CP is the equivalent of cP, DP, and C′P that of c'P, DP, by the preceding prop. Therefore the components CP, C′P, are the ſame in effect as the oppoſite ones cP, c'P, together with DP, DP; that is, the equivalent ſought is 2DP or RP, the diagonal of the parallelogram. Q. E. D.

COR. Since RP : rad. :: CP : cos. CPR :: C′P : cos. C′PR, we have the reſultant equal to either component divided by the coſine of the angle which it makes with the reſultant.

42. PROP. *Any two forces whatever have their equivalent expreſſed in magnitude and direction by the diagonal RP of the parallelogram conſtructed on the lines CP, C′P, which repreſent theſe forces.*

Having completed the parallelogram CPC′R (fig. 7. pl. I.) on the given ſides, draw cc' perpendicular, and Cc, C′c', parallel to the diagonal, demit alſo CD, C′D′, perpendicular to the diagonal: then will CcPD, C′cPD′, be rectangles, and the triangles CRD, C′PD′, equal in all reſpects; conſequently Cc = DP, RD = D′P, and cP = c'P. The addition of the equal forces c, c', acting in the oppoſite directions cP, c'P, will make no difference as to the ſtate of the ſyſtem: and ſince the components D P, c P, have C P for their reſultant, and the components D′P, c'P, the reſultant C′P (by the preceding prop.), we may inſtead of the original forces C P, C′P, ſubſtitute the forces c P, c'P, D P, D′P, of which the two former deſtroy each other's effects, and the latter D P, D′P, are manifeſtly equal to R P; that is, the reſultant of the two forces C P, C′P, is equal to the diagonal R P of the parallelogram. Q. E. D.

43. Thus have we, by a ſeries of connected propoſitions, demonſtrated that which is juſtly reckoned the moſt important in the theory of Statics, and which is now commonly ſpoken of under the title of the *Parallelogram of Forces.* The demonſtration here given is commenced upon the ſame principle (38.) as that propoſed by *D'Alembert* in the Memoirs of the French Academy for 1769: it was ſomewhat ſimplified by *Francœur* in his Mechanics; but what is here offered, at the ſame time that it is more conciſe than the demonſtration of Francœur, is freed, it is hoped, from ſome objectionable poſitions into which that author has certainly fallen. This demonſtration would manifeſtly be the ſame in every ſtep, ſuppoſing the forces to act in the directions P C, P C′: ſo that the proof is general for the compoſition of any ſyſtems of forces whether preſſing, or impulſive, or attracting, or repelling, &c. Profeſſor *Robiſon*, by ingeniouſly blending the Demonſtrations of Bernouilli, D'Alembert, and Paul Friſi, deduces the ſame concluſion by a ſeries of propoſitions commencing with the caſe of two equal forces acting at right angles. See art. DYNAMICS. Supp. Ency. Britan. The demonſtrations by Laplace and Lagrange, though confeſſedly curious, depend upon principles too abſtruſe to be admitted, without impropriety, into this work. In order to facilitate the application of the Parallelogram of Forces, it will be proper to ſpecify the moſt uſeful propoſitions and corollaries which immediately flow from it.

44. COR. 1. *If two component forces are repreſented by the two ſides of a triangle, the reſultant will be repreſented by the double of the line drawn from the point where the forces act, to the middle of the third ſide of the triangle.*

Thus, in figs. 3, 6, 7. the reſultant P R = 2 P D; becauſe the diagonals of the parallelogram C P C′R biſect each other.

45. COR. 2. *If the angle at which two given forces act be diminiſhed, their reſultant will be increaſed, and* vice verſa.

For in the triangle C P C′, in which the line P D biſects the baſe, it is known that $CP^2 + CP^2 = 2\,DP^2 + 2\,CD^2$ (Hutton's Geom. th. 38.) Whence $2\,DP^2 = CP^2 + C'P^2 - 2\,CD^2$, or $4\,DP^2 = RP^2 = 2\,CP^2 + 2\,C'P^2 - C'C^2$. But C P, and C′P remaining conſtant, the ſide C C′ will vary as the meaſure of the angle C P C; and ſince the ſquare of C C′ is ſubtractive in the above value of $R P^2$, the latter will increaſe as the former decreaſes, and *vice verſa.*

46. COR. 3. *Two given forces produce the greateſt effect when they act in the ſame direction, and the leaſt when they act in oppoſite directions.*

For, in the former caſe the diagonal of the parallelogram becomes equal to the ſum of its ſides, and in the latter equal to their difference. The ſame concluſion may alſo be deduced immediately from art. 34. The conception of the ſtudent may be aſſiſted by referring to figs. 8 and 9.

46. PROP. *If a material point be kept in equilibrio by the ſimultaneous action of three forces, thoſe forces will be repreſented in magnitude and direction by the three ſides of a triangle reſpectively parallel to the directions of the forces.*

For, when a ſyſtem of forces ſuſtains a point in equilibrio, any one of the forces, as that repreſented by O P (fig. 10. pl. I.) is equal and oppoſite to R P the reſultant of the others (35). But the reſultant of the two forces repreſented by C P and C′P, is repreſented by the diagonal of the parallelogram conſtituted upon C P, C′P, (42.) and conſequently if P R be equal to O P, C R will be parallel to P C′. Whence, it is obvious that the three ſides of the triangle P C R are reſpectively equal to the three forces O, C, C′, and parallel to the directions in which they act.

47. COR. 1. *If a body be kept in equilibrio by three forces, and two of them be repreſented in magnitude and direction by two ſides of a triangle, the third ſide, taken in order, will repreſent the magnitude and direction of the other force.*

48. COR. 2. Since the ſides of triangles are as the ſines of the oppoſite angles, it follows that *when three forces keep a point in equilibrio, each force is proportional to the ſine of the angle made by the directions of the other two.* Thus, in the forces repreſented in fig. 10.

O : C : C′ : : ſin. CPC′ : ſin. OPC′ : ſin. OPC : : ſin. PCR : ſin. CRP : ſin. CPR.

49. COR. 3. *If a material point be kept at reſt by three forces, and lines be drawn at right angles to the directions in which they act, and produced till they form a triangle, the ſides of this triangle will be proportional to the reſpective forces: or, if lines be drawn each making the ſame given angle with the directions of the forces, the ſides of the triangle formed by theſe lines will likewiſe be proportional to the forces.*

For it may eaſily be ſhewn that each of theſe triangles will be ſimilar to the triangle formed by the directions of the forces.

50. PROP. *If, when three given forces O, C, C′, acting upon a point P keep it in equilibrio, right lines PO, PC, PC′, be ſet off from P proportional to the ſeveral forces and in their reſpective di-*

rections, the point P will be the centre of gravity of the triangle O C C′.

Produce OP to R (fig. 11. pl. I.) making PR = OP, then will PR be equal to the equivalent of the components C, C′, and conſequently it will be the diagonal of the parallelogram CPC′R: wherefore PD, its half, is likewiſe equal to half OP, and falls upon the middle of the ſide CC′ of the triangle OCC′. But when OP = 2PD, CD being = C′D, P is the centre of gravity of the triangle. (See Art. 114.) Q. E. D.

51. Prop. *To find the reſultant of any number of forces acting upon a point, their directions all lying in the ſame plane.*

This is done with much facility by an obvious application of the parallelogram of forces: for ſince the united action of any two forces upon a point puts it in the ſame ſtate as if their equivalent had acted upon it, we may compound any two of the forces, and ſubſtitute their reſultant for them; this compounded with a third force will give a new reſultant; and this blended with a fourth force, another; and ſo on as far as neceſſary. Thus ſuppoſe the reſultant of the five forces repreſented in magnitude and direction by PC, PC′, Pc, Pc', and PC″ (fig. 12. pl. I.), were required. Firſt complete the parallelogram CPC′R, and inſtead of the component forces PC, PC′, make uſe of the reſultant PR: then take PR, Pc, as components and complete the parallelogram RPcR′, ſo will PR′ be the reſultant of the two forces PR, Pc, or of the three PC, PC′, Pc: again, take PR′, Pc', as components, and complete the parallelogram PR$c'r$: next, take the new reſultant Pr, and the fifth force PC″ as components, and complete the parallelogram Prr'C″; ſo will the new reſultant Pr, be the reſultant of the five original forces: and of conſequence (35.) a force equal and oppoſite to Pr will keep the equilibrium in the whole ſyſtem.

52. Cor. 1. Since CR is both equal and parallel to PC′, RR′ equal and parallel to Pc, R′r to Pc', &c. the reſultant when the forces are all in one plane may be readily determined, by drawing conſecutive lines, each from the end of the preceding, and having the directions and proportions of the ſeveral forces. Thus draw from C (fig. 12.) CR equal and parallel to PC′, RR′ to Pc, R′r to Pc', rr' to PC″, and Pr' will be the reſultant, as before.

53. Cor. 2. *If any number of forces repreſented in magnitude and direction by the ſides of a polygon taken in order, act ſimultaneouſly upon the ſame point of a body at reſt, they will keep it at reſt.*

Let PC, CR, RR′, R′r, rr', r'P (fig. 12.), repreſent the forces; then, ſince the firſt five of theſe have Pr' for their equivalent (52.), and r'P the fifth force is equal and oppoſite to Pr', it will preſerve the equilibrium of the ſyſtem (35.).

54. PROP. *A ſingle force may be reſolved into any number of forces.*

Since any ſingle force as PR (fig. 10.) is the equivalent of the two forces PC, PC′, which are two contiguous ſides of a parallelogram whoſe diagonal is PR, and ſince it will likewiſe be the reſultant of any two forces proportional to, and in the direction of any parallelogram which has PR for a diagonal; it follows that the ſingle force PR may be decompoſed into as many pairs of forces as there can be deſcribed parallelograms having the ſame common diagonal, that is, into an indefinite number of ſuch pairs. And ſince either, or both, of theſe component forces may again be decompoſed, in like manner, into two, and either of theſe again into two; it is manifeſt that the original force may in this way be conceived to be reſolved into any number of others, *ad libitum*; and theſe either into the ſame plane, or any others.

55. SCHOLIUM. The preceding propoſition is uſeful in what is called the *Reduction of forces*, or eſtimating their effects in any given direction. Thus, if it were required to eſtimate the effect of the force PC (fig. 7. pl. I.) in the direction of the given line PR: conſtruct the rectangular parallelogram P*c* CD, and PC will manifeſtly be the equivalent of the two forces P*c*, and PD, of which the former is perpendicular to the given direction, and therefore neither promotes nor obſtructs the effect of the force in that direction, ſo that PD is the aggregate of force in the direction PR. Of this ſpecies of reduction many examples will occur hereafter.

56. PROP. *The effects of forces, when eſtimated in given directions, are not altered by compoſition or reſolution.*

Let the forces PC, PC′, and their reſultant PR, be eſtimated in the propoſed directions PX, PY (fig. 14. pl. I.). Draw the various lines reſpectively parallel to PX, PY, as in the figure. Then, the force PC, when referred to the given directions, will furniſh the components PD, P*d*; the force $CR = PC$, the components C*e*, CE; and the force PR gives the components PF, P*f*. But the forces PD, C*e*, being in oppoſite directions will have oppoſite ſigns (+ and —), and by reaſon of the parallels DE, FR, P*f*, and PD, *d*C, *f*E, it will be $PD - Ce = PD - DF = PF$, the effect of the two forces PC, PC′, eſtimated upon PY; alſo $Pd + CE = Pd + df = Pf$, the effect of the ſame two forces eſtimated upon PX; and theſe, it is evident, are likewiſe the effects of the reſultant PR, when eſtimated in the ſame directions. In a ſimilar manner the truth of the propoſition may be ſhewn, for any number of forces, by compounding each pair, and tracing their ſimultaneous effects, in the propoſed directions. Q. E. D.

Cor. If a ſyſtem of equilibrated forces is reduced to any one direction, the reduced forces will be in equilibrio.

57. **Prop.** *To find the reſultant of a ſyſtem of forces applied to one point and acting in the ſame plane, by the method of rectangular co-ordinates.*

Let the three forces C, C′, C″ (fig. 13. pl. I.), ſolicit the point P, with energies proportional to P C, P C′, P C″, and in thoſe directions: through any point A draw two lines X X′, Y Y′, in any directions reſpectively perpendicular to each other; through P draw P d'' parallel to X X′ and P D parallel to Y Y′, and reſolve each of the forces into components reſpectively parallel to theſe co-ordinates; for inſtance, P C into P D, P d; P C′ into P D′, P d'; and P C″ into P D″, P d: ſet off A Y and A X reſpectively equal to $PD + PD' + PD''$, and $Pd + Pd' + Pd''$; then complete the rectangle A R X Y, and its diagonal A R will repreſent the magnitude of the reſultant of the ſyſtem, and will be parallel to its direction. The truth of which is manifeſt, from the foregoing propoſition and ſcholium.

58. **Cor.** In the caſe of an equilibrium A R being oppoſed by an equal and contrary force, the reſultant will be nothing, or zero: or when the ſeveral forces are reduced to the rectangular co-ordinates, the ſum of the forces upon A Y′ which may be conſidered as negative, muſt be equal to the ſum falling upon A Y, and the ſum upon A X′ in like manner conſidered negative, equal to thoſe upon A X.

59. **Scholium.** In caſes where accuracy is required, the ſtudent will find it neceſſary to aim at ſomething further than a graphical ſolution; to aſſiſt his progreſs, we, therefore, add a few algebraical formulæ which may often be advantageouſly applied.

Let the angles which the directions of the component forces C, C′, C″, make reſpectively with the axis A X (fig. 13.), be a, a', and a''; and let the angle which R the reſultant makes with A X be denoted by r. Then when each of the forces is reduced to rectangular co-ordinates, we ſhall have (41 *cor.*) $C = \frac{Pd}{\cos. a}$, $C' = \frac{Pd'}{\cos. a'}$, $C' = \frac{Pd''}{\cos. a''}$, whence $C \cos. a$, $C' \cos. a'$, $C'' \cos. a''$, are equal to Pd, Pd', Pd'', the various components of theſe forces parallel to A X; and, in a ſimilar manner may be obtained C ſin. a, C' ſin. a', C'' ſin. a'', the components of the ſame forces, parallel to A Y: the former of theſe acting in the ſame line, are equivalent to a ſingle force expreſſed by their ſum [uſing the word *ſum* in the ſenſe of the algebraiſts], that is, $C \cos. a + C' \cos. a' + C'' \cos. a''$, + &c.: and the latter are likewiſe equivalent to a ſingle force expreſſed by their ſum, or

$C\,\text{ſin.}\,a + C'\,\text{ſin.}\,a' + C''\,\text{ſin.}\,a'' + \&c.$ The two components of the reſultant R referred to the ſame axis, are evidently $R\,\text{cos.}\,r$ and $R\,\text{ſin.}\,r$: conſequently (56.) the following equations obtain:

$$\text{(I.)}\ .\quad \left\{\begin{array}{l} C\,\text{cos.}\,a + C'\,\text{cos.}\,a' + C''\,\text{cos.}\,a'' + \&c. = R\,\text{cos.}\,r \\ C\,\text{ſin.}\,a + C'\,\text{ſin.}\,a' + C''\,\text{ſin.}\,a'' + \&c. = R\,\text{ſin.}\,r \end{array}\right.$$

If we denote the ſum of the components in the direction of the axis A X by x, and the ſum of thoſe falling upon AY by y, the preceding will be repreſented in an abridged form, thus:

$$\text{(II.)}\ .\ .\ .\ .\ .\ .\ R\,\text{cos.}\,r = x,\ .\ .\ .\ .\ .\ R\,\text{ſin.}\,r = y.$$

When the ſyſtem is in a ſtate of equilibrium, the reſultant is zero (58): thus our firſt equations give for the conditions of equilibrium:

$$\text{(III.)}\left\{\begin{array}{l} x = 0 \\ y = 0 \end{array}\right\}\text{ or }\left\{\begin{array}{l} C\,\text{cos.}\,a + C'\,\text{cos.}\,a' + C''\,\text{cos.}\,a'' + \&c. = 0. \\ C\,\text{ſin.}\,a + C'\,\text{ſin.}\,a' + C''\,\text{cos.}\,a'' + \&c. = 0. \end{array}\right.$$

In thoſe caſes where the equilibrium has not place, it is ſtill eaſy to find the magnitude and direction of the reſultant: for, adding together the ſquares of the equations (II.) we ſhall have

$R^2\,(\text{ſin.}^2\,a + \text{cos.}^2\,a) = x^2 + y^2$, wherefore, becauſe
$\text{ſin.}^2\,a + \text{cos.}^2\,a = \text{rad.}^2 = 1$, we ſhall have

$$\text{(IV.)}\ .\ .\ .\ .\ .\ R = \sqrt{x^2 + y^2}$$

Which is, in fact, the well-known property of the right-angled triangle.

Our ſecond equations likewiſe give, by a ſimple diviſion,

$$\text{(V.)}\ .\ .\ .\ \text{cos.}\,r = \frac{x}{R}\ .\ .\ .\ \text{ſin.}\,r = \frac{y}{R}.$$

Or if we divide the one of our ſecond equations by the other, another equation will be obtained, by which we may determine the direction of the reſultant: for, ſince ſin. ÷ cos. = tang. we have

$$\text{(VI.)}\ .\ .\ .\ .\ .\ .\ \text{tang.}\,r = \frac{y}{x}.$$

When only one of the equations (III.) obtains, there will not be an equilibrium, but the reſultant will be parallel to one of the axes: thus, if we have only $x = 0$, this will give $R\,\text{cos.}\,r = 0$, of courſe $\text{cos.}\,r = 0$, or $r = 180°$: that is, the reſultant is parallel to AY or perpendicular to AX. If only $y = 0$, the reſultant will, in like manner, be parallel to A X.

60. We may deduce from the equations (I.) in the preceding article, two or three curious conſequences; which may likewiſe be found of utility in ſome ſubſequent inveſtigations.

Let us take in the plane of the forces C, C′ C″, any point S at pleaſure, and let the capital S denote its diſtance from P the point acted upon by the ſyſtem, while the ſmall s repreſents the angle formed by SP and XA (fig. 15. pl. I.). Now multiplying the firſt of the equations juſt referred to, by $S\,\text{ſin.}\,s$, the ſecond by $S\,\text{cos.}\,s$, and ſubtracting, we have

$CS\,(\text{cos.}\,a.\ \text{ſin.}\,s \backsim \text{ſin.}\,a\,\text{cos.}\,s) + C'S\,(\text{cos.}\,a'.\ \text{ſin.}\,s \backsim \text{ſin.}\,a'\,\text{cos.}\,s)$

$+ C''S\,(\text{cos.}\,a''\,\text{ſin.}\,s \backsim \text{ſin.}\,a''\,\text{cos.}\,s) + \text{\&c.} = RS\,(\text{cos.}\,r\,\text{ſin.}\,s \backsim \text{ſin.}\,r\,\text{cos.}\,s)$.

But it is known (See *Crakelt*'s *Mauduitt*, pa. 48. *Hutton*'s *Dict.* art Sine) that $\text{cos.}\,a\,\text{ſin.}\,s \backsim \text{ſin.}\,a\,\text{cos.}\,s = \text{ſin.}\,(a \backsim s,)$ and the like of all the other expreſſions between the parentheſes; whence the equation becomes

$$CS\,\text{ſin.}\,(a \backsim s) + C'S\,\text{ſin.}\,(a' \backsim s) + C''S\,(\text{ſin.}\,a'' \backsim s) + \text{\&c.} = RS\,\text{ſin.}\,(r \backsim s).$$

Here the angle $a \backsim s = CPS$, and $r \backsim s = RPS$, RP being the reſultant: wherefore, if upon PC the perpendicular Ss be demitted, we ſhall have in the right-angled triangle PSs, $Ss = S\,\text{ſin.}(a \backsim s)$; and in like manner $Ss' = S\,\text{ſin.}\,(a' \backsim s)$ the perpendicular let fall upon PC′, and ſo on. If, therefore, we call the perpendiculars from S upon the ſeveral directions of C, C′ C″, &c. p, p', p'', &c. and the perpendicular upon the reſultant ϱ, the preceding equation will be transformed to this:

$$\text{(VII.)} \;\ldots\; Cp + C'p' + C''p'' + \text{\&c.} = R\varrho.$$

Hence, if we uſe the term *moment* in the ſenſe of art. 31. this equation will furniſh the following theorem.

The moment of the reſultant of a ſyſtem of forces, diſpoſed in the ſame plane, and directed to one point, is equal to the ſum of the moments of the components.

By the *ſum* of the moments is here meant their aggregate when incorporated according to their ſigns, uſing the affirmative ſign for the moments of thoſe powers which lie on one ſide of the point S, the negative ſign for thoſe which are found on the other ſign. Or if the point S be conſidered as fixed, and the lines Ss, &c. as inflexible rods, the action of each of the forces upon the point M will tend to make it turn about S: viewing the matter thus, the *poſitive* moments will be thoſe of ſuch forces as tend to produce a rotation in one ſenſe, and the *negative* moments, of thoſe which tend to produce motion in the oppoſite ſenſe. The previous deduction may, therefore, be ſtated thus:

When ſeveral forces in the ſame plane are applied to a particle of matter, the moment of the reſultant is equal to the exceſs of the ſum of the moments of thoſe forces which tend to produce a rotation in one direction, over that of the moments of ſuch as tend to produce rotation in the contrary direction.

It muſt be obſerved, however, that the idea of *rotation* introduced here, is merely called in for the purpoſe of aſſiſting the ſtudent in determining the ſigns; but is no eſſential part of the principle.

61. The equation (VII.) will in two caſes aſſume this form:

$$\text{(VII. 2.)} \ldots Cp + C'p' + C''p'' + \text{\&c.} = 0.$$

That is, 1. When $R = 0$; or when the ſyſtem is in equilibrio. 2. When $\rho = 0$; that is, when the point S is taken on the line of direction of the reſultant. Hence it appears, that,

The ſum of the moments of forces which tend to produce rotation in one direction, is equal to the ſum of the moments of forces which tend to produce rotation in a contrary direction, 1. when the forces are in equilibrio; 2. when the moments are taken with relation to ſome point in the direction of the reſultant.

62. To illuſtrate the uſe of the formulæ for finding the reſultant of a ſyſtem of forces, we will now propoſe an inſtance:

Let the forces C, C', C'' (*fig.* 13. *pl.* I.) *be repreſented by the numbers* 30, 25, *and* 20, *and* P *being the point acted upon, let the angles made by their directions, be* $CPC' = 30^\circ$, *and* $C'PC'' = 28^\circ$; *it is propoſed to find the magnitude and direction of the equivalent of theſe three forces.*

If we ſuppoſe the direction $C''P$ to make an angle of 20° with the aſſumed axis AX, then will the angles which the ſeveral forces make with that axis, be 78°, 48° and 20°. Wherefore, uſing a table of natural ſines, &c. we ſhall have $C \cos. a + C' \cos. a' + C'' \cos. a'' = 30.\,\text{ſin.}\,12^\circ + 25.\,\text{ſin.}\,42^\circ + 20.\,\text{ſin.}\,70^\circ = (30 \times {\cdot}2079117) + (25 \times {\cdot}6691306) + (20 \times {\cdot}9396926) = 41{\cdot}759468 = R \cos. r = x$; and $C\,\text{ſin.}\,a + C'\,\text{ſin.}\,a' + C''\,\text{ſin.}\,a'' = 30.\,\text{ſin.}\,78^\circ + 25.\,\text{ſin.}\,48^\circ + 20.\,\text{ſin.}\,20^\circ = (30 \times {\cdot}9781476) + (25 \times {\cdot}7431448) + (20 \times {\cdot}3420201) = 54{\cdot}76345 = R\,\text{ſin.}\,r = y$. Then $R = \sqrt{x^2 + y^2} = 68{\cdot}86857$ the magnitude of the reſultant; and $\frac{y}{x} = 1{\cdot}31140203 = \text{tang.}\,52^\circ\,40'\,\frac{30}{79}$ the angle which the reſultant makes with AX; this leſſened by 48°, leaves $4^\circ\,40'\,\frac{30}{79}$ = the angle $C'PR$ between $C'P$ and CP. Or when the direction of the reſultant is determined, its magnitude may be found without the extraction of roots: for, (V.) ſince $\text{ſin.}\,r = \frac{}{R}$ we have $R = \frac{y}{\sin. r} = 68.86857$, as before.

To ſhew that the ſame reſult may be obtained although the poſition of the co-ordinates AX, AY, be changed, ſuppoſe AX to coincide with $C''P$: then $C \cos. a + C' \cos. a' + C'' \cos. a'' = 30.\,\text{ſin.}\,32^\circ + 25\,\text{ſin.}\,62^\circ + 20.\,\text{rad.} = 57{\cdot}971269 = x$, and $C\,\text{ſin.}\,a + C'\,\text{ſin.}\,a' + C''\,\text{ſin.}\,a'' = 30.\,\text{ſin.}\,58^\circ + 25\,\text{ſin.}\,28^\circ + 20.\,\text{ſin.}\,0^\circ = 3717.8233 = y$. Hence $\frac{y}{x} = {\cdot}6413332 = \text{tang.}\,32^\circ\,40'\,\frac{30}{79}$, which leſſened by $C''PC'$ gives $4^\circ\,40'\,\frac{30}{79} = C'PR$, and $y \div \text{ſin.}\,32^\circ\,40'\,\frac{30}{79} = 68{\cdot}86857$, the reſultant: agreeing with the former.

63. To be convinced of the ſimplicity of this method of finding the magnitude and direction of the reſultant, in caſes where the forces meet in a point and are diſpoſed in one plane, it is only neceſſary to compare it with the common method,

Now, if the reſultant of forces were determined in the uſual way, the proceſs would be this: Suppoſe P C, P C′ (fig. 16. pl. I.), to repreſent the magnitude and direction of two of the forces: then letting fall C E and R F perpendicularly upon P C, we ſhould have rad. : C P : : ſin. P : C E : : cos. P : P E; and P F (= P C′ + P E) : rad. : : R F (= C E) : tang. R P F; alſo rad. : P F : : ſec. R P F : P R. Then taking P R, and P C″, for the two forces of given magnitude and directions, four more proportions would determine their reſultant: and in like manner four others would be neceſſary for a fourth force, and ſo on: whereas, by adopting the method juſt explained there is but little more labour neceſſary to determine the reſultant of five or ſix forces than of two.

II. *Of Forces directed to one Point, but not confined to one Plane.*

64. Prop. *If three forces are repreſented in magnitude and direction by the three edges contiguous to the ſame angle of a parallelopiped, their equivalent will be repreſented in magnitude and direction by the diagonal drawn from that angle of the ſolid.*

Let the three components ſoliciting the particle P be repreſented by P C, P C′, P C″ (fig. 17. pl. I.), and let the parallelopiped be completed. The equivalent of the two forces C and C′ will be P *r*; we may, therefore, ſubſtitute P *r* for thoſe two forces: but ſince the plane in which are the parallels C″P, R *r*, cuts the two parallel planes B D, C C′, the lines C″R, P *r*, in which it interſects them will be parallel (Euc. XI. 16.); and conſequently C″P *r* R is a parallelogram: wherefore, compounding the two forces P C″, P *r*, we have P R the diagonal of the parallelogram C″*r*, or of the parallelopiped B D *r*, for the equivalent of the three forces C, C′, C″. Q. E. D.

65. Cor. *Hence, if four forces act upon a particle in different planes, and keep it in equilibrio, they are to each other in magnitude and direction as the three edges and the diagonal of a parallelopiped, conſtructed upon lines reſpectively parallel to the directions of the forces.*

For a parallelopiped may be conſtructed ſo that its three edges contiguous to any one angle, ſhall have magnitudes and directions analogous to any three of the four forces; the diagonal of this ſolid will, by the prop. be the equivalent of thoſe three forces: and ſince in the caſe of an equilibrium the fourth force (35.) muſt be equal and oppoſite to the equivalent of the other three, it may be repreſented by the diagonal of the ſolid.

66. Cor. 2. Hence, likewiſe, *any number of forces whatever,*

acting upon a particle in different planes, may be compounded by pairs till there remain either two forces in the ſame plane, or three in different planes, and of theſe the reſultant may be found either by art. 42. *or by this propoſition.*

67. Cor. 3. And hence, converſely, a ſingle force may be reſolved into three forces, in different planes, and each of theſe into other forces in the ſame or other planes, each of theſe again into others; the reſolution admitting of being carried on, *ad infinitum.*

68. Prop. *If three forces in any one plane are in equilibrio, and theſe are reduced to any other plane, the reduced forces will likewiſe be in equilibrio.*

Let the three forces O, C, C′ (fig. 1. pl. II.), repreſented by OP, CP, C′P, in the plane OCC′ be in equilibrio, and let them be reduced by perpendiculars to the plane LM, then ſhall the reduced forces op, cp, $c'p$, be in equilibrio. For the lines Oo, Pp, Cc, $C'c'$, Rr, being all perpendicular to the ſame plane are parallel to each other; and becauſe CP, RC′, are equally inclined to the plane LM, they are equally inclined to the lines cp, rc', in that plane; hence $CP : cp :: RC' : rc'$, and by permutation $CP : RC' :: cp : rc' :: 1 : 1$; in like manner it may be ſhewn that cr is to pc' in a ratio of equality; and conſequently $pcrc'$ is a parallelogram: thus alſo, ſince $OP = PR$, $op = pr$: whence, op being equal and oppoſite to the diagonal of the parallelogram of forces $cpc'r$, the reduced ſyſtem is in equilibrio. Q. E. D.

69. Prop. *To find the reſultant of a ſyſtem of forces applied to one point, but diſpoſed in different planes; by the method of rectangular co-ordinates.*

Here the proceſs will be ſimilar to that deſcribed in art. 57, and may be explained by a reference to the ſame figure (fig. 13. pl. I.): in which let the three forces C, C′, C″, directed to the ſame point but not in the ſame plane, be repreſented by PC, PC′, PC″, ſuppoſed not in the ſame plane: through an aſſumed point A draw any two lines XX′, YY′, perpendicular to each other, and ſuppoſe another line AZ drawn through A perpendicular to the plane of the figure; theſe will be the co-ordinates to which the forces are to be referred. Through P the point of application of the forces, let lines PD, Pd, $P\delta$, be drawn parallel to the reſpective co-ordinates: to each of theſe lines refer the component forces, by perpendiculars CD, Cd, $C\delta$, $C'D'$, $C'd$, $C'\delta'$, &c. and the ſums $PD + PD' + PD''$, $Pd + Pd' + Pd''$, $P\delta + P\delta' + P\delta''$, will denote the equivalents of the three original forces, upon the edges of the rectangular parallelopiped; whence the diagonal of the parallelopiped whoſe ſides are expreſſed by theſe ſums repreſents the magnitude and direction of the reſultant ſought.

70. Schol. In this caſe of many forces in different planes, the algebraic method of ſolution will often be peculiarly uſeful; and it will be eaſily underſtood, being an obvious extenſion of the formulæ applied to forces in one plane. Thus, let
the component forces which we call . . . . C, C', C'', &c.
form with the axis A X (x) the angles . . a, a', a'', &c.
with the axis A Y (y) . . . . b, b', b'', &c.
with the axis A Z (z) . . . . c, c', c'', &c.
Each of theſe forces being decompoſed into three others, of which the directions are parallel to each axis, it has been ſhewn (59.) that for components parallel to theſe axes, we have
Of x . . . . $C \cos. a$, $C' \cos. a'$, $C'' \cos. a''$, &c.
y . . . . $C \cos. b$, $C' \cos. b'$, $C'' \cos. b''$, &c.
z . . . . $C \cos. c$, $C' \cos. c'$, $C'' \cos. c''$, &c.
Theſe groups of forces are each equivalent to a ſingle force, repreſented by their ſum, acting in the ſeveral lines to which the original forces are referred; ſo that we have for the components of the reſultant parallel to each axis, as below:

$$x = C \cos. a + C' \cos. a' + C'' \cos. a'' + \&c.$$
$$y = C \cos. b + C' \cos. b' + C'' \cos. b'' + \&c.$$
$$z = C \cos. c + C' \cos. c' + C'' \cos. c'' + \&c.$$

Now denote by r, r', r'', &c. the unknown angles formed by the direction of the reſultant and each of the co-ordinates, and $R \cos. r$, $R \cos. r'$, $R \cos. r''$, will repreſent the equivalents of this reſultant in the ſeveral directions of the axes: hence, we have

$$\text{(ii.)} \quad \ldots\ldots \quad \left\{ \begin{array}{l} R \cos. r = x. \\ R \cos. r' = y. \\ R \cos. r'' = z. \end{array} \right.$$

When the ſyſtem is in a ſtate of equilibrium, we have $R = 0$, ſo that the equations expreſſing the equilibrium are

$$\text{(iii.)} \quad . \quad \left\{ \begin{array}{l} x = C \cos. a + C' \cos. a' + C'' \cos. a'' + \&c. = 0. \\ y = C \cos. b + C' \cos. b' + C'' \cos. b'' + \&c. = 0. \\ z = C \cos. c + C' \cos. c' + C'' \cos. c'' + \&c. = 0. \end{array} \right.$$

If an equilibrium does not obtain in the ſyſtem, the magnitude and direction of the reſultant may be deduced from the three equations (ii.), for by adding together their ſquares, we have $R^2 (\cos. r + \cos. r' + \cos. r'') = x^2 + y^2 + z^2$. But when r, r', & r'', are angles made by any line, and three rectangular co-ordinates, it is known that the ſum of their ſquares is equal to unity: therefore,

$$\text{(iv.)} \quad \ldots \quad R = \sqrt{x^2 + y^2 + z^2}.$$

Or, referring to fig. 17. pl. I. and conceiving the parallelopiped rectangular, and P C = x, P C′ = y, P C″ = z, we have (Euc. I. 47.) $Pr^2 = PC^2 + Cr^2 = x^2 + y^2$, and $PR^2 = Pr^2 + rR^2 = x^2 + y^2 + z^2$, whence R is found, as above. And with reſpect to the poſition of the reſultant, that is eaſily determined; for from the equations (ii.) we deduce

$$\text{(v.)} \quad \ldots \quad \cos. r = \frac{x}{R} \quad \ldots \quad \cos. r' = \frac{y}{R} \quad \ldots \quad \cos. r'' = \frac{z}{R}.$$

When either of the quantities in equation (ii.) become = 0, the conſequences are ſimilar to thoſe we traced with reſpect to forces diſpoſed in one plane: thus, if the equations take at once this form,—

$$x = R\cos. r = 0, y = R\cos. r', z = R\cos. r'':$$

it is then manifeſt that cos. r = 0, or that r is a right angle; or that the reſultant is ſituated in a plane perpendicular to the axis A X. If we have, at one and the ſame time,

$$x = \cos. r = 0, y = \cos. r' = 0, z = \cos. r'',$$

it will be obvious that the direction of the reſultant is perpendicular to A X, A Y, and parallel to A Z.

III. *Of Forces ſituated in one Plane, but applied to different Points of a Body.*

71. PROP. *If two parallel forces act perpendicularly upon a right line, in the ſame direction, their reſultant is parallel to them, equal to their ſum, acts in the ſame direction, and divides the line of application into two parts which are reciprocally proportional to the components.*

Let the two forces C, C′ acting in the directions C P, C′P′ (fig. 2. pl. II.), perpendicular to P P′, be thoſe whoſe equivalent is ſought. Conceive any two forces c c', equal to each other to act in the oppoſite directions c P, c'P′, oppoſite to each other; and it is obvious they will cauſe no change in the ſtate of the ſyſtem: therefore, if r the reſultant of C and c act in the direction r P, and r' the reſultant of C′, c', in the direction r'P′, the lines r P, r'P′, when produced, will interſect in a point A, through which R p, the direction of the reſultant of the components C, C′, muſt likewiſe paſs. This granted, through A, the point of concourſe of r P, r'P, draw B D, A p, reſpectively perpendicular and parallel to the directions of the original forces, and decompoſe each of the forces r, r', into two others acting in the directions B A, p A, and D A, p A. Now, ſince the circumſtances of the reſolution of r, and r', are the ſame in A as in P, P′, the force r acting upon A will be decompoſed into the two, c acting in B A, and C in p A; and the force r' acting upon A will be reſolved into c' acting in D A and C′ acting in p A. But the two forces c, c', being equal and oppoſite are annihilated, therefore R the equivalent of C and C′ acts in a parallel direction R A and is equal to their ſum.

Again, to determine the point p in P P′ through which the reſultant paſſes: ſince r is the equivalent of c, and C, it follows (59. *vi.*) that the angle r P C has $\frac{c}{C}$ for its tangent; while in the triangle P p A, we have tang. P A p = tang. r P C $= \frac{Pp}{Ap}$: thus $\frac{c}{C} = \frac{Pp}{Ap}$, and $c = \frac{C.Pp}{Ap}$. Proceeding in a ſimilar manner, we have $\frac{c'}{C'} = \frac{P'p}{Ap}$, and $c' = \frac{C'.P'p}{Ap}$. But, by hypotheſis $c = c'$; conſequently $C.Pp = C'.P'p$, and $Pp : P'p :: C' : C$. Q. E. D.

72. Cor. 1. *When three parallel forces acting perpendicularly upon a right line keep it in equilibrio, one of them will act in a direction oppoſite to the other two, and it will be equal to their ſum; and any two of them will be to each other inverſely as their diſtances from the point to which the third force is applied.*

For the third force A muſt be equal and oppoſite to the reſultant R of C, C′, and muſt be applied at the ſame point p. And ſince $C \propto pP'$, and $C' \propto Pp$, $A = C + C'$ will $\propto Pp + pP' = PP$. That is, $A : C : C' :: PP' : P'p : Pp$, as affirmed in the corollary.

73. Cor. 2. *When two component forces act in contrary directions, their reſultant is equal to their difference, and is applied at the ſame point as the power which eſtabliſhes the equilibrium.*

Thus the reſultant of the forces A and C (fig. 2. pl. II.) acts at the point C′ and is equal and oppoſite to CP′.

74. Cor. 3. *The equivalent of any number of parallel forces acting perpendicularly upon a line will be equal to their ſum when they act the ſame way; or to the exceſs of the ſum of thoſe which act in one direction above thoſe which act in the contrary direction, when they act contrarily:* and to find their point of application compound them, two by two, as in the propoſition.

75. Cor. 4. *If a right line be kept in equilibrio by any number of forces acting perpendicularly either at the ſame, or at different points, the ſum of all the forces acting on one ſide, will be equal to the ſum of all thoſe acting on the other.*

76. Prop. *The moment of the reſultant of two parallel forces taken with relation to any point whatever in the ſame plane is equal to the ſum of the moments of the components.*

1. Let C, and C′ (fig. 3. pl. II.), be two forces acting in the parallel directions C P, C′P, and S any point taken in their plane: perpendicular to C P, C′P′, draw the line S P P′, and conſider P, P′, as the points of application of the forces. The reſultant R being directed towards Q, we have (71.) C × P Q

$= C' \times P'Q$. Now $R \times SQ = (C + C') \times SQ = (C \times SQ) + (C' \times SQ)$, and $SQ = PQ + SP = SP' - P'Q$: therefore, by ſubſtitution, $R \times SQ = C \cdot PQ + C \cdot SP + C' \cdot SP' - C' \cdot P'Q = C \cdot SP + C' \cdot SP'$, becauſe $C \cdot PQ - C' \cdot P'Q = 0$.

2. If inſtead of taking S out of the ſpace included by CP, C′P′, we take a point S′ between thoſe lines, the ſame thing will hold, uſing the word *ſum* in the extenſive algebraic ſenſe (60.). For, in that caſe, we have again, $R \times S'Q = (C + C') \times S'Q = (C \times S'Q) + (C' \times SQ)$. But, as we have $S'Q = S'P' - P'Q = PQ - P'S'$, we obtain by ſubſtitution, $R \times S'Q = C \times (PQ - PS') + C' \times (S'P' - P'Q)$, and becauſe $C \cdot PQ - C' \cdot P'Q = 0$, we have $R \cdot S'Q = C' \cdot S'P' - C \cdot S'P$.

77. If we denote as in art. 60. the perpendiculars from S upon the directions of C, C′, and R, by p, p', ρ, we ſhall have the equation $R\rho = Cp + C'p'$, correſponding with the equation (VII) in that article. So that the conſequence ſtated there, has equally place here, and the obſervations relative to the poſitive and negative ſigns, are equally uſeful in both caſes.

78. Since a force O equal and oppoſite to the reſultant R, acting at the point Q will ſuſtain the ſyſtem in a ſtate of equilibrium, we have $O = -R$, and the preceding equation aſſumes this form:

$$R\rho + Cc + C'c' = 0.$$

Conſequently, *when three parallel forces are in equilibrio, the ſum of their moments with reſpect to any point in the ſame plane, is equal to zero.*

79. Drawing from the point S (fig. 3. Pl. II.) any right line whatever Spp', and ſuppoſing the three forces C, C′, O, which are in equilibrio, applied at the points p, p', q, we have, by reaſon of the parallels, $SQ : SP : SP' :: Sq : Sp : Sp'$. But the equation of the moments (78.) is $O \cdot SQ + C \cdot SP + C' \cdot SP' = 0$: in which, ſubſtituting the conſequents of the proportion for the antecedents, there reſults $O \cdot Sq + C \cdot Sp + C' \cdot Sp' = 0$; whence it follows, *generally*, *that the reſultant of two parallel forces divides any right line to the extremities of which they are applied into parts reciprocally as the forces*: and moreover that the reſults in arts. 71 . . . . 75. are applicable to all parallel forces acting upon one line, without regarding the angle that line makes with the direction of the forces.

80. PROP. *When a given power acts upon a certain point in a line, to determine its effects upon any other two points in the ſame line.*

Let the force R act upon the point q, it is required to determine the effort exerciſed upon the points p, p' (fig. 3. pl. II.). This is nothing elſe than to reſolve the force R into the two parallel components C, C′, acting at the propoſed points. Conſequently we muſt have $C + C' = R$, and $C : C' :: qp' : qp$: whence, $C + C' : C :: qp' + qp\ (= pp') : qp'$, and $C = \frac{R \cdot qp'}{pp'}$. In like manner we find $C' = \frac{R \cdot qp}{pp'}$.

81. Scholium. We may now deduce from what is gone before, a few theorems, which will be of utility in enquiries reſpecting parallel forces in one plane. Thus, let C, C', C'', &c. conſtitute a ſyſtem of parallel forces acting in one plane, R their reſultant, p, p', p'', ρ, &c. the perpendiculars demitted from any point in the ſame plane, upon the directions of the ſeveral forces. Then with relation to two forces C, C', we have $R = C + C'$, and $R\rho = Cp + C'p'$, for the equations by which we may determine the magnitude and poſition of R. Subſtituting for the two forces C, C', their equivalent R, and compounding this with the third force, we find the new reſultant, by means of the equations $R' = R + C''$ and $R'\rho' = R\rho + C''p''$, or,

(I.) $R' = C + C' + C'' + \&c.$ and $R'\rho' = Cp + C'p' + C''p'' + \&c.$

And thus may we proceed with other forces.

When the forces are in a ſtate of equilibrium, the equations become

(II.) … $C + C' + C'' + \&c. = 0$, … $Cp + C'p' + C''p'' + \&c. = 0$.

When the equilibrium does not obtain, the firſt equation (1) determines the magnitude of the reſultant: its direction will evidently be parallel to thoſe of the components, and its poſition will be determined from this equation:

$$\rho = \frac{Cp + C'p' + C''p'' + \&c.}{C + C' + C'' + \&c.} = \frac{Cp + C'p' + C''p'' + \&c.}{R}$$

In theſe equations we conſider as negative the forces which act in a contrary direction to thoſe which we reckon poſitive: and if R come out negative, the reſultant of the ſyſtem will be a force acting in a contrary direction to the forces we account affirmative: if ρ be negative, the reſultant muſt be diſpoſed, with regard to the origin of the moments, on the ſide oppoſite to that on which the forces are whoſe diſtances from this point we conſidered as poſitive.

* 81. Prop. *To find the reſultant of any number of forces acting upon different points of a body, their directions being all in the ſame plane.*

This is performed very readily by a graphic proceſs ſimilar to that deſcribed in art. 51. Thus, let any two of the forces be taken, and, prolonging their directions, conceive them to be applied at their point of concourſe (32.) and find their reſultant by means of the parallelogram of forces (42.). Compound this, in like manner with any one of the remaining forces; and ſo on: the magnitude and direction of the diagonal of the laſt parallelogram, will ſhew the magnitude and direction of the force equivalent to the ſeveral components. For an example, let it be propoſed to find the equivalent of the three forces C, C', C'', acting upon the body A B (fig. 4. pl. II.) in the directions C P, $C'P'$, $C''P''$, and with intenſities proportional to thoſe lines. Produce C P, $C'P'$, till they meet in D, and on thoſe directions ſet off D E, D F, reſpectively equal to C P, $C'P'$: complete the parallelogram E F, and its diagonal will repreſent in magnitude and direction the equivalent of the two forces C, C'. Then produce G D till it meet $C''P''$ produced in d: make $de = DG$, and $df = P''C''$, and complete the parallelogram ef: Laſtly, produce the diagonal dg, till Q R is equal to it; ſo will R Q ſhew the magnitude, poſition, and direction of the force equivalent to the three components C, C', and C''.

82. PROP. *When three forces in the ſame plane, acting upon different points of a body (conſidered as void of gravity) keep it in equilibrio, they are ſuch as would balance if applied to one point; their directions continuing parallel.*

If the forces C, C', O, (fig. 5. pl. II.) which act upon the body AB, at the points P, P', Q, in the directions CP, $C'P'$, OQ, with energies $\propto$ to thoſe lines, keep it in equilibrio, they would balance if applied at one point. For, producing two of the directions *e. g.* C P, $C'P'$, till they meet at a point D, making E D, F D, equal to C P, $C'P'$, reſpectively and completing the parallelogram E F, its diagonal G D will repreſent the magnitude and direction of the equivalent of the components C, C'. And ſince no ſingle force but one that is equal and oppoſite to the reſultant of C, C', can keep them in equilibrio, the direction of the third force O muſt paſs through D, and its magnitude O Q muſt be equal to gD, or D G. Q. E. D.

83. COR. 1. *Any two of theſe forces are inverſely proportional to the perpendiculars demitted upon their reſpective directions, from the point of application of the third force, or from any point in its direction.*

For (48.) $O : C : C' :: \text{ſin. } CDC' : \text{ſin. } ODC' : \text{ſin. } ODC$ (fig. 6. pl. II.) And if we produce the directions of any two of the forces as CP, $C'P'$, and upon them let fall from Q, the point of application of the third force, the perpendiculars Q K, Q I, or from any other point q in the direction of that force, the perpendiculars qk, qi, theſe perpendiculars will be to each other as the ſines of the angles QDK, QDI, or as the ſines of the ſupplemental angles O D C, O D C'. Wherefore $C : C' :: QI : QK :: qi : qk$.

84. COR. 2. *When four forces are in equilibrio by the intervention of a ſolid body, they are ſuch as would balance each other if applied to one point.*

Thus (fig. 5. pl. II.) the four forces C, C', C'', C''', which when the parallelograms of forces are conſtituted on the reſpective directions, give the equal and oppoſite reſultants G D, D g, are in equilibrio whether the points of application be P, P′ P″ P‴, or whether they are blended in D.

85. Cor. 3. *When any number of forces are in equilibrio by the intervention of a ſolid body, they would balance each other if applied at one point.* For either the lines of directions of all the forces will meet in one point when produced, or they will meet in different points forming ſo many ſyſtems in equilibrio; and all theſe ſyſtems when applied to one point, are in equilibrio, by art. 37.

86. Prop. *If a body A B (fig. 7, pl. II.) be ſuſtained in equilibrio by the ſimultaneous action of ſeveral forces C, C′, C″, &c. in one plane, their quantities and directions being repreſented by C P, C′ P,′ C″ P″, &c. cutting any line A X drawn through the body, in the points P, P′, P″, &c. and if lines C S, C′ S′, C″ S″, &c. be drawn parallel to each other from the points C, C′, C″, till they interſect the line A X: then, 1. the ſums of the portions P S, P s, P′ S′, and P s′, P″ S″, p″ s″, eſtimated in contrary directions muſt be equal. 2. The ſums of the parallel forces on each ſide A X, i. e. C S, + C′ S′ + C″ S″, and c s + c′ s′ + c″ s″, muſt be equal. 3. The ſums of the moments on each ſide from any point A, i. e. A P. C + A P′. C′ + A P″. C″ and A P. c + A P. c′ + A p″. c″, muſt be equal.*

Conceive the various parallelograms of forces to be conſtructed, as in the figure, then will the force repreſented by C P, be the equivalent of D P, S P, the force C′ P′, the equivalent of D′ P′, S′ P′, and ſo of the others. And it is evident that the forces P S, P′ S′. &c. muſt make up equal ſums in the contrary directions, otherwiſe, inſtead of the ſyſtem being at reſt, it would move either in the direction of A X or of X A. And with reſpect to the parallel forces D P, D′ P′, &c. or their equals C S, C′ S′, &c. they fall under the deductions in arts. 75 .... 79. Whence the truth of the propoſition is manifeſt.

87. Cor. If we conſider the forces reduced to A X to have either poſitive or negative ſigns according as they act towards or from A; and the parallel forces on different ſides of A X to have contrary ſigns: then will the ſum of the forces reduced to A X, the ſum of the parallel forces C S, C′ S′, &c. and the ſum of the moments of the forces with reſpect to any point on A X, be each equal to nothing.

88. Schol. Since the preceding propoſition is true, whatever direction the parallel lines C S, C′ S′, &c. make with the aſſumed line A X, we may readily deduce from hence the formulæ for ſeveral forces acting at different points in various directions in the ſame plane, according to the method of rectangular co-ordinates. To this end, let C S, C′ S′, &c. (fig. 7.)

be perpendicular to A X, and let the ordinate A Y be drawn parallel to C S. Then if the angles which the directions of the forces C, C′, C″, &c. make with AX, be denoted by a, a', a'', &c. and the angle made by R the reſultant with the ſame be r; alſo if the ⊥s from A upon C P, C′ P′, C″ P″, &c. be repreſented by p, p', p'', &c. we ſhall have $\mathrm{S\,P} = \mathrm{C} \cos. a$, $\mathrm{S'\,P'} = \mathrm{C'} \cos. a'$, &c. and $\mathrm{C\,S} = \mathrm{C}\ \text{ſin.}\ a$, $\mathrm{C'\,S'} = \mathrm{C'}\ \text{ſin.}\ a'$, &c. And in the caſe of an equilibrium, we ſhall have the following equations, the reſultant being abſolutely nothing.

$$(\mathrm{i.})\quad . \quad \begin{cases} \mathrm{C} \cos. a + \mathrm{C'} \cos. a' + \mathrm{C''} \cos. a'' + \&\text{c.} = x = 0. \\ \mathrm{C}\ \text{ſin.}\ a + \mathrm{C'}\ \text{ſin.}\ a' + \mathrm{C''}\ \text{ſin.}\ a'' + \&\text{c.} = y = 0. \\ \mathrm{C}\ p + \mathrm{C'}\ p' + \mathrm{C''}\ p'' + \&\text{c.} = \pi = 0. \end{cases}$$

Theſe equations are (like thoſe in arts. 59. 70. 81.) each compoſed of as many terms as there are forces, unleſs one of the forces ſhould have its direction coinciding either with A X, or A Y, or parallel to either of them, or paſſing through A their aſſumed origin, — when one term will manifeſtly vaniſh, in the correſponding equation.

89. If the ſyſtem of forces is not in equilibrio, the ſubſequent equations will aſſiſt in finding the reſultant, its magnitude, and its point of application. Beſides the above characters, let ρ denote the perpendicular A n from the origin of the axes upon the direction of the reſultant: then, ſince an equal force to the reſultant applied in a contrary direction reſtores the equilibrium, making it to ſubſiſt between C, C′, &c. and − R, by introducing the expreſſions $-\mathrm{R}\cos. r$, $-\mathrm{R}\ \text{ſin.}\ r$, and $-\mathrm{R}\rho$, into the equations, we ſhall get,

$$(\mathrm{II.})\quad .\ .\quad \mathrm{R}\cos. r = x,\ .\ .\quad \mathrm{R}\ \text{ſin.}\ r = y,\ .\ .\ .\quad \mathrm{R}\rho = \pi.$$

Adding together the ſquares of the two firſt of theſe, there will be found

$$(\mathrm{III.})\quad .\ .\ .\ .\ .\quad \mathrm{R} = \sqrt{x^2 + y^2}.$$

And from the equations (II) we deduce, by diviſion,

$$\left.(\mathrm{IV.})\ .\ \cos. r = \frac{x}{\mathrm{R}}\ .\ .\ \text{ſin.}\ r = \frac{y}{\mathrm{R}}\ .\ .\ \text{tang.}\ r = \frac{y}{x}\ .\ .\ \rho = \frac{\pi}{\mathrm{R}}.\right\}$$

Of theſe equations III determines the magnitude; either of the firſt three of IV aſcertains its direction, and the laſt gives the perpendicular diſtance of its direction from the origin of the ordinates. Thus, drawing a line A $n = \rho$ making with AY an angle $= r$, and drawing $n\,q$ perpendicular to A n, the line $n\,q$ R will be the poſition of the reſultant; any point in this line may be conſidered as the point of application: the ſigns of r and of ρ, will determine on which ſide of A Y and of A X, the line A n muſt be drawn.

90. When the three equations (88. I.) all obtain at the ſame time, they denote that a ſyſtem of forces diſpoſed in the ſame plane, but not concurring in the ſame point, is perfectly in

equilibrio: but it will not be entirely unintereſting to conſider what will be the conſequence when all three have not place at once. When only the firſt, or the ſecond exiſts, we may apply the remarks made at the end of art. 59. ſo that it only remains for us to examine here, what will occur when the third only exiſts. Thus, if inſtead of the ſyſtem being in abſolute equilibrium, we have the three following equations:

$$R\,\text{cos.}\,r = x, \qquad R\,\text{ſin.}\,r = y,$$

$$Cp + C'p' + C''p'' + \&c. = R\rho = 0.$$

Here $R\rho = 0$, gives $\rho = 0$, that is to ſay, the reſultant paſſes through the origin A of the co-ordinates: it is obvious that if that point in the ſyſtem is fixed this reſultant ought to be annihilated, and that there is an equilibrium whatever x and y may be. The equation $Cp + C'p' + C''p'' + \&c. = 0$, therefore, ſuffices ſolely to indicate that there is an equilibrium, on the ſuppoſition that the origin is a fixed point: and hence

In order that a ſyſtem of forces diſpoſed in the ſame plane be in equilibrio about any fixed point, it will be ſufficient if the ſum of the moments of the forces, with reſpect to this point, be equal to zero.

Suppoſing that a body ſubjected to the action of ſeveral powers is retained by a fixed point, it follows neceſſarily that it can only have a motion of rotation about this point: if, therefore, the equilibrium does not exiſt, neither can the reſultant paſs by this point, nor can we have $Cp + C'p' + C''p'' + \&c. = 0$. Hence, when $\varpi = 0$, there can be no rotatory motion, and when both $x = 0$, and $y = 0$, there can be no rectilinear motion.

IV. *Of Forces not confined to one Plane, and directed to various Points of a Body.*

91. Def. *Any point in a ſyſtem of parallel forces, through which the reſultant paſſes, and which retains the ſame place, although all the forces change their directions, provided they continue reſpectively parallel, may be called the Centre of Parallel Forces.*

92. Prop. *To find, by a graphic proceſs, the reſultant of any number of parallel forces, however diſpoſed.*

Conceive any plane to be interſected by the directions of the ſeveral forces, and the points of interſection to be joined by right lines falling on the plane. Then, by the method already given (71. 79.) find the reſultant of any two of the forces and its point of application at the imaginary plane: then take this reſultant and its point of application, and in like manner compound with a third force, and aſcertain their reſultant and its point of application: let this new reſultant be compounded with a fourth force, and ſo on through the whole. Thus for example, let the five forces repreſented by C, C', C'', C''', c, have P, P', P'', &c. for their points of application in the plane $PP'P''$ (fig. 8. pl. II.). In the right line PP' make $Pq : qP' :: C' : C$, and q will be the point of application of $R=C+C'$, the reſultant of C and C'. Join the points q, P', in the ſame plane, and make $qq' : q'P'' :: C'' : R$, then will q' be the point of application of the new reſultant $R'=R+C''=C+C'+C''$. Again, for the fourth force C''', join $q'P'''$, and make $q'q'' : q''P'' :: C''' : R'$, then q'' will be the point of application of the third reſultant $R''=R'+C'''=C+C'+C''+C'''$. Laſtly, join $q''p$, and make $q''q''' : q'''p :: c : R''$, then will q''' be the point of application of the reſultant $R'''=R''+c=C+C'+C''+C'''+c$. Had any one or more of theſe powers been exerted in an oppoſite direction, as for inſtance C'' and c, the point of application of the reſultant R''' would ſtill have been the ſame; but its magnitude would then have been $C+C'-C''+C'''-c$: as is evident from art. 73.

93. Cor. 1. If the components C, C', C'', &c. all change their directions in ſuch a manner as to remain parallel to each other, their reſultant will ſtill be applied to the ſame point in the plane $PP'P''$; or, if all the powers change their magnitudes in one and the ſame ratio, although the reſultant will undergo a correſponding mutation in its magnitude, its point of application will ſtill be the ſame; and will, therefore, be the *centre of parallel forces* in the ſyſtem whoſe variations are thus regulated.

94. Cor. 2. Hence it will be eaſy to put any ſyſtem of parallel forces into an equilibrated ſtate: for it requires nothing more than to find, by this propoſition, the magnitude and point of application of the reſultant; and then to apply an equal force to the ſame point in an oppoſite direction.

95. Scholium. Here alſo it may be proper to deduce a few general theorems which may be uſeful in future enquiries relative to parallel forces however diſpoſed. For which purpoſe let us begin with three parallel forces C, C', C'', in a ſtate of equilibrium, which, it is obvious, muſt neceſſarily be in one plane: let YAX (fig. 9. pl. II) be the plane of the rectangular co-ordinates to which we mean to refer; this plane muſt be cut (Euc. xi. 3.) by the plane in which are the directions of the forces, in a right line, ſuppoſe $ACCC''$ in the figure; in this line let C, C', C'', be the points where thoſe three forces are applied in the plane of the axes. From any point A in the line AC'' draw any rectangular co-ordinates AX, AY, and on theſe let fall from C, C', C'', the perpendiculars $Cd, C'd', C''d''$, and $CD, C'D', C''D''$: let $Ad=d$, $Ad'=d'$, $Ad''=d''$, $AD=D$, $AD'=D'$, $AD''=D''$. Then ſince what has been ſhewn

in art. 81. is applicable here, we have, in addition to the condition that the points C, C′, C″, are in one right line, theſe equations:

$$C.AC+C'.AC'+C''.AC''=0, \text{ and } C+C'+C''=0.$$

But by reaſon of the parallel lines, we have $AC : AC' : AC'' :: Ad : Ad' : Ad'' :: AD : AD' : AD''$, and ſubſtituting the conſequents of theſe for their reſpective antecedents in the firſt preceding equation, we obtain the following equations of condition for three parallel forces in equilibrio:

$$\text{(I.)}\ldots\ldots \left\{\begin{array}{l} Cd + C'd' + C''d'' = 0 \\ CD + C'D' + C''D' = 0 \\ C + C' + C'' = 0 \end{array}\right.$$

Some perſons, from a ſlight conſideration, might conclude, that ſince the quantities in the two firſt of theſe three equations are proportional, they need not both be taken at the ſame time: but it ſhould be recollected, that when any one of them is taken together with the third, this does not enſure the eſſential condition of the three points C, C′, C″, being in one right line; for while d, d', d'', remained the ſame, the points might not be found in AC″, but in any unlimited poſitions upon Cd, $C'd'$, $C''d''$; whereas the contemporaneous exiſtence of all the three equations eſtabliſhes the proper magnitudes of the forces, and confines their points of application to the interſections of DC, dC, &c. upon AC″.

96. A force −R which is equal and applied in an oppoſite direction to the reſultant R of two forces C, C′, will manifeſtly eſtabliſh the equilibrium. Therefore if x and y be put for the diſtances at which perpendiculars from the point of application of the reſultant will cut the axes AX, and AY; and −R, x, and y, being ſubſtituted for C″, d'' and D″, in the equations I, they will be transformed to

$$\begin{array}{l} Rx = Cd + C'd' \\ Ry = CD + C'D' \\ R = C + C' \end{array}$$

And theſe equations will evidently ſerve to determine the magnitude and poſition of the reſultant of two forces.

Having four forces C, C′, C″, C‴, if we compound two of them, as C, C′, into one force R, we may eſtabliſh the equilibrium between this and the other two by cauſing them to ſatisfy the equations (I): we ſhall have, therefore,

$$\begin{array}{l} Rx + C''d'' + C'''d''' = 0 \\ Ry + C''D'' + C'''D''' = 0 \\ R + C'' + C''' = 0 \end{array}$$

But the firſt terms of each of theſe equations being already known, we may ſubſtitute for them their values, and we ſhall thence obtain the equations for four forces in equilibrium:

$$\text{viz.} \ldots\ldots \begin{cases} C\,d + C'\,d' + C''\,d'' + C'''\,d''' = 0 \\ C\,D + C'\,D' + C''\,D'' + C'''\,D = 0 \\ C + C' + C'' + C''' = 0 \end{cases}$$

Here, however, it is not neceſſary that the directions of all the four forces ſhould be in one plane : it is merely requiſite that the three forces R, C, C′, ſhould act in one plane, and R, C″, C‴, in one plane ; or that the plane which contains the directions of C, C′, ſhould interſect that which contains the directions of C″, C‴, in a right line which will be the reſultant of both theſe pairs of forces. Now, on comparing the equations laſt exhibited with the equations (95. I.) it will appear, that they are exactly the ſame, excepting that each of the latter compriſes one term more than the former : and, if a ſimilar proceſs were adopted with reſpect to a fifth force, ſimilar equations would be obtained with another additional term in each. Hence, we may generaliſe without heſitation, and for as many parallel forces as we pleaſe may lay down the following equations, as conditions of equilibrium.

$$\text{(II.)} \ldots\ldots \begin{cases} C\,d + C'\,d' + C''\,d'' + \&c. = 0 \\ C\,D + C'\,D' + C''\,D'' + \&c. = 0 \\ C + C' + C'' + \&c. = 0 \end{cases}$$

Each of theſe equations will compriſe as many terms as there are forces, unleſs one of theſe forces have its direction paſſing through the origin A of the axes, in which caſe one of the terms will vaniſh from each of the firſt two equations : as to the ſigns, they will be poſitive or negative according as the forces to which they are attached act in the ſame or a contrary direction, or as their points of application fall on the ſame or different ſides of either of the co-ordinates.

97. When the equilibrium does not obtain, we ſhall by proceeding as in the former part of art. 96. have theſe equations

$$\text{(III.)} \ldots\ldots \begin{cases} R\,x = C\,d + C'\,d' + C''\,d'' + \&c. \\ R\,y = C\,D + C\,D' + C''\,D'' + \&c. \\ R = C + C' + C'' + \&c. \end{cases}$$

If the plane X A Y be perpendicular to the directions of the forces, a ſuppoſition which will not diminiſh the univerſality of the deductions ; and if we conceive AX and AY to be two planes perpendicular to X A Y, the rectangles R x, and R y, will then become *moments* (31.) of the reſultant, taken with regard to each of thoſe two planes ; and C d, C D, will in like manner be moments of C ; and ſo of the others. Hence then, taking the word *ſum* in the ſenſe we have ſo often explained, we may deduce the following theorem for all parallel forces, however ſituated :

The reſultant of any number of parallel forces whatever is parallel to them, equal to their ſum, and has its moment eſtimated with reſpect *to every parallel plane, equal to the ſum of the moments of the components.*

Thus the reſultant is determined in magnitude and direction, and the point in which it interſects the plane X A Y may be readily found by means of theſe equations:

$$\text{(IV.)} \ldots \begin{cases} x = \dfrac{C\,a + C\,d + C''\,d'' + \&c.}{R} \\ y = \dfrac{C\,D + C'\,D' + C''\,D'' + \&c.}{R} \end{cases}$$

98. The equilibrium will be abſolute when all the three equations (II.) obtain at once: but it will not be uſeleſs if we enquire what will be the conſequence when the equations of condition are taken partly from equa. II. and partly from equa. III. If, for example, we have the following equations:

$$R = C + C' + C''$$
$$R\,x = C\,d + C'\,d' + C''\,d'' + \&c. = 0$$
$$R\,y = C\,D + C\,D' + C''\,D'' + \&c.$$

Here $R\,x$ being $= 0$, and R an abſolute quantity, we conclude that $x = 0$; in this caſe the direction of the reſultant will interſect the plane X A Y ſomewhere in A Y. If therefore this axis is fixed in the ſyſtem, whatever R and y may be, the only neceſſary condition of equilibrium is indicated by the equation

$$C\,d + C'\,d + C''\,d'' = 0$$

In like manner the equation

$$C\,D + C'\,D' + C''\,D'' = 0$$

has place when the direction of the reſultant paſſes through ſome one of the points in A X; and when this axis is fixed in the ſyſtem, this equation is alone neceſſary to indicate an exiſting equilibrium.

If we ſuppoſe that the plane X A Y is perpendicular to the directions of the forces, the terms of theſe two equations denote the moments of theſe forces with relation to planes paſſing through A X and A Y perpendicular to their plane. Hence it follows that,

In order to have a ſyſtem of parallel forces in equilibrio about any fixed axis, it is neceſſary that the ſum of the moments of theſe forces with reſpect to a plane parallel to their forces, and paſſing through that axis, ſhall be equal to zero.

If the two equations $R\,x = 0$, and $R\,y = 0$, exiſt together, there will reſult $x = 0$, $y = 0$; that is, the reſultant will paſs both through ſome point in A X and ſome point in A Y; that is, it will paſs through A the origin of the axes: if, therefore, this point be fixed, theſe two equations denote an equilibrium.

Again, if we have only this equation

$$R = C + C' + C'' + \&c. = 0,$$

we are not to conclude that becauſe $R = 0$, the equilibrium obtains: for this iſolated equation merely denotes that the ſum of the forces which act in one direction, is equal to the ſum of thoſe which are exerted in the contrary direction; and this may take place independently of the poſitions of the forces: but the equilibrium exiſts only when the equal and oppoſite forces are applied at the ſame point. In this caſe, however, we may always eſtabliſh the equilibrium by means of one additional force; or we may reſtore it in an endleſs variety of ways by means of two additional forces: but it is not worth while to go further into theſe minutiæ.

99. PROP. *To find the reſultant, or reſultants, of a ſyſtem of forces applied to different parts of a body, and acting in various directions in different planes.*

The beſt method of performing this will be by the method of rectangular co-ordinates, as follows. Suppoſe the directions of the ſeveral forces prolonged till they meet the plane Y A X (fig. 9. pl. II. or fig. 13. pl. I), and conceive them all applied at their points of interſection with that plane: then each of theſe forces may be reſolved into two others, the one perpendicular to the plane Y A X, the other ſituated in the plane. The perpendicular forces will have for their reſultant a force in like manner perpendicular to the plane, and the forces ſituated in the plane will obviouſly have their reſultant lying in the ſame plane. If the directions of theſe reſultants meet in a point, they may be compounded by art. 41. and the ſyſtem will have *one reſultant*: but in many, indeed, in moſt caſes, theſe directions will *not* meet; and then, as the effects of perpendicular forces not meeting in a point are independent, the ſyſtem will have *two diſtinct reſultants.*

COR. In the firſt of the above caſes a ſingle additional force may reſtore the equilibrium; in the latter, two at leaſt will be requiſite.

100. PROP. *If a body be kept in equilibrio by ſeveral forces acting at different points and in various directions not in one plane, the forces are ſuch as would be in equilibrio if applied to one point, and in directions reſpectively parallel to the former.*

For in any aſſumed plane the forces parallel and perpendicular to any line, will be the ſame whether applied at one point or many: and when the directions of any of the forces are out of this plane, ſuch extraneous forces may be reduced to others, one ſet acting in the plane, the other perpendicular to it; and both theſe will be equal in quantity in each caſe; therefore if the equilibrium obtains in the one caſe, it muſt in the other, both with reſpect to the perpendicular and parallel forces, all which will be ſufficiently obvious after recollecting what was ſhewn with regard to ſeveral forces in one plane. (arts. 82...86.)

If the weight of the body is taken into the conſideration it may be regarded as a ſingle force applied vertically at the centre of gravity: the reaſon of which will appear from the diſcuſſions in the next chapter.

SCHOLIUM.

101. We might now proceed to deduce the equations of equilibrium, &c. for forces acting at different points and in various planes: but as the proceſs would be complex and intricate, and after all but of trifling utility in elementary mechanics, except in the caſe of parallel forces already treated in arts. 95....98. it is thought beſt to omit them. The ardent enquirer may, however, be referred to the treatiſes on mechanics by *Lagrange* and *Prony*, for further information on this point. We merely add here, what follows at once from a compariſon of the preceding propoſition with art. 70. namely that

When a ſyſtem of forces acting upon different points of a body in various planes keeps it in equilibrio, the ſum of the moments taken relatively to each of three rectangular co-ordinates, of the components eſtimated in a plane perpendicular to that ordinate, is equal to zero.

CHAPTER III.

Centre of Gravity.

102. Def. *THE Centre of Gravity of any body or ſyſtem of bodies is that point about which the body or ſyſtem, acted upon only by the force of gravity, will balance itſelf in all poſitions: or it is a point which when ſupported, the body or ſyſtem will be ſupported, however it may be ſituated in other reſpects.*

The centre of gravity of a body is not always *within* the body itſelf: thus the centre of gravity of a ring is not in the ſubſtance of the ring, but in the axis of its circumſcribing cylinder; and the centre of gravity of a hollow ſtaff, or of a bone, is not in the matter of which it is conſtituted, but ſomewhere in its imaginary axis. Every body, however, has a centre of gravity, and ſo has every ſyſtem of bodies, as will ſoon be made evident: but it will be proper to premiſe a few brief remarks with reſpect to gravity itſelf, and its effect upon bodies ſubjected to its operation.

103. It is a fact eſtabliſhed by general obſervation in all ages and all countries, that whenever bodies are unſupported or left to themſelves, they begin to move downwards in vertical lines, and continue thus to move until they meet with ſomething which interrupts their motion or prevents their further deſcent. This is obſerved to take place not only with reſpect to large and weighty bodies, but to ſmaller ones, and even to the moſt minute particles into which they can be ſeparated, provided they are not ſo ſmall as to elude the obſervation of our ſenſes. And if certain ſubſtances, ſuch as ſmoke, and vapours, &c. ſeem to contradict this univerſal fact; it is becauſe they are only in *appearance* left to themſelves, while in reality they are ſupported, and put into an aſcending motion, by the action of the fluids, &c. that compoſe the atmoſphere which ſurrounds the earth. All bodies, and their moſt intimate particles tend towards a point which is either accurately or very nearly the centre of the terraqueous globe; yet this tendency is certainly not eſſential to matter, it is an effort which matter of itſelf is not able to make, being indifferent to either motion or reſt (18. 26.): we are authoriſed, then, to conclude that this tendency to motion is cauſed by a power not exiſting in the matter on which our obſervations are made, but in ſomething exterior; and this

force, without attempting to explain its nature and eſſence, we deſignate by the term *Gravity:* the general fact or event of bodies falling is denoted by the verbal noun *Gravitation;* and it is a part or conſequence of a more univerſal property, not here entered upon,—that of the mutual *Attraction* of the different bodies in the univerſe towards each other.

104. Since gravity impreſſes, or has a tendency to impreſs, on every particle of bodies, in an inſtant, a certain velocity with which they would begin to fall, if they were not ſupported; and ſince, abſtracting the influence of the air, this velocity would be the ſame for each of the moleculæ of bodies, whatever be their ſubſtance, it will not be difficult to attach a juſt and ſcientific meaning to that which is commonly called *weight:* it is the effort neceſſary to prevent a body from falling. But bodies fall in conſequence of the action of the force of gravity upon each of their particles, and they can be prevented from falling by a force equal and oppoſite to the reſultant or equivalent of all theſe actions. Hence, we may readily diſtinguiſh between the effect of gravity and that of weight, by adopting the language of *Condorcet*, when he ſays "the former is the power of tranſmitting, "or a tendency to tranſmit into every particle of matter a certain "velocity which is abſolutely independent on the number of "material particles; and the ſecond is the effort which muſt be "exerciſed to prevent a given maſs from obeying the law of "gravity. *Weight*, accordingly, *depends on the maſs, but gravity* "*has no dependence at all upon it.*"

105. Every particle of which bodies are compoſed receiving from gravity equal ſolicitations towards the centre of the earth, it follows that if the ſupports of bodies, whether large or minute, were taken away, and they were permitted to fall from equal altitudes, they would arrive at the ſurface of the earth after equal portions of time: and this is confirmed by experience; for under the exhauſted receiver of the Air-pump (where the reſiſtance of the air is removed) the heavieſt metals and the lighteſt feathers, or down, fall in the ſame time. If, therefore, a body is divided into ever ſo many parts, each of them left to itſelf would arrive at the ſurface of the earth, after the ſame time as would have been employed by the whole body in deſcending. All bodies being more or leſs porous, and poſſeſſing different degrees of denſity (10.) they will contain a greater or leſs number of equal moleculæ in the ſame volume or bulk: hence all bodies of equal bulk are not equal in weight. But ſince the weight is equal to the ſum of all the efforts exerciſed by gravity upon the conſtituent moleculæ of a body, it is proportional to its *denſity* or to its *maſs*. If p, p', p'', &c. be the ſeveral particles of which a body is compoſed, and M its maſs, then will $M = p + p' + p'' + \&c.$

and if g repreſent the force of gravity ſoliciting each particle, we ſhall have the weight $= g\text{M} = gp + gp' + gp'' + \&c.$

106. When bodies are compoſed of moleculæ, which are of the ſame ſize and ſubſtance, and ſimilarly poſited throughout, they are ſaid to be *homogeneous:* ſuch are the bodies which we ſhall conſider in this chapter; and in which the maſs will manifeſtly be proportional to the extenſion or the magnitude, ſo that the one may be ſubſtituted for the other in our inveſtigations. The vertical lines which would be deſcribed by bodies if ſubjected to the free action of gravity, are frequently called *lines of direction.* Since they would, if produced, meet at the centre of the earth, they cannot, ſtrictly ſpeaking, be parallel: but, with reſpect to any body or any ſyſtem of bodies connected for mechanical purpoſes, the whole ſpace occupied by all their particles muſt be ſo very minute compared with the magnitude of the earth, that their ſeveral lines of direction may be conſidered as parallel without any danger of ſenſible errour; juſt as we ſpeak of a moderate portion of the earth's ſurface as a plane, although it is, in fact, nearly ſpherical. Hence, then, the actions of gravity upon a body, or ſyſtem, may be conſidered as thoſe of parallel forces applied to their various particles; and, of conſequence, the concluſions and theorems which were deduced, arts. 92....98. with regard to ſuch forces, may be adopted in our preſent inveſtigations relating to the *centre of gravity.* This being admitted, the enſuing particulars are without difficulty inferred.

I. By the definition of the centre of gravity, when *it* is ſupported the body is in equilibrio; and from the nature of equilibrium it can only be produced ſingly by the exerciſe of a force equal and oppoſite to the reſultant of all the other forces acting upon the ſeveral particles of the body, that is, ſince in this caſe the forces are parallel, by a force (104.) equal to the weight of the body applied at the centre of parallel forces (91.): conſequently the centre of gravity coincides with the centre of parallel forces.

II. Varying the poſition of the body, will not cauſe any change in the centre of gravity; ſince any ſuch mutation will be nothing more than changing the directions of the forces, without their ceaſing to be parallel; and if the forces do not continue the ſame, in conſequence of the body being ſuppoſed at different diſtances from the earth, ſtill the forces upon all the moleculæ vary proportionally, and their centre remains unchanged (93.).

III. When a heavy body is ſuſpended by any other point than its centre of gravity, it will not reſt unleſs that centre is in the ſame vertical line with the point of ſuſpenſion: for in all other poſitions the force which is intended to enſure the equilibrium, will not be directly oppoſite to the reſultant of the parallel forces of gravity upon the ſeveral particles of the body, and of courſe the equilibrium will not be obtained.

IV. If a heavy body be ſuſtained by two or more forces, their directions muſt meet either at the centre of gravity of that body, or in the vertical line which paſſes through it. Thus, let B (fig. 10. pl. II.) be a heavy body whoſe centre of gravity is G, it will be at reſt if the ſtring *a* B by which it is ſuſpended hang vertically, whether the ſtring itſelf be fixed at *a*, or be attached to two other ſtrings whoſe lengths are *b a*, *c a*, and fixed at *b* and *c*. If the ſtring *a* B were either longer or ſhorter, the point *a* continuing fixed and the poſitions of *b* and *c* being unchanged, the body would ſtill hang at reſt, and the ſtrings *ba*, *ca*, under the ſame tenſion. If the body were removed vertically to the dotted ſituation in the diagram, it would be ſuſtained in that poſition by the ſtrings *bd* and *ce*, which would ſuffer the ſame tenſion as when they were united in the point *a*. If inſtead of the ſtrings we applied props *fg*, *hi*, in the ſame directions on the oppoſite ſides of the body, their feet *f*, *h*, being fixed; or, if other props *m n*, *o p*, were applied, either parallel to the former, or having their directions meeting in the vertical line *q* G, the body would ſtill be ſupported: and if the directions of the props and of the ſtrings were parallel, the compreſſion in the one caſe would be equal to the tenſion in the other. Either the compreſſion or the tenſion may in every caſe be readily eſtimated by means of the parallelogram of Forces, &c. (46.) the weight of the body being known.

V. When a body ſtands upon a plane, if a vertical line paſſing through the centre of gravity fall within the baſe on which the body ſtands, it will not fall over; but if that vertical line paſſes without the baſe, the body will fall, unleſs it be prevented by a prop or a cord. When the vertical line falls upon the extremity of the baſe, the body *may* ſtand, but the equilibrium may be diſturbed by a very trifling force; and the nearer this line paſſes to any edge of the baſe the more eaſily may the body be thrown over; the nearer it falls to the middle of the baſe, the more firmly the body ſtands.

VI. The various motions of animals, if attentively conſidered, will appear to be regulated conſiſtently with the principles juſt ſtated.

Thus, when a man endeavours to riſe from his ſeat he thruſts forward his body, and draws his feet backward till the vertical line from the centre of gravity falls juſt before his feet; this enables, or indeed compels him to riſe; and to prevent falling forwards he advances one of his feet, till the vertical line of direction is brought between his feet, in conſequence of which he may ſtand firmly. In walking, he firſt extends his hindmoſt leg and foot almoſt to a right line, and at the ſame time bends the knee of his fore leg a little; by this means his body is thruſt forward, and the line of direction from its centre of gravity falls beyond the fore foot, on which account he is ready to fall, but prevents it by immediately taking up the other foot, and putting it forward beyond the line of direction. After the ſame manner, he thruſts himſelf forward by the leg which is now the hindmoſt, till the line of direction from the centre of gravity be beyond his fore foot, when he again ſets his hind foot forward: and thus he continues the motion of walking at pleaſure. While walking a man always ſets down one foot before the other is taken up; ſo that at each ſtep he has both feet upon the ground. But in running he takes one up before he ſets the other down; ſo that his feet touch the ground alternately for moments of time, and in the intermediate portions he does not touch it at all.

In walking up hill a man bends his body more forward than in walking on a horizontal road, that the line of direction may be thrown before his feet: in walking down hill he rather leans backwards to prevent the line of direction from being too forward, which would occaſion his fall. In carrying a burthen a man always leans the contrary way to that in which the burthen lies, in order that the common centre of gravity of both may have its line of direction between his feet.

When a quadruped, as a horſe, moves, he leans forward, all at once lifting up one of his fore feet and one of his hind feet; when the right leg before is puſhed forward the left leg behind is moved on at the ſame time; and this motion being made, the left leg before takes its turn conjointly with the right leg behind, and ſo on: as the body when ſtanding is ſupported by four props which form a rectangle, the moſt commodious mode of moving is to change the poſitions of two feet at a time diagonally, and thus to cauſe the centre of gravity of the animal's body to make but a ſmall movement, and to remain always very nearly in the direction of the two points of ſupport. This rule of motion is always obſerved, but with theſe differences: in the pace there are four times in the complete movement; if the fore right leg be moved firſt, the left leg behind follows the inſtant after; then the left fore leg has its turn, which is followed the next inſtant by the right foot behind. Thus the fore right foot comes firſt to the earth, the left foot behind next, the left fore foot third, and the right hind foot laſt: ſo that there are four motions, and three intervals, of which the firſt and laſt are ſhorter than the middle one. In trotting, he takes up two feet together, and ſets down two together, diagonally oppoſite. In galloping he takes up his feet one by one, and ſets them down one by one; but he ſtrikes with the two fore feet nearly at once, and the two hind feet nearly at once: all the time the line from the centre of gravity lies forward, ſo that the animal requires the fore feet to come to the ground at ſhort intervals to prevent a fall. If the various motions of other animals be conſidered, they will all be found conformable to the ſame principles. But it would be tedious to enter further into the detail in this place.

VII. To find the centre of gravity mechanically, it is only requiſite to diſpoſe the body ſucceſſively in two poſitions of equilibrium, by the aid of two forces in vertical directions, applied in ſucceſſion to two different points of the body; the point of interſection of theſe two directions will ſhew the centre required.

This may be exemplified by particulariſing a few methods. If the body have plane ſides, as a piece of board, hang it up by any point, then a plumb-line ſuſpended from the ſame point will paſs through the centre of gravity; therefore mark that line upon it: and after ſuſpending the body by another point, apply the plummet to find another ſuch line, then will their interſection ſhew the centre of gravity.

Or thus: hang the body by two ſtrings from the ſame point fixed to different parts of the body; then a plummet hung from the ſame tack will fall on the centre of gravity.

Another method: Lay the body on the edge of a triangular priſm, or ſuch like, moving it to and fro till the parts on both ſides are in equilibrio, and mark a line upon it cloſe by the edge of the priſm: balance it again in another poſition, and mark the freſh line by the edge of the priſm; the vertical line paſſing through the interſection of theſe lines, will likewiſe paſs through the centre of gravity. The ſame thing may be effected by laying the body on a table, till it is juſt ready to fall off, and then marking a line upon it by the edge of the table: this done in two poſitions of the body, will in like manner point out the centre of gravity.

107. When a plane or a line can be ſo drawn as to divide a ſolid or a plane into two parts equal and ſimilar, or ſo that its moleculæ ſhall be diſpoſed two by two, in the ſame manner, with reſpect to ſuch plane or ſuch line, we may call the body ſymmetrical with regard to that plane or axis. And in all ſuch bodies, it is obvious that the ſum of the moments of its ſeveral moleculæ with relation to ſuch plane or axis, will be nothing: for, if we take two particles diſpoſed in the ſame manner but on different ſides, their moments will be equal but with contrary

ſigns; and, conſequently, their ſum will be equal to *zero*: and the ſame may be ſhewn of every other pair of moleculæ, ſimilarly ſituated: whence, as (by hyp.) there are none but what are ſimilarly ſituated, the reſultant of the ſyſtem will be in ſuch plane, or line, (97.) and, of conſequence, its centre of gravity will be there alſo. The ſame reaſoning may be extended to the centre of figure or of magnitude, that is, the point with reſpect to which a whole body ſhall be ſymmetrical. Hence we conclude that the centre of gravity of a right line, or of a parallelogram priſm or cylinder, is in its middle point; as is alſo, that of a circle, or of its circumference, or of a ſphere, or of a regular polygon; that the centre of gravity of a triangle is ſomewhere in a line drawn from any angle to the middle of the oppoſite ſide; that of an ellipſe, a parabola, a cone, a conoid, a ſpheroid, &c. ſomewhere in its axis. And the ſame of all ſymmetrical figures whatever.

108. PROP. *To deduce ſome general theorems which may be uſeful in finding the centre of gravity of any propoſed body.*

The determination of the centre of gravity, being reduced to that of the centre of parallel forces, we may here adopt the theorem for the moments laid down at the end of art. 97. From which it will follow, that if p, p', p'', &c. (fig. 11. pl. II.) be equal material particles, and g the point through which the reſultant R of the gravitating forces upon theſe particles always paſſes; and ABCD be a vertical plane, on which perpendiculars from $p, p'\, p''$, and g are let fall, then will the ſum of the products of the forces upon p, p', p'', into their reſpective diſtances from A B C D, be equal to the product of the reſultant R into its diſtance, where the force R would be equal to thoſe upon $p+p'+p''$. The ſame would likewiſe obviouſly hold with reſpect to perpendiculars upon the other plane A E C G: and ſince the ſame will alſo obtain with relation to any vertical plane, although the poſition of p, p', and p'' be changed, provided they retain their *relative* ſituations, it will of courſe obtain when the poſition of the ſyſtem is ſo varied that A E B F becomes a vertical plane: conſequently the equality of the products may be affirmed with regard to all the three planes at the ſame time, and if the diſtances from the ſeveral planes be referred to the rectangular co-ordinates A X, A Y, A Z, we may readily appropriate the equations (97. III.) to our preſent purpoſe. Denote, as before, the force of gravity by g, the diſtances referred to A X by d, d', d'', &c. the diſtances referred to A Y, by D, D′, D″, &c. and thoſe referred to Z by $\delta, \delta', \delta''$, &c. the diſtances from the centre of parallel forces to the ſame axis being denoted by X, Y, and Z: then we ſhall have

$$\mathrm{R}\,\mathrm{X} = g p d + g p' d' + g p'' d'' + \&c.$$

$$RY = gpD + gp'D' + gp''D'' + \&c.$$
$$RZ = gp\delta + gp'\delta' + gp''\delta'' + \&c.$$

But $R = gp + gp' + gp'' + \&c.$ and $M = p + p' + p'' + \&c.$ whence

$$\text{(I.)} \ldots \left\{ \begin{aligned} X &= \frac{pd + p'd' + p''d'' + \&c.}{M} \\ Y &= \frac{pD + p'D' + p''D'' + \&c.}{M} \\ Z &= \frac{p\delta + p'\delta' + p''\delta'' + \&c.}{M} \end{aligned} \right.$$

Here, if we adopt the language of fluxions, and put x, y, and z, for the variable diſtances from A upon AX, AY, and AZ, reſpectively, we may convert theſe equations into the following form, which will render it more uſeful in many inveſtigations.

$$\text{(II.)} \ldots \left\{ \begin{aligned} X &= \frac{\text{fluent of } x\dot{M}}{\text{fluent of } \dot{M}} = \frac{\text{fluent } x\dot{M}}{M}. \\ Y &= \frac{\text{fluent of } y\dot{M}}{\text{fluent of } \dot{M}} = \frac{\text{fluent } y\dot{M}}{M}. \\ Z &= \frac{\text{fluent of } z\dot{M}}{\text{fluent of } \dot{M}} = \frac{\text{fluent of } z\dot{M}}{M}. \end{aligned} \right.$$

As theſe values together determine only one point, we ſee that a body has but one centre of gravity; of which the three equations determine the three co-ordinates, and of conſequence the diſtances of the centre from three planes reſpectively perpendicular to each other.

Theſe reſults being entirely independent of g, that is, of the force of gravity, ſome philoſophers have preferred the term *centre of inertia* to that of *centre of gravity*: other philoſophers, have, on account of other properties, preferred different terms, which will be mentioned as we proceed.

When it is required to find the centre of gravity of any line whatever, it is conſidered as compoſed of a ſeries of material heavy particles contiguous to each other, and connected by a law which is expreſſed by the equation of the curve, with reſpect to any two rectangular co-ordinates x and y. In this caſe the centre of gravity will manifeſtly be in the ſame plane as the propoſed line, ſo that the plane YAX may contain the centre of gravity, whence $Z = 0$, and the value of y being deduced, from the equation of the curve in terms of y, the centre of gravity may be determined by theſe two equations:

$$\text{(III.)} \ldots X = \frac{\text{flu. } x\dot{M}}{M} \cdots Y = \frac{\text{flu. } y\dot{M}}{M}$$

If the curve have two legs ſymmetrical with relation to any axe, then we may reckon the vertex of that axe the origin of the co-ordinates, and y being $= 0$, we ſhall only require $X =$

$\frac{\text{flu. } x\dot{M}}{M}$; but in this caſe, the fluxion of M the curve being $= \sqrt{\dot{x}^2+\dot{y}^2}$, we have alſo,

$$\text{(IV.)} \ldots X = \frac{\text{flu. } x\sqrt{\dot{x}^2+\dot{y}^2}}{M}$$

If the figure is a plane, its centre of gravity will be in the ſame plane, and of courſe we may take $Z=0$; and becauſe $\dot{M}=y\dot{x}$, our equations become,

$$\text{(V.)} \ldots \begin{cases} X = \frac{\text{flu. } xy\dot{x}}{\text{flu. } y\dot{x}} = \frac{\text{flu. } xy\dot{x}}{M} \\ Y = \frac{\text{flu. } y^2\dot{x}}{\text{flu. } y\dot{x}} = \frac{\text{flu. } y^2\dot{x}}{M} \end{cases}$$

Here again, if the plane be ſymmetrical with reſpect to the axe, the equation for X will alone be wanted.

When the figure is the ſuperficies of a body generated by the rotation of any line about an axis, then will $Y=0$, and $Z=0$: and putting $\pi=3{\cdot}14159$, &c. $2\pi y$ will denote the circumference of the generating circle, and $2\pi y\dot{M}$ the fluxion of the ſurface, wherefore

$$\text{(VI.)} \ldots\ldots X = \frac{\text{flu. } 2\pi yx\dot{M}}{\text{flu. } 2\pi y\dot{M}} = \frac{\text{flu. } yx\dot{M}}{\text{flu. } y\dot{M}}.$$

When the figure is a ſolid of revolution, the centre of gravity being upon its axis, we have $Y=0$, $Z=0$: and πy^2 denoting the area of the circle whoſe radius is y, and $\pi y^2\dot{x}=\dot{M}$ the fluxion of the ſolid, we readily find,

$$\text{(VII.)} \ldots X = \frac{\text{flu. } \pi y^2 x\dot{x}}{\text{flu. } \pi y^2\dot{x}} = \frac{\text{flu. } y^2 x\dot{x}}{\text{flu. } y^2\dot{x}}$$

COR. When $X=0$, $Y=0$, $Z=0$; that is, when the centre of gravity is at the origin of the co-ordinates, the equations (I.) will give $pd+p'd'+p''d''+\&c.=0$, or $d+d'+d''+\&c.=0$; in like manner, $D+D'+D''+\&c.=0$, and $\delta+\delta'+\delta''+\&c.=0$: and the ſame will hold with reſpect to any other co-ordinates whoſe origin is the centre of gravity; that is, the ſum of perpendiculars from all the particles affected with contrary ſigns as they lie on different ſides of either axis is then equal to *zero*: and conſequently, *if on any plane paſſing through the centre of gravity of a body perpendiculars be let fall from each of its moleculæ, the ſum of all the perpendicular diſtances on one ſide of the plane will be equal to the ſum of all thoſe on the other ſide.*

109. PROP. *The poſition, diſtance, and motion of the centre of*

gravity of any body is a medium of the poſitions, diſtances, and motions of all the particles in the body.

Firſt, the *diſtance* of the centre of gravity of any body from a given plane, is an average of the diſtances of each of its conſtituent particles from the ſame plane: For, let A B (fig. 1. pl. III.) be a body whoſe centre of gravity is G, and C D any plane, from which the diſtances are to be eſtimated: then is the diſtance of any particle as p, beyond the plane A B (drawn through the body parallel to C D), equal to $p\mathrm{E}+\mathrm{EH}=p\mathrm{E}+\mathrm{GK}$; and if n be the number of particles p on the ſame ſide of A B, then will the ſum of all the diſtances p H, be equal to the ſum of all the perpendiculars p E, added to all the diſtances E H or G K, that is, $n\,.\,p\,\mathrm{H}=(n\,.\,\mathrm{GK})+(n\,.\,p\,\mathrm{E})$. Again, let p' be a particle of the body between the parallel planes AB, CD, its diſtance from the plane C D will be equal to $\mathrm{FI}-\mathrm{F}p'=\mathrm{GK}-\mathrm{F}p'$; and, if n' be the number of all the particles p' of the body between A B and C D, we ſhall have the ſum of all their diſtances, that is, $n'\,.\,\mathrm{P'I}=(n'\,.\,\mathrm{GK})-(n'\,.\,\mathrm{F}p')$: Hence $(n\,.\,p\,\mathrm{H})+(n'\,.\,p'\mathrm{I})=(n+n').\,\mathrm{GK}+(n\,.\,p\,\mathrm{E})-n'\,.\,\mathrm{F}p')$. But by the corollary to the preceding propoſition, the ſum of all the perpendiculars, on one ſide of the plane AB, is equal to the ſum of all thoſe on the other: conſequently $n\,.\,p\,\mathrm{E}-n'\,.\,\mathrm{F}p'=0$, and $n\,.\,p\,\mathrm{H}+n'.\,p'\mathrm{I}=(n+n')\,\mathrm{GK}$; that is, GK is the mean of all the diſtances from every particle of the body to the plane CD. And the like may be ſhewn of any other plane, or, of the body in any other poſition.

Secondly, the *motion* of the centre of gravity is an average of the motions of the ſeveral particles of which the body is compoſed: For, ſuppoſe the body to have moved towards the plane CD, its centre of gravity having paſſed from G to g, along the right line GK, and the body itſelf now ſituated as in *a b*. Here, it might be ſhewn, in the ſame manner as in the firſt caſe, that $(n+n')\,g\mathrm{K}$, is the ſum of the perpendiculars from all the particles in the body to the plane C D: conſequently, $(n+n')\,\mathrm{GK}-(n+n')\,g\,\mathrm{K}=(n+n')\,\mathrm{G}g$, is the ſum of the approaches of all the particles towards C D, and Gg being the $\frac{1}{n+n'}$ of this ſum is evidently their mean. And if the motion of G was along a curvilinear path, the ſame concluſion would be deduced, if we conceive the curve to be ſeparated into its infinitely ſmall elements, and the motions with reſpect to each determined. The concluſion will likewiſe be the ſame although the body may have turned round ſome centre or axis.

This property of the centre of gravity has occaſioned it to be called, by ſome authors, the *centre of poſition*; by others, the *centre of mean diſtances.* The celebrated French mathematician *Carnot*, in his *Geometrie de Poſition*, makes uſe of the latter term, and has endeavoured to ſhew "that the theory of this centre belongs to geometry, and that it would be very advantageous for "the progreſs of that ſcience, to re-eſtabliſh in this reſpect the "natural order of ideas."

110. PROP. *The common centre of gravity, or of poſition, of two bodies divides the right line drawn between the reſpective centres of the two bodies in the inverſe ratio of their maſſes.*

Let A and B (fig. 12. pl. II.) be two bodies whoſe centres of gravity are united by the inflexible line AB, then, if A G : G B : : B : A, G will be the common centre of gravity of thoſe two bodies, that is, if G be ſupported, thoſe two bodies actuated by the force of gravity will be in equilibrio in any poſition: for, through G let the vertical line CD be drawn, on which let fall the perpendiculars A F, B E, from the centres of gravity of the two bodies, then, becauſe of the ſimilar triangles B E G, A F G, we have A G : GB : : A F : B E : : B : A, whence A . A F= B . BE. But A . AF, is equal to the ſum of the products of all the particles in A into their reſpective diſtances from C D, by the laſt prop. and, in like manner B . B E, is equal to the ſum of the products of the particles in B into their reſpective diſtances; therefore g . A . AF=g . B . BE; that is, the ſums of the moments of the forces of gravity upon A and B, with reſpect to CD are equal. If A and B be removed to any other poſition as a, b, the point G remaining fixed, it will appear in like manner that $g \, . \, a \, . \, af = g \, . \, b \, . \, be$: ſo that G is the centre of the forces of gravity with reſpect to A and B, that is, it is their common centre of gravity. In a manner but very little different G may be ſhewn to be their common centre of poſition: and the two bodies, if conſidered as united by their centres of poſition at G, will then, as well as when their centres are ſeparated by AB, have the ſum of the perpendiculars from the ſeveral particles on one ſide of any plane paſſing through G, equal to the ſum of all the perpendiculars on the other ſide of it.

COR. The centre of gravity of three or more bodies may, hence, be found, by conſidering the firſt and ſecond as a ſingle body equal to their ſum and placed in their common centre of gravity, determining the centre of gravity of this imaginary body, and a third. Theſe three again being conceived united at their common centre, we may proceed, in like manner, to a fourth; and ſo on, *ad libitum.*

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111. It may not be altogether uſeleſs to ſhew that the centre of gravity of three or more bodies, as determined by this pro-

poſition and corollary, will be the ſame by whatever ſteps the proceſs is conducted. Let, therefore, a, b, c, denote the maſſes of the three bodies placed at A, B, C (fig. 2. pl. III.); and let G be the centre of gravity of A and B: make $EG : EC :: c : a+b$, and draw AEF; then F will be the centre of gravity of B and C, and $AE : EF :: b+c : a$. Draw GD parallel to BC, and FH parallel to BA; then will $AG : AB :: GD : BF = GD \cdot \frac{AB}{AG} = GD \cdot \frac{a+b}{b}$; and $GE : EC :: GD : CF = GD \cdot \frac{EC}{GE} = GD \cdot \frac{a+b}{c}$: wherefore, $BF : CF :: \frac{a+b}{b} : \frac{a+b}{c} :: c : b$, and F is the centre of gravity of B and C. Again, $CB : CF :: BG : FH = BG \cdot \frac{CF}{CB} = BG \cdot \frac{b}{b+c}$; but $AG = BG \frac{b}{a}$: and $FH : GA :: \frac{b}{b+c} : \frac{b}{a} :: a : b+c :: EF : AE$, whence E is the ſame point as was determined from A and E. From this the ſame may be ſhewn true in caſes where the number of bodies is greater, following the changes, ſtep by ſtep. Thus, in 4 bodies a, b, c, d, the order a, b, c, d, will give the ſame reſult as c, a, d, b; ſince $(a, b, c,)\ d$, is ſhewn to give the ſame as $(c, a, b,)\ d$; and $(c, a,)\ b$, d, the ſame as $(c, a,)\ d$, b, or as c, a, d, b.

112. *If the particles or bodies of any ſyſtem be moving uniformly and rectilineally, with any velocities and directions whatever, the centre of gravity is either at reſt, or moves uniformly in a right line.* For, let one of the bodies as C (fig. 3. pl. III.) move uniformly from C to D: then, g being the centre of gravity of the remaining bodies, join Dg, and take gE to ED, as the maſs D, to the ſum of the other maſſes; then, is GE obviouſly parallel to CD, and $CD : GE :: A+B$ &c. $: C$, GE being in this caſe the path of the common centre. And thus may the motion of the centre of gravity be found, which would be produced by the uniform rectilinear motion of each body in the ſyſtem. Then, becauſe each correſponding motion of the centre of gravity is uniform and rectilinear, the reſult of the whole will be either a uniform and rectilinear motion, or no motion at all.

Hence, if a rotatory motion be communicated to a body and it be then left to move freely, the axis of rotation will paſs through the centre of gravity: for, that centre, either remaining at reſt or moving uniformly forward in a right line, has no rotation.

Here too it may be remarked, that *a force applied at the centre of gravity of a body, cannot produce a rotatory motion.* For every particle reſiſts, by its inertia, the communication of motion, and in a direction oppoſite to that in which the force applied tends to communicate the motion; the reſiſting forces, therefore, act in parallel lines, in the ſame manner as the gravitating forces: conſequently, ſince the latter balance each other on the centre of gravity, the former will do ſo likewiſe.

The truths thus briefly ſhewn in this article, are often diſcuſſed more at large; but a bare ſtatement of them muſt ſuffice in this place, ſince they fall naturally under the province of Dynamics.

113. Prop. *To find the centre of gravity of the perimeter of any right-lined figure.*

If the figure be a regular polygon, the centre of gravity of its perimeter will be the centre of its circumſcribing or inſcribed circle. But if it be irregular, we conceive the particles of each line to be all placed at their reſpective centres of gravity, that is, at the middle of each line (107.), and proceed thus. Join the middle points of any two of the ſides, as F B, B C, (fig. 4. pl. III.) by the line fb : make $fG : Gb :: BC : BF$, or $fG : fb :: BC : BC+BF$, and G will be the common centre of gravity of the two ſides B C, B F. Then join G c, c being the middle point of a third ſide C D; make $GG' : cG' :: CD : BC+BF$, and G′ will be the centre of gravity of the three ſides, F B, B C, C D. In like manner, join G′ and the middle point d of a fourth line, and find the new centre of gravity G″: and ſo on, for all the ſides of the figure.

Or, drawing through any point A in the ſame plane, the rectangular co-ordinates AX, AY, and denoting the lines F B, BC, C D, &c. by p, p', p'', &c. the diſtances of their middle points from AX, by d, d', d'', &c. and the diſtances of the ſaid middle points from AY, by D, D′, D″, &c. we ſhall have (108. I.)

$$X=AI=\frac{p\,d+p'\,d'+p''\,d''+\&c.}{p+p'+p''+\&c.}\cdot\quad Y=AH=\frac{p\,D+p'\,D'+p''\,D''+\&c.}{p+p'+p'+\&c.}\cdot$$

Then, drawing H G, I G, parallel to AX and AY, their interſection G‴ will be the centre of gravity required.

114. Prop. *To find the centre of gravity of a plane triangle.*

Let ABC (fig. 5. pl. III.) be any triangle: draw AE from one of its angles to the middle of its oppoſite ſide, then will AE divide every line which can be drawn in the triangle parallel to BC into two equal parts; conſequently the ſurface of the triangle is ſymmetrically diſpoſed with reſpect to AE, and the centre of gravity will be found in that line (107.). For a like reaſon, if from any other angle as C we draw CD to the middle of its oppoſite ſide, the centre of gravity of the triangle will be ſomewhere upon that line: it will, therefore, be at G the interſection of thoſe lines.

Now, ſince the points D, and E, divide the ſides BA, BC, of the triangle proportionally, the line DE which joins them muſt be parallel to the third ſide AC: hence the triangles BDE, BAC are ſimilar, and ſo are the triangles G D E, G C A. Conſequently, GD : GC :: GE : GA :: DE : AC :: BD : BA :: 1 : 2.

Therefore $AG=\frac{2}{3}AE$, and $CG=\frac{2}{3}CD$, alſo $BG=\frac{2}{3}BF$.

Cor. If AB, BC, CA, be denoted by a, b, and c, and AG, BG, CG, by m, n, and r, we have, by a well-known theorem in geometry, the three following equations:

$$\begin{cases} AB^2+AC^2=2\,BE^2+2\,AE^2, & i.\,e.\quad a^2+c^2=\frac{1}{2}b^2+\frac{9}{2}m^2 \\ AB^2+BC^2=2\,CF^2+2\,BF^2, & \qquad\quad a^2+b^2=\frac{1}{2}c^2+\frac{9}{2}n^2 \\ AC^2+BC^2=2\,AD^2+2\,CD^2, & \qquad\quad c^2+b^2=\frac{1}{2}a^2+\frac{9}{2}r^2 \end{cases}$$

Adding together theſe equations and clearing them of fractions, there reſults $a^2+b^2+c^2=3\,m^2+3\,n^2+3\,r^2$; which gives this curious theorem: *In any plane triangle the ſum of the ſquares of the three ſides, is equal to thrice the ſum of the ſquares of the diſtances of each of its angles from the centre of gravity.*

Cor. 2. From the three equations in the preceding corollary, we readily find $m=\frac{1}{3}\sqrt{2\,a^2+2\,c^2-b^2}$, $n=\frac{1}{3}\sqrt{2\,a^2+2\,b^2-c^2}$, and $r=\frac{1}{3}\sqrt{2\,b^2+2\,c^2-a^2}$: by means of which the diſtance of the centre of gravity from either angle of any given triangle may be ſoon found.

115. Prop. *To find the centre of gravity of a trapezium.*

Let ABCD (fig. 6. pl. III.) be any trapezium. Divide it into two triangles by the diagonal AC: find their centres of gravity H and I, by the laſt propoſition; then join IH, and make IG, to GH, as the triangle ABC, to the triangle ADC, and G will be the centre of gravity of the trapezium.

Or, divide the trapezium into two other triangles by the diagonal BD; find their centres of gravity E, and F, and draw EF. Then the centre of gravity of the trapezium muſt be in the line HI; and it muſt likewiſe be in the line EF; conſequently it muſt be in G their point of interſection.

When two ſides of the quadrilateral, as AD and BC, (fig. 7. pl. III.) are parallel, the centre of gravity is ſomewhere upon the line KL joining their middle points: and it is ſomewhere upon the line EF joining the centres of gravity of the two component triangles ABD, CBD: it is, therefore, at G the point of interſection of EF and KL. And in this caſe $KG=\frac{1}{3}KL\cdot\frac{BK+AD}{BK+AL}$.

For, drawing the lines Ee, Ff, parallel to AD, BC; ſince $EL=\frac{1}{3}BL$, and $KF=\frac{1}{3}KD$, we have $Ee=\frac{1}{3}BK=\frac{1}{6}BC$, and $Ff=\frac{1}{3}LD=\frac{1}{6}AD$. Alſo $Le=\frac{1}{3}KL$, and $Kf=\frac{1}{3}KL$. Whence $ef=\frac{1}{3}KL$. Now, the ſimilar triangles GeE, GfF give $Ee : Ge :: Ff : Gf$, therefore $Ee+Ff : Ge+Gf :: Ff : Gf$; that is, $\frac{1}{6}BC+\frac{1}{6}AD : \frac{1}{3}KL :: \frac{1}{6}AD : Gf=\frac{\frac{1}{3}KL\cdot AD}{BC+AD}$. Hence then, $KG=Kf+fG=\frac{1}{3}KL+\frac{\frac{1}{3}KL\cdot AD}{BC+AD}=\frac{1}{3}KL\cdot\frac{BC+2\,AD}{BC+AD}$ $=\frac{1}{3}KL\cdot\frac{BK+AD}{BK+AL}$.

116. Thus alſo, to find the centre of gravity of the area of any rectilinear figure, divide it into triangles, and find their reſpective centres of gravity, by art. 114. then conceiving each of the triangles collected into their reſpective centres of gravity, their common centre of gravity may be found, either by the method deſcribed in art. 111. or by the theory of moments (equa. I. 108.)

117. PROP. *To find the centre of gravity of any triangular pyramid.*

Let A B C D (fig 8. pl. III.) be a triangular pyramid: on one of its faces, as B C D, draw D I from the angle D to the middle of its oppoſite ſide B C; ſet off $I\gamma = \frac{1}{3}$ I D, and γ will (114.) be the centre of gravity of that face. From A the vertex of the pyramid, draw A γ, and it will paſs through the centres of gravity of every ſection of the pyramid parallel to B C D: conſequently, it will paſs through the centre of gravity of the pyramid. Again, on the face A B C, draw A I from A to the middle of the oppoſite ſide; ſet off $I g = \frac{1}{3}$ I A, and join g D: the centre of gravity of the pyramid will, it is obvious, be found upon this line alſo. But the two lines A γ, D g, being both in the plane of the triangle A D I, muſt interſect each other, and G their point of interſection muſt neceſſarily be the centre of gravity of the pyramid.

Now, if we conceive $g\gamma$ drawn, it will be parallel to A D, ſince $Ig : IA :: I\gamma : ID$: hence the triangles $Gg\gamma$, GDA, are ſimilar, and,

$$Gg : GD :: G\gamma : GA :: g\gamma : AD :: 1 : 3.$$

Therefore, $Gg = \frac{1}{3} GD = \frac{1}{4} gD$; $G\gamma = \frac{1}{3} AG = \frac{1}{4} A\gamma$, &c.

COR. 1. If AB be put $=a$, AC $=b$, AD $=c$, BC $=d$, BD $=e$, CD $=f$; and the diſtances from the angles to the centre of gravity, AG $=m$, BG $=n$, CG $=r$, DG $=s$: then, by a well known theorem $AI^2 = \frac{AB^2 + AC^2 - 2\,BI^2}{2} = \frac{a^2 + b^2 - \frac{1}{2}d^2}{2}$; and, by the ſame, $DI^2 = \frac{e^2 + f^2 - \frac{1}{2}d^2}{2}$: again, ſince $AG = \frac{3}{4} A\gamma$, we have $AG^2 = \frac{9}{16} A\gamma^2 = \frac{9}{16}\left(\frac{AI^2 . D\gamma + AD^2 . I\gamma - ID . I\gamma . \gamma D}{ID}\right)$. (See the Lemma at pa. 128. Simpſon's Select Exerciſes.) Subſtituting in this expreſſion the literal values of AI, D γ, &c. and reducing, we at length obtain $AG^2 = \frac{1}{16}(\overline{3a^2 + 3b^2 + 3c^2 - d^2 + e^2 + f^2}$. A like equation being in the ſame manner deducible for BG^2, CG^2, DG^2, we have the following general theorem: *In any triangular pyramid the diſtance of any one of the angles of the pyramid, from the centre of gravity, is equal to one-fourth of the ſquare root of the difference of thrice the ſum of the ſquares of the three edges meeting at that angle, and the ſum of the ſquares of the other three edges.*

Cor. 2. Adding together the four equations found by the preceding corollary, viz.

$$AG^2=\tfrac{1}{16}(3\,AB^2+3\,AC^2+3\,AD^2)-\tfrac{1}{16}(BC^2+BD^2+CD^2)$$
$$BG^2=\tfrac{1}{16}(3\,BA^2+3\,BC^2+3\,BD^2)-\tfrac{1}{16}(AC^2+AD^2+CD^2)$$
$$CG^2=\tfrac{1}{16}(3\,CA^2+3\,CB^2+3\,CD^2)-\tfrac{1}{16}(AB^2+AD^2+BD^2)$$
$$DG^2=\tfrac{1}{16}(3\,AD^2+3\,DC^2+3\,BD^2)-\tfrac{1}{16}(AB^2+AC^2+BC^2)$$

there ariſes the following:

$$AG^2+BG^2+CG^2+DG^2=\tfrac{1}{4}(AB^2+AC^2+AD^2+BC^2+BD^2+CD^2).$$

That is, *the ſum of the ſquares of the diſtances from the angles of any triangular pyramid to its centre of gravity, is equal to a fourth of the ſum of the ſquares of the ſix edges of the pyramid.*

Theſe two corollaries are manifeſtly analogous to thoſe of art. 114.

Cor. 3. If the ſides of the baſe are equal, as $BD=BC=CD$, we have $AG^2=\tfrac{3}{16}(AB^2+AC^2+AD^2-BC^2)$.

Cor. 4. If, moreover, $AB=AC=AD$, then $AG^2=\tfrac{3}{16}(3\,AB^2-BC^2)$.

Cor. 5. If all the edges of the pyramid are equal, we have for the regular tetraedron, $AG^2=BG^2=CG^2=DG^2=\tfrac{3}{8}AB^2$, whence $AG=\tfrac{1}{4}AB\sqrt{6}$.

Cor. 6. A pyramid whoſe baſe is any polygon, will have its centre of gravity upon the line drawn from the vertex to the centre of gravity of the baſe, and at the diſtance of $\tfrac{3}{4}$ of its length from its vertex. Which will be ſufficiently obvious if it be conſidered, that any ſuch pyramid is compoſed of triangular pyramids, whoſe centres of gravity all lie in one plane parallel to the baſe; conſequently their common centre of gravity muſt be in the ſame plane, and muſt likewiſe be in the line drawn from the vertex to the common centre of gravity of all the triangles which conſtitute the baſe.

118. Prop. *To find the centre of gravity of a circular arc.*

From the middle point of the propoſed arc M A M (fig. 9. pl. III.) conceive the line A C drawn through C the centre of the circle, on which let AP be denoted by x, the variable ordinate P M by y, and the radius A C of the circle by r, the half arc being denoted by z. Then accounting A the origin of the co-ordinates, ſince the curve is ſymmetrical with reſpect to A C, we need only make uſe of the equation (108. IV.) which, ſubſtituting z for M will become, $X=\dfrac{\int x\sqrt{\dot{x}^2+\dot{y}^2}}{z}$*. Now the equation expreſſing the relation between the rectangular co-ordinates of a circle is $y^2 = 2rx - x^2$; by means of which we have $X = \frac{r}{z}\int\frac{x\dot{x}}{\sqrt{(2ax - x^2)}}$. The fluent of this is $X = \frac{r}{z}(z - y) = r - \frac{ry}{z}$: which needs no correction, becauſe when $y = 0$, $z = 0$. Hence then, G being the centre of gravity, we have $AG = r - \frac{ry}{z} = AC - \frac{ry}{z}$. Conſequently $CG = AC - AG = \frac{ry}{z}$: that is, *the diſtance of the centre of gravity of a circular arc from its centre, is a fourth proportional to the arc, the radius, and the chord of the arc.*

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* Here, (as well as in the ſubſequent parts of this work, we make the character $\int$ denote the fluent of the expreſſion which ſtands after it.

COR. 1. When the arc is a ſemicircle, the chord is double the radius, and $CG = \frac{2r}{3{\cdot}141593} = \frac{r}{1{\cdot}57079} = {\cdot}63662\,r$.

COR. 2. When $x = 2r$, $y = 0$, and conſequently $CG = \frac{ry}{z} = 0$: that is, the centre of gravity coincides with the centre of the circle; as is ſufficiently obvious independent of the fluxional proceſs.

119. PROP. *To find the centre of gravity of a circular ſegment.*

Let MAM′P in the figure laſt referred to be the ſegment propoſed, and let the parts be denoted as before. Here we take the firſt of the equations (108. V.), that is $X = \frac{\text{flu. } xy\dot{x}}{M}$, where M denotes the area of APM. Now, ſince $y = \sqrt{2rx - x^2}$, we have $XM = \int xy\dot{x} = \int x\dot{x}\sqrt{2rx - x^2} = -\frac{(2rx - x^2)^{\frac{3}{2}}}{3} + rM$: this divided by M gives $X = AG = r - \frac{PM^3}{3M} = CA - \frac{PM^3}{3 \text{ area } APM}$. Conſequently $CG = \frac{PM^3}{3 \text{ area } APM} = \frac{MM^3}{12 \text{ area seg.}}$.

COR. When the ſegment becomes a ſemicircle, we have $M'M = 2r$, and $CG = \frac{r}{2{\cdot}356194} = {\cdot}42441\,r$.

120. PROP. *To find the centre of gravity of any parabola.*

Here the general equation is $y = x^n \div a^{n-1}$. And by ſubſtitution $\frac{\text{flu. } xy\dot{x}}{\text{flu. } y\dot{x}} = \frac{\int x^{n+1}\dot{x}}{\int x^n\dot{x}} = \frac{n+1}{n+2} \times x = X$.

COR. 1. If $n = \frac{1}{2}$, which is the caſe in the common or Apollonian parabola, $\frac{n+1}{n+2}x = \frac{3}{5}x$. That is, $AG = \frac{3}{5}AC$. (fig. 10. pl. III.)

COR. 2. If $n = 1$, the figure becomes a triangle, and then $AG = \frac{2}{3}x$: which agrees with art. 114.

120. Prop. *To find the centre of gravity of any ſemiparabola.*

In this caſe the diſtance on the abſciſs, or the value of X is determined by the foregoing problem; we have now to find Gg (fig. 10. pl. III.): in order to which we take the ſecond equation (108. V.) $Y=\frac{\text{flu. } y^2\dot{x}}{M}$. By ſubſtituting for y^2 in the numerator of this expreſſion, its value in the equation of the curve, we have $YM=\int\frac{x^{2n}\dot{x}}{a^{2n-2}}$, which, by the known rules for the determination of fluents is $=y^2\times\left(\frac{h}{2}-\frac{nx}{n2+1}\right)$, where $h=AC$ the height of the parabola. And this when $x=h$, becomes $y^2\times\frac{h}{4n+2}$: wherefore, dividing by $M=\frac{h\times CD}{n+1}$, we obtain $Y=\frac{(n+1)CD}{4n+2}=Gg$.

Cor. 1. If $n=\frac{1}{2}$, as in the common parabola, $Gg=\frac{3}{8}CD$.

Cor. 2. In the triangle, where $n=1$, $Gg=\frac{1}{3}CD$.

122. Prop. *To find the centre of gravity of the ſegment of a ſpheroid.*

Here the figure may be divided into two parts ſymmetrical with reſpect to the axis: ſo that if we account the vertex A the origin of the co-ordinates, we ſhall need the equation for X (108. VII.). Let the fixed axis of the ſpheroid be called a, the revolving axis c, then the equation of the curve is $y^2=\frac{c^2}{a^2}\times(ax-x^2)$: conſequently $\frac{\text{flu. } y^2x\dot{x}}{\text{flu. } y^2\dot{x}}=\frac{\int(ax-xx)\cdot x\dot{x}}{\int(ax-xx\cdot\dot{x})}=\frac{\frac{1}{3}ax^3-\frac{1}{4}x^4}{\frac{1}{2}ax^2-\frac{1}{3}x^3}=\frac{4a-3x}{6a-4x}x=AG$.

Cor. 1. When the ſegment becomes a hemiſpheroid, $x=\frac{1}{2}a$, and $\frac{4a-3x}{6a-4x}x=\frac{5}{8}x$ for the diſtance of the centre of gravity from the vertex, therefore $\frac{3}{8}x$ is its diſtance from the centre of the baſe.

Cor. 2. When $c=a$, the ſpheroid becomes a ſphere, and as the theorem is independent of c, it is alike applicable to both ſolids, and to their correſponding ſegments.

Cor. 3. Since the equation to the hyperbola is $y^2=\frac{c^2}{a^2}(ax+x^2)$ which differs from that to the ellipſis only in the ſign of the ſecond term, we get by a ſimilar proceſs $\frac{4a+3x}{6a+4x}x$, for the diſtance of the centre of gravity from the vertex of a hyperboloid.

123. PROP. *To find the centre of gravity of a ſector of a ſphere.*

A ſpheric ſector may be conſidered as a ſolid of rotation; for it may be generated by the rotation of a circular ſector: thus, in fig. 9. pl. III. if the circular ſector CMA make a complete revolution upon AC as an axis, it will deſcribe the ſpheric ſector of which CMAM will be a ſection. This ſolid is compoſed of a ſpheric ſegment and a cone: the centre of gravity of the firſt of theſe we ſuppoſe at G, that of the ſecond at g. If AP be denoted by x, and AC by r, we ſhall have $AG = \frac{8r - 3x}{12r - 4x} x$, and, the cone being a kind of pyramid, $Cg = \frac{3}{4} CP = \frac{3}{4}(r - x)$: conſequently $Ag = AC - Cg = \frac{3x + r}{4}$. Let π denote, as before, the number 3·141593, and by well known theorems the ſolid contents of the reſpective figures will be thus expreſſed:

The content of the ſpheric ſector $= \frac{2}{3} \pi r^2 x$.
the ſegment . $= \frac{1}{3} \pi x^2 (3r - x)$.
the cone . . $= \frac{1}{3} \pi (2rx - x^2) . (r - x)$.

Now, if we conceive each ſolid to be condenſed into its centre of gravity, we muſt firſt take the ſum of the moments with reſpect to A, which are,

For the ſegment, $\frac{1}{3} \pi x^2 (3r - 2x) \frac{8r - 3x}{12r - 4x} x = \frac{1}{3} \pi x^3 . \frac{8r - 3x}{4}$

the cone . $\frac{1}{3} \pi (2rx - x^2) . (r - x) \frac{3x + r}{4}$.

Then, dividing the ſum of theſe moments by the ſum of the maſſes, or $\frac{2}{3} \pi r^2 x$, the content of the ſector, we ſhall have

$$X = \frac{\frac{1}{4} x^2 (8r - 3x) + \frac{1}{4}(2r - x) . (r - x) . (3x + r)}{2r^2} = \frac{1}{8}(2r + 3x);$$

an equation expreſſing the diſtance of the centre of gravity of the ſpheric ſector from A the centre of its baſe.

SCHOLIUM.

124. Since the knowledge of the poſition of the centre of gravity in any body or ſyſtem of bodies is of very great importance in almoſt every part of Mechanics, we have thought it right to be tolerably copious in the diſcuſſion of this ſubject; and have, therefore, given inſtances of various methods of aſcertaining the ſituation of the centre of gravity, one or other of which may be applied in moſt caſes which can ariſe. To ſave, however, the trouble of a diſtinct inveſtigation in ſome inſtances which often occur, we may now ſtate in addition to the preceding propoſitions, a few known reſults.

1. In a circular ſector, the diſtance from the centre of the circle is $\frac{2cr}{3a}$; where a denotes the arc, c its chord, and r the radius.

2. The centres of gravity of the ſurface of a cylinder, of a cone, and of a conic fruſtum, are reſpectively at the ſame diſtances from the origin as are the centres of gravity, of the parallelogram, triangle, and trapezoid, which are vertical ſections of the reſpective ſolids.

3. The centre of gravity of the ſurface of a ſpheric ſegment, is at the middle of its verſed ſine or height.

4. In a cone, as well as any other pyramid, the diſtance from the vertex is $\frac{3}{4}$ of the axis.

5. In a conic fruſtum, the diſtance on the axis from the centre of the leſs end, is $\frac{1}{4} h \cdot \frac{3R^2+2Rr+r^2}{R^2+Rr+r^2}$: where h denotes the height, and R and r, the radii of the greater and leſs ends.

6. The ſame theorem will ſerve for the fruſtum of any regular pyramid, taking R and r, for the ſides of the two ends.

7. In the paraboloid, the diſtance from the vertex is $\frac{2}{3}$ of the axis.

8. In the fruſtum of the paraboloid, the diſtance on the axis from the centre of the leſs end, is $\frac{1}{3} h \cdot \frac{2R^2+r^2}{R^2+r^2}$: where h denotes the height, R and r the radii of the greater and leſs ends.

Of the Centrobaryc Method.

Among the ſeveral uſes to which the doctrine of the centre of gravity may be appropriated, one, which for its elegance and ſimplicity deſerves being mentioned here, is that which is called the Centrobaryc method, and by which the magnitudes of ſurfaces and ſolids may often be determined with great facility. The relation between the centre of gravity and the figure generated by the revolution of any line or plane, which is the foundation of this method, was firſt diſtinctly ſtated by *Pappus* in the preface to his 7th book: but it was not completely diſcuſſed till the time of father *Guldin*; who, in the 2d and 3d books of his Treatiſe on the Centre of Gravity, treated this method very fully, and exhibited its utility in a variety of examples. The doctrine is compriſed in the following propoſition, and the corollaries which naturally flow from it.

125. Prop. *If any line, right or curved, or any plane figure, whether it be bounded by right lines or curves, revolve about an axis in the plane of the figure, the ſurface or ſolid generated will be reſpectively equal to the ſurface or ſolid whoſe baſe is the given line or figure, and its height equal to the arc deſcribed by the centre of gravity of the ſaid generating line or figure.*

Let the line or plane B C D (fig. 11. pl. III.) be that by whoſe motion a certain ſurface or ſolid is generated, and let QAR be the axis about which the rotation is performed: through G the centre of gravity of B C D draw A B G D perpendicular to Q R, and either ſuppoſe the various particles in B C D to be referred, by perpendiculars, to Q R, or, which will come to the ſame thing, eſtimated upon A B D by lines drawn from the ſeveral moleculæ parallel to Q R; and let B and D be two of thoſe reduced points. Now, if ABD after performing a rotation, or part of a rotation, be brought to the poſition AEH, BDHE will repreſent the figure generated, and GF the path of the centre of gravity; the generated figure being equal to all the BE, GF, DH, &c. But the arcs BE, GF, DH, &c. are ſimilar (being equal parts of concentric circles), and therefore their ſums are as all the AB, AG, AD, &c. and, by art. 109. theſe ſums are as ſo many times AG: conſequently the ſum of all the BE, GF, DH, &c. is equal to as many times FG, or equal to BCD × FG; that is, the figure BEHD is equal to the baſe drawn into the path deſcribed by its centre of gravity. Q. E. D.

The ſame thing may be ſhewn very conciſely by means of the two values of Y in formulæ III. and V. art. 108. Annexing the co-efficient 2π to both numerator and denominator will not change the value of the fractions; they may, therefore, be expreſſed thus:

$$Y=\frac{\int 2\pi y\dot{M}}{2\pi M}\quad.\quad.\quad.\quad.\quad.\quad.\quad Y=\frac{\int 2\pi y^2\dot{x}}{2\pi M}.$$

The firſt of theſe equations expreſſes the diſtance from the axe AX of the centre of gravity of any line: a ſimple transformation gives us $2\pi YM=\int 2\pi y\dot{M}$, where $2\pi Y$ is the circumference of which Y is the radius, and is that which is deſcribed by the centre of gravity of M if that curve be made to turn about AX as an axis: moreover $\int 2\pi y\dot{M}$ is the expreſſion for the area of the ſurface which is generated by the arc M during this rotation. Conſequently, the ſurface generated by the rotation of any given curve about an axis is equal to the product of the generating arc into the circumference deſcribed by its centre of gravity.

From the ſecond of theſe formulæ we have $2\pi YM=\int 2\pi y^2\dot{x}$; where if we ſuppoſe a revolution upon the axis AX, the area of which the algebraic expreſſion is M or $y\dot{x}$, generates a ſolid denoted by $\int 2\pi y^2\dot{x}$, and the centre of gravity will deſcribe the circumference $2\pi Y$. Therefore the ſolid generated by the rotation of any plane figure about an axis, has for its capacity *the product of the generating area into the circumference deſcribed by its centre of gravity.*

If inſtead of 2π we introduced any fraction, $\frac{1}{n}\pi$, into the numerators and denominators of the ſame formulæ, it would thence be equally obvious that the ſame property was applicable to the curve ſurfaces or the capacities of figures generated by a partial revolution about a fixed axis. And if many curves compriſed in the ſame plane generate at once as many ſurfaces and ſolids, we may apply the ſame propoſition, by taking the common centre of gravity of the ſyſtem. From the whole, then, we may deduce a few corollaries.

Cor. 1. If the diſtance of the centre of gravity of any line or ſurface from the axis of rotation, and the magnitude of the line or ſurface be given, the value of the ſurface or ſolid generated by either a total or partial revolution may be found, and conſequently any two of the three being given, the third may be eaſily determined.

Thus, if the rectangle whoſe ſides are m and n revolve upon m as an axis, it will generate a cylinder whoſe altitude is m and radius of its baſe n: now the centre of gravity of the rectangle will be at the diſtance of $\frac{1}{2}n$ from m, and will therefore deſcribe in a complete rotation the circumference $\frac{1}{2}n \times 2\pi = \pi n$: conſequently, by the propoſition, $mn \times \pi n = \pi m n^2$, is the capacity of the ſolid; and this is equal to $\pi n^2 \times m$, the baſe of the cylinder into its altitude, as it ought to be.

Again, putting b the baſe and a the axis of a ſemiparabola, which by its rotation generates a paraboloid; we have $\frac{3}{8}b$ for the diſtance of the centre of gravity of the ſemiparabola from the axis, and $\frac{3}{8}b \times 2\pi = \frac{3}{4}b\pi$, the circum. deſcribed by the centre of gravity: and, ſince the area of the parabola is $\frac{2}{3}ab$, we have $\frac{3}{4}b\pi \times \frac{2}{3}ab = \frac{1}{2}ab^2\pi$, for the content of the paraboloid; which is manifeſtly equal to half the circumſcribing cylinder, as it ought to be.

For another example, let us take a ſphere whoſe radius is r, and capacity $\frac{4}{3}\pi r^3$, being generated by the rotation of the ſemicircle whoſe area is $\frac{1}{2}\pi r^2$. Here $\frac{4}{3}\pi r^3 \div \frac{1}{2}\pi r^2 = \frac{8}{3}r$, the circumference deſcribed by the centre of gravity; and this divided by 2π, gives $\frac{4r}{3\pi} = \cdot 42441\, r$, for the diſtance of the centre of gravity of a ſemicircle from its centre: agreeing with the reſult in art. 112. By a ſimilar proceſs the centre of gravity may be found in ſeveral figures, more eaſily than by the uſual direct method.

Cor. 2. If the generating or revolving lines or ſurfaces are equal, but the diſtances of their centres of gravity unequal, then the generated ſurfaces or ſolids will be directly as the diſtances of their centres of gravity from the axis of motion.

Thus, if two ſolid rings of iron have both the ſame thickneſs of metal, but have different diameters, the capacities of the rings, or their weights, will be as the arithmetical means between their reſpective inner and outer diameters.

Cor. 3. If the diſtances of the centres of gravity of the generating or revolving lines or ſurfaces from the axis of motion be equal, but the generating lines or ſurfaces unequal, then the generated ſurfaces and ſolids will be as the generating lines and ſurfaces.

Cor. 4. If neither the generating lines or ſurfaces nor the diſtances of their centres of gravity from the axis of motion be equal, the generated ſurfaces or ſolids will be to each other in a ratio compounded of the ratio of the generating lines or ſurfaces, and the ratio of the diſtances of their centres of gravity from the axis of motion.

Thus, for example, the triangle is to its circumſcribed parallelogram as 1 to 2, and the diſtance of the centre of gravity of the triangle from its vertex is $\frac{2}{3}$ of the axis; but the diſtance of the centre of gravity of the parallelogram from its ſide is $\frac{1}{2}$ of the other ſide, or of the axis, therefore the diſtance of the centre of gravity of the triangle from the axis of rotation is to the diſtance of the centre of gravity of the parallelogram, as 4 to 3. So that, if both the ſpaces are ſuppoſed to revolve about a line touching the vertex of the triangle, and parallel to its baſe, the ſolid generated by the triangular ſpace will be to that generated by the parallelogram as 4 to 6, or as 2 to 3.

Again, the common parabola is to its circumſcribing parallelogram as 2 to 3; and the diſtance of the centre of gravity of the parabola is to the diſtance of that of the parallelogram from the ſame point of ſuſpenſion, as 6 to 5; wherefore the ſolid generated by the parabola is to that generated by the parallelogram, each revolving about the ſame common axis paſſing through the vertex of the parabola, and parallel to its baſe, as 12 to 15, or as 4 to 5: and conſequently, the ſolid generated by the triangle is to that generated by the parabola in a ratio compounded of the ratios of 5 to 4, and of 4 to 6; that is, of 20 to 24, or 5 to 6.

After a ſimilar manner may the value of any ſurface or ſolid be found, and thoſe of all kinds of ſurfaces and ſolids be compared together; for various other examples of which we may refer to *Dr. Wallis*'s Treatiſe *de Calculo Centri Gravitatis*, and *Dr. Hutton*'s *Menſuration.*

CHAPTER IV.

On the Mechanical Powers.

126. Def. A *machine* is any thing that ſerves to augment or to regulate moving forces or powers; or it is any body deſtined to produce motion ſo as to ſave either time or force; or, we may in general apply the term *machines* to the material agents by the aid of which forces operate one upon another, and which when employed to ſecond the efforts of certain powers enable them to overcome others which may be more conſiderable.

127. Machines are diſtinguiſhed into *ſimple* and *compound.* The only machines which we think can be ſtrictly called ſimple, are the *lever*, the *inclined plane*, and *cords:* it has been cuſtomary, however, to conſider as ſimple machines thoſe of which all compound machines are found to be conſtituted, and into which when their combined powers are eſtimated they muſt be divided in the inveſtigation; theſe are in number ſix, viz. the *lever*, the *wheel and axle*, the *pulley*, the *inclined plane*, the *ſcrew*, and the *wedge*. Theſe are moſt commonly known by the name of the *mechanical powers*. In all theſe the mechanical advantage ariſes from diſtributing the power to be overcome among the different parts of the machine, ſo that the part ſuſtained by the contrary power ſhall bear but a ſmall ratio to the whole: thus a power incapable of communicating motion to a body, or of ſupporting its preſſure, without mechanical aſſiſtance, may effect its deſigned purpoſe by transferring a part of the weight upon a fulcrum, diſtributing it among a ſyſtem of pulleys, placing it upon an inclined plane, ſcrew, &c.

128. Forces of different kinds may, it is obvious, be made to operate ſo as to counteract each other's effects, by the intervention of machines: as for example, the muſcular force of men and animals may be applied to machines ſo as to oppoſe and counteract the force of gravity; but in any ſuch caſes, ſince the juſt and adequate meaſures of theſe forces are their ſimultaneous effects in ſimilar ſituations, the theoretic proceſſes muſt be conducted in the ſame manner whether the forces exerted are unlike or the ſame, as when the force of gravity ſoliciting two bodies

cauſes them to act as moving powers by means of their weights. Hence then, whatever is ſhewn to obtain with relation to any two forces whoſe effects are known, while their nature is undefined, may be ſafely applied to all which are reſpectively equal and operate in a ſimilar manner, whether they are dead weights, living powers, or forces exiſting naturally.

129. In the theory of ſimple machines, and indeed of machines in general, they are firſt conſidered as in a ſtate of balanced reſt: and when the ratio of the forces is aſcertained which by the intervention of any given machine produces ſuch equilibrium, it is manifeſt that a ſlight change in the magnitude of either of the forces, or in their points of application, by cauſing a preponderance on one ſide or other, will produce motion. In actual practice the change muſt always be made in favour of the power which is to produce the motion: and this may eaſily be accompliſhed when the conditions of the equilibrium are once eſtabliſhed. When forces act upon each other by means of machines, they meet with various obſtacles on account of the roughneſs of ſurfaces, the ſtiffneſs of cords, &c. which if introduced at the commencement of our inveſtigations would render them very embarraſſing and intricate: we therefore, at firſt, ſuppoſe ſurfaces to be perfectly ſmooth, cords to be perfectly flexible, &c. and afterwards blend with the deductions from the theory thoſe which have reſulted from experiment with regard to theſe obſtacles. As to what relates to the diſpoſition of powers upon machines ſo that when in motion their effects ſhall be greateſt or leaſt, it is a diſtinct branch of the theory, and falls under the head of *Dynamics*.

130. Writers on the ſubject of mechanics have often attempted to demonſtrate the properties of the ſeveral ſimple machines, by means of a celebrated theorem, which is this: *When two heavy bodies counterpoiſe each other by means of any machine, and are then made to move together,* the products of each maſs into its velocity, or, as it is technically expreſſed, *the quantities of motion with which one body deſcends and the other aſcends perpendicularly, will be equal* [*]. Since an *equilibrium* always accompanies

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[*] This theorem is due to *Galileo*, who is juſtly reckoned the father of the ſcience of *Dynamics*. It is nearly analogous to the celebrated principle of the *virtual velocities*, from which M. La Grange has deduced many ingenious and elegant ſolutions of ſome difficult problems in his *Mécanique Analytique*. The principle is thus enunciated: " *Si un ſyſtême quelconque eſt ſollicité par des puiſſances en equilibre, et qu'on donne à ce ſyſtême un petit mouvement quelconque, en vertu duquel chaque point parcoure un eſpace infiniment petit, la ſomme des puiſſances multipliées chacune par l'eſpace que le point où elle eſt appliquée parcourt ſuivant la direction de cette même puiſſance, ſera toujours égale à zero: en regardant comme poſitifs les petits eſpaces parcourus dans le ſens des puiſſances, et*

this equality of motions; it bears ſuch a reſemblance to the caſe wherein two moving bodies ſtop each other when they meet together with equal quantities of motion, that many have thought that the cauſe of an equilibrium in the ſeveral machines might be immediately aſſigned by ſaying, that becauſe one body always loſes as much motion as it communicates to another, two heavy bodies counteracting each other muſt continue at reſt when they are ſo circumſtanced that one cannot deſcend without cauſing the other to aſcend at the ſame time, and with velocities inverſely proportional to their maſſes; for then, ſhould one of them begin to deſcend, it muſt inſtantly loſe its whole motion by communicating it to the other. But this argument, however plauſible it may ſeem, is (as Dr. Hamilton remarked) by no means ſatisfactory; for when we ſay that one body communicates its motion to another, we muſt neceſſarily ſuppoſe the motion to exiſt firſt in the one, and then in the other; but in the preſent caſe, where the two bodies are ſo connected that one cannot poſſibly begin to move before the other, the deſcending body cannot be ſaid to communicate its motion to the other, and thereby make it aſcend: but whatever we ſhould ſuppoſe cauſes one body to deſcend, muſt be alſo the immediate cauſe of the other's aſcending, ſince, from the connection of the bodies, it muſt act upon them both together, as if they were really but one. And therefore, without contradicting the laws of motion, we might ſuppoſe the ſuperior weight of the heavier body, which is in itſelf more than able to ſuſtain the lighter, would overcome the lighter, and cauſe it to aſcend with the ſame quantity of motion with which the heavier deſcends; eſpecially as both their motions, taken together, may be leſs than what the difference of the weights, which is here ſuppoſed to be the moving force, would be able to produce in a body falling freely. For theſe reaſons, and various others which might eaſily be aſſigned, we are of opinion that all proofs founded upon this theorem as a baſis are neceſſarily unſatisfactory: we have, nevertheleſs, thought it right to notice it; and, as it may ſerve as a good *index* of an *equilibrium* in many machines, and admits in ſome inſtances of a uſeful application, we

comme négatifs les eſpaces parcourus dans un ſens opposé." This principle, or rather, univerſal fact, La Grange expreſſes by a very ſimple formula, from which he deduces his whole ſyſtem algebraically, without a ſingle diagram: a mode of procedure which, though it diſplays aſtoniſhing analytical ſkill, is certainly far from the beſt way of *teaching* mechanical philoſophy; ſince the whole may be gone over, without either a diſtinct or even indiſtinct idea of the things repreſented by the ſymbols. M. Carnot in his *Geometrie de Poſition*, pa. 339, demonſtrated a purely geometrical theorem which is very ſimilar to this celebrated mechanical principle.

may again refer to it in the practical part of this Treatiſe. Our preſent inveſtigations will be made to depend upon the principles of equilibrium explained in the ſecond and third chapters.

I. *Of the Lever.*

131. DEF. A *lever* is an inflexible bar or rod, which may be ſubjected to the action of two or more forces or powers, at different points, while it is ſupported on a fulcrum or prop, about which it may move freely.

Of levers there are two kinds which differ eſſentially: *i. e.* thoſe in which the forces act on *contrary* ſides of the centre of motion, or fulcrum, and thoſe in which they act on the *ſame* ſide. They are, however, uſually diſtinguiſhed into *three* kinds, according to the reſpective diſpoſitions of the prop, the power, and the reſiſtance. In levers of the *firſt* kind, the fulcrum is between the power and the reſiſtance: ſuch are, the balance, ſteelyards, ſciſſars, pokers, pincers, ſnuffers, &c. In levers of the *ſecond* kind the reſiſtance is between the fulcrum and the power: ſuch are, cutting knives fixed at one end, the oars and rudders of boats, which may be conſidered as having their fulcra in the water, doors whoſe hinges ſerve as a fulcrum, &c. In levers of the *third* kind the power acts between the prop and the reſiſtance; as in tongs, ſheers for ſheep, a ladder while raiſing up, &c. This too is the kind of lever which is found moſt to obtain in the animal functions: for the muſcles, by means of which the bones are turned upon their joints, are inſerted much nearer to the centre of motion than the point in which is the centre of gravity of the weight to be raiſed. The mechanical contrivances in different parts of animal frames furniſh a very curious and intereſting topic: but as we cannot, without widely digreſſing, enlarge upon it here, we would beg to refer to Chapters 7 and 8 of Dr. Paley's Natural Theology, where it is diſcuſſed very admirably.

132. PROP. *Any two weights acting perpendicularly upon an horizontal lever (conſidered as void of gravity), and at contrary ſides of the fulcrum, will balance each other if they be reciprocally proportional to the diſtances from the fulcrum at which they act.*

Conceive any cylinder or priſm, as AB (fig. 12. pl. III.), hung by its ends to the inflexible line HI, by means of the cords AH, BI. Now if the whole be ſuſpended by the cord C placed in the middle of HI, it will remain in equilibrio, from the nature of the centre of gravity. And if we ſuppoſe the priſm divided into two unequal parts by a vertical plane paſſing through D, then if a cord as ED keep the parts AD, DB, in the ſame poſition with reſpect to HI, the equilibrium will not be affected.

Or, if inſtead of the three cords HA, ED, IB, the two pieces were attached to the line HI, by cords GL, FM, fixed to their middle points L, M, the equilibrium will ſtill obtain, in like manner as at firſt. So that we may conſider two heavy bodies AD, and DB, ſuſpended from the terms G, and F of the beam, to be in equilibrio upon the fulcrum C. Here, ſince GE $= \frac{1}{2}$ EH, and EF $= \frac{1}{2}$ EI, we ſhall have GF $= \frac{1}{2}$ HI, and conſequently $=$ CI: whence, taking away CF which is common, there remains GC $=$ FI $=$ FE. Therefore, adding CE, we have GE $=$ CF. Hence then, GE : EF : : FC : CG : : 2 GE : 2 EF : : AD : DB; wherefore, convertendo, GC : CF : : the weight BD : the weight AD. And if theſe two priſms AD, DB, were reduced to cubes, or ſpheres, or any other figures, the equilibrium would ſtill remain, provided the points of ſuſpenſion G and F continued the ſame: whence the propoſition is manifeſt. Had the priſm AB been ſuppoſed laid upon HI, the ſteps of the demonſtration would ſtill have been the ſame.

This demonſtration, which, in our opinion, is far more ſimple and obvious than that of Archimedes for the ſtraight lever, is given by *Galileo* in his ſecond Dialogue on the Reſiſtance of Solids. It appears but little known: for *Dr. Matthew Young*, the late learned biſhop of Clonfert, has, in his Analyſis of the Principles of Natural Philoſophy, deduced this property in a manner nearly ſimilar, without ſeeming at all aware that the ſame thing had been long ſince effected by the illuſtrious Piſanian philoſopher.

133. Prop. *When three forces act upon an inflexible lever, void of gravity, in any parallel directions in the ſame plane, and keep it in equilibrio, any two of them will be to each other inverſely as their diſtances from the point to which the third force is applied.*

This propoſition has been already demonſtrated in Chap. II. For all that was ſhewn there (in arts. 71...81.) with reſpect to parallel forces in one and the ſame plane, will evidently apply to the caſe of the lever. So that, inſtead of repeating the demonſtration, we refer to thoſe articles; and ſhall here deduce from them a few evident corollaries more immediately applicable to what is before us.

Cor. 1. If two weights, or parallel forces, balance each other upon any ſtraight lever in any poſition, they will balance each other in any other poſition of the lever while the ſame fulcrum is retained: for in this caſe the fulcrum correſponds with what is named the centre of parallel forces, (art. 91.)

Cor. 2. In the ſtraight lever of the firſt order (fig. 1. pl. IV.) we have $P \times AF = W \times BF$, and the preſſure upon the fulcrum in the equilibrated ſtate equal to the ſum of the two weights, or of the power and the reſiſtance $P + W$.

COR. 3. In the ſtraight lever of the ſecond kind (fig. 2.), we have likewiſe $P\cdot AF = W\cdot BF$: but the preſſure upon the fulcrum is $= W - P$.

COR. 4. In the ſtraight lever of the third kind (fig. 3.), $P\cdot AF$ is again $= W\cdot BF$: but the preſſure upon the fulcrum is $= P - W$.

COR. 5. If a given weight U be moved along either arm of a ſtraight lever (as FA. fig. 1.) the weight W which it will balance at B on the other arm is proportional to FA, the diſtance from the fulcrum at which the given weight acts. For, in the caſe of equilibrium, $W\cdot BF = U\cdot FA$; wherefore, ſince FB and U are invariable, $W \propto FA$.

COR. 6. If a weight W (fig. 5.) be placed upon a lever which is ſupported by two props A and B, in an horizontal poſition, the preſſure on A : preſſure on B :: BC : CA. Or preſſure upon $A = \frac{W\cdot BC}{AB}$. Preſſure upon $B = \frac{W\cdot AC}{AB}$. (art. 80.)

COR. 7. In the common balance (fig. 4.) where the arms of the lever either are equal or ought to be ſo, the weights in the two ſcales muſt be equal alſo.

COR. 8. In the falſe balance, where arms of unequal lengths are made to balance each other when the ſcales are empty, *the true weight is a geometrical mean proportional between the apparent weights.*

For if X be the unknown weight, W the weight which puts it in equilibrio, x and w being the correſponding arms of the balance, we have $Xx = Ww$. If we put the weight X into the baſon at the end of the arm w, ſome new weight as W' will reſtore the equilibrium, if put in the contrary baſon, and we ſhall have $Xw = W'x$. By multiplying theſe two equations together we obtain $X^2 xw = WW' xw$, whence $X = \sqrt{WW'}$.

In all caſes where great accuracy is required, we may apply this rule advantageouſly, even though the balance be thought correct: if the difference between the apparent weights when tried in the oppoſite ſcales is but trifling, the arithmetical mean may ſuffice.

COR. 9. When ſeveral weights, as T, U, V, W, (fig. 6.) keep a lever in equilibrio upon a fulcrum F, the preſſure upon that fulcrum is equal to $T + U + V + W$, and the ſums of the moments of the weights on different ſides of the fulcrum are equal, *i. e.* $W\cdot AF + U\cdot CF = T\cdot BF + V\cdot DF$.

COR. 10. If a ſtraight lever be kept at reſt by any number of parallel forces acting either at the ſame or different points, the ſum of all the forces acting on one ſide will be equal to the ſum of all the forces on the other.

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134. The obſervation in the firſt corollary above, is equally

applicable to ſeveral forces as to two; that is, if they balance each other in any one poſition of a ſtraight lever, they will likewiſe balance in any other poſition, provided the fulcrum is not changed. But it muſt be remarked that in any other poſition than that in which all the forces are perpendicular to the lever, and conſequently their reſultant perpendicular to it, the lever will have a tendency to ſlide along and change its fulcrum, unleſs prevented. For, ſince it is evident that a force F ſoliciting a material point, or body, in a direction perpendicular to any plane AB may be entirely deſtroyed by the reſiſtance of the plane, and that a force acting in the ſame direction as the plane muſt have its entire effect for aught that depends upon that plane; it follows that, in order that when a ſingle force acts upon a body it ſhould remain immoveable, this force muſt be exerted in a direction perpendicular to the plane on which the body is ſupported. If any other force as F′ acting obliquely ſhould be affirmed to keep the body at reſt, we might reſolve it into two, the one in the direction of the plane, the other perpendicular to it; of theſe the firſt producing its entire effect, the body muſt move in conſequence of its action, which is contrary to the hypotheſis.

135. Prop. *To eſtabliſh the equilibrium on a ſtraight lever, having regard to its weight.*

When we take the weight of the lever into our inveſtigation, we may conſider it as acting at its centre of gravity; or in levers of the firſt kind we may conſider the weight of each arm of the lever as a new power acting at its centre of gravity; and in either caſe the equilibrium may be eſtabliſhed as in *Cor.* 9. of the foregoing propoſition. Let us conſider each of the levers as priſms or cylinders; then will their centres of gravity reſide in their middle points, and the centres of gravity of the arms in *their* middle points eſtimated from the fulcrum. Hence, in the lever of the firſt kind (fig. 1. pl. IV.) we have in the caſe of an equilibrium, $P \times AF + \text{weight } AF \times \frac{1}{2} AF = W \times BF + \text{weight } BF \times \frac{1}{2} AF$. In the ſecond kind of lever (fig. 2.) $P \times AF = W \times BF + \text{weight } AF \times \frac{1}{2} AF$. And in the third kind of lever, (fig. 3.) $P \times AF = W \times BF + \text{weight } BF \times \frac{1}{2} BF$. Here we have ſuppoſed throughout, that the forces are all parallel to that of gravity.

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136. As it is often requiſite to pay a regard to the weights of levers, we ſhall here exhibit a few algebraic formulæ for each kind of ſtraight lever, which may be found of utility in various calculations.

In all the figures (1, 2, 3, pl. IV.) let $FA = p$, and $FB = w$, and let the weight of the lever divided by its length be denoted by m; then will m repreſent the weight of a unit of AB, in the ſame terms as P and W are expreſſed in. And we ſhall have,

For a lever of the firſt kind: $\mathrm{P}p+\frac{1}{2}mp^2=\mathrm{W}w+\frac{1}{2}mw^2$.

From which we get (I.) . . $\mathrm{P}=\frac{\mathrm{W}w+\frac{1}{2}mw^2-\frac{1}{2}mp^2}{p}$.

$$(\text{II.})\ldots \mathrm{W}=\frac{\mathrm{P}p+\frac{1}{2}mp^2-\frac{1}{2}mw^2}{w}.$$

$$(\text{III.})\ldots m=\frac{2\mathrm{P}p\backsim 2\mathrm{W}w}{w^2\backsim p^2}.$$

$$(\text{IV.})\ldots w=\tfrac{1}{m}\sqrt{m^2p^2+2m\mathrm{P}p+\mathrm{W}^2}-\tfrac{1}{m}\mathrm{W}.$$

$$(\text{V.})\ldots p=\tfrac{1}{m}\sqrt{m^2w^2+2m\mathrm{W}w+\mathrm{P}^2}-\tfrac{1}{m}\mathrm{P}.$$

When $\mathrm{W}=0$ . . (VI.) . . $p=\sqrt{w^2+\frac{\mathrm{P}^2}{m^2}}-\frac{1}{m}\mathrm{P}$.

Or, when $\mathrm{P}=0$ . . (VII.) . . $p=\sqrt{w^2+\frac{2}{m}\mathrm{W}w}$.

The latter formula manifeſtly has place, when the longer arm of the lever balances the ſhorter arm together with the weight attached to it: ſo that P has no proper *minimum* here.

For a lever of the ſecond kind: $\mathrm{P}p=\mathrm{W}w+\frac{1}{2}mp^2$.

Whence, ... (i.) . . . $\mathrm{P}=\frac{\mathrm{W}w}{p}+\frac{mp}{2}$.

$$(\text{ii.})\ldots \mathrm{W}=\frac{\mathrm{P}p-\frac{1}{2}mp^2}{w}.$$

$$(\text{iii.})\ldots m=\frac{2\mathrm{P}p-2\mathrm{W}w}{p^2}.$$

$$(\text{iv.})\ldots w=\frac{\mathrm{P}p-\frac{1}{2}mp^2}{\mathrm{W}}.$$

$$(\text{v.})\ldots p=\frac{1}{m}\mathrm{P}\pm\frac{1}{m}\sqrt{\mathrm{P}^2-2m\mathrm{W}w}.$$

When $\mathrm{W}=0$, that is, when the power juſt ſuſtains the lever, we have (vi.) . . . $p=\frac{2\mathrm{P}}{m}$.

In this caſe it is evident the equation for P is ſuſceptible of a *minimum:* for, adopting the fluxionary proceſs, ſince

$$\mathrm{P}=\frac{\mathrm{W}w}{p}+\frac{mp}{2}=\mathrm{W}w\times p^{-1}+\frac{m}{2}p, \text{ we have}$$

$\frac{m}{2}\dot{p}=\mathrm{W}wp^{-2}\dot{p}$, and conſequently when P is a *minimum* we have

$$(\text{vii.})\ldots p=\sqrt{\frac{2\mathrm{W}w}{m}}.$$

For a lever of the third kind: $\mathrm{P}p=\mathrm{W}w+\frac{1}{2}mw^2$.

Wherefore, . . . (1.) . . $\mathrm{P}=\frac{\mathrm{W}w+\frac{1}{2}mw^2}{p}$.

$$(2.)\ldots \mathrm{W}=\frac{\mathrm{P}p}{w}-\frac{mw}{2}.$$

$$(3.)\ldots m=\frac{2Pp-2Ww}{w2}.$$

$$(4.)\ldots w=\tfrac{1}{m}\sqrt{W^2+2mPp}-\tfrac{1}{m}W.$$

When $W=0$, or the power juſt ſuſtains the lever,

we have $$(5.)\ldots w=\tfrac{1}{m}\sqrt{2mPp}=\sqrt{\frac{2Pp}{m}}.$$

$$(6.)\ldots p=\frac{Ww+\frac{1}{2}mw^2}{P}.$$

Or, when the power juſt ſuſtains the lever alone

$$(7.)\ldots p=\frac{mw^2}{2P}.$$

137. Def. The Roman *ſtatera*, or *ſteelyard*, is a lever of the firſt kind, having unequal arms; and is ſo contrived that one weight alone may ſerve to counterpoiſe a great variety of others, by ſliding it to different diſtances from the point of ſuſpenſion upon the longer arm of the lever.

138. Prop. *Upon the ſteelyard the diſtances at which the conſtant weight muſt be hung from ſome certain point will vary as the weights ſuſpended from the ſhorter arm.*

Let W, W', W'', &c. denote the weights which are ſuſpended ſucceſſively from B (fig. 7. pl. IV.), w the invariable diſtance FB, p, p', p'', &c. the variable diſtances FD, FD′, &c. at which the conſtant weight P muſt be hung from the fulcrum to keep the equilibrium: let alſo N, N', denote the weights of the brachia BF, FA, reſpectively, n, n', the diſtances of their centres of gravity from F. Then, by the laſt propoſition, we have theſe equations:

$$Ww+Nn=Pp+N'n'. \qquad W''w+Nn=Pp''+N'n'.$$
$$W'w+Nn=Pp'+N'n'. \qquad W'''w+Nn=Pp'''+N'n'. \text{ \&c.}$$

Now ſubtracting in ſucceſſion the firſt equation from the ſecond, the ſecond from the third, &c. we thence deduce,

$$p'-p=\frac{W'-W}{P}w \quad \ldots \quad p'-p'=\frac{W''-W'}{P}w.$$

$$p'''-p''=\frac{W'''-W''}{P}w. \text{ \&c.}$$

Here, if the weights are in arithmetical progreſſion, we have $W'-W=W''-W'=W'''-W''$, &c. conſequently $p'-p=p''-p'=p'''-p''$ &c. that is, the diviſions on the graduated arm are then equal to each other, or the diſtances from a certain point on the ſteelyard will vary as the weights W, W', &c.

The beam, with its appendages of ſcales or hooks, is either ſo contrived as to be in equilibrium upon the point F, without the weights P, W, &c. or ſo that the arm FA when both are unloaded ſhall preponderate In the former caſe F is the point from which the equal diviſions on the beam are to be eſtimated: in the latter, to find the point where the diviſions are to commence, proceed thus. Find experimentally the point C, at which if a ſingle weight equal to the conſtant weight P be ſuſpended, it will keep the lever at reſt; that point will be the one from which the graduations muſt be meaſured. For, when W and P are ſuſpended upon the beam and the whole is in equilibrio, W ſuſtains P together with a weight which would ſupport P if placed at C: hence $W w = P p + P \times F c$, or $W . BF = P . DF + P . FC = P . DC$, where, becauſe BF and P are invariable, W muſt vary as DC.

139. This inſtrument is very convenient, becauſe it requires but one weight, is eaſily carried, and the preſſure on the fulcrum is leſs than in ſcales, when the ſubſtance to be weighed is heavier than the conſtant weight. But when the conſtant weight is greater than that of the ſubſtance to be weighed, the preſſure on the fulcrum is greater in the ſteelyard than in the common balance; for which reaſon, together with the following, it is not ſo accurate as ſcales in determining the quantity of ſmall weights. 1ſt. Becauſe, the length of the beam being given, the arms in the ſteelyard will be ſhorter than in the balance, whenever the ſubſtance to be weighed is equal to, or leſs than, the conſtant weight. 2dly, The balance admits of a more nice adjuſtment. 3dly, The ſubdiviſion of weights for the balance can be effected with greater preciſion than the ſubdiviſion of the arm of the ſteelyard.

140. PROP. *Any two forces acting upon a bent lever (conſidered as without weight) in different directions but in the ſame plane, are in equilibrio, when they are to each other inverſely as the perpendiculars let fall from the fulcrum upon their directions.*

Let the two powers W, P, (fig. 8. pl. IV.) act upon the lever WFP, whoſe centre of motion is F, in the directions WM, PL, in the ſame plane; and let FM, FL, be the perpendiculars upon the directions of the powers: from F as a centre with radius FL equal to the longer perpendicular, deſcribe a circular arc interſecting the direction of the force W in D: and, becauſe the efficacy of forces is the ſame (32.) to whatever points of their directions they may be applied, we may conceive the two forces applied at D and L. Now, the forces being in equilibrio when acting at D and L, may be conceived the ſame in effect as forces acting at the periphery of a wheel DHL, perpendicular to its radii DF, LF, and preventing it from turning upon the centre F, and in this caſe the forces acting in the direction of the circumference at their points of application muſt manifeſtly be equal. Hence, if DE in the direction of W repreſent the magnitude of that force, and be reſolved into the two, DG in the direction of radius FD, and GE perpendicular to it, the line GE will repreſent the part of W which tends to produce the motion on the centre, and is therefore equal to P. Conſequently, by the nature of equilibrated forces and ſimilar triangles W : P : : DE : EG : : DF (=FL) : FM. Q. E. D.

This demonſtration is the ſame in principle as the one given by *Newton* in the *Principia:* we have adopted it, becauſe, notwithſtanding the various objections which have been urged againſt it, we think it more conciſe, and at the ſame time more ſatisfactory, than any other iſolated demonſtration extant. The two principles aſſumed are, 1ſt. That any force W will have the ſame energy in the direction EDM whether it be applied at D, W, or M; a truth which cannot well be denied (art 32.). And, 2dly, That equal and contrary forces acting perpendicularly to any two radii of a wheel at equal diſtances, will prevent it from turning about its centre: and this preſents itſelf to the mind as remarkably evident. With this view of the matter, we ſhall not attempt to controvert the arguments of Dr. Hamilton and others; who do not ſeem to have attended very carefully to the nature of Newton's demonſtration.

Cor. 1. *If* DW *and* PL, *the directions of the forces (figs.* 9. 10. *pl.* IV. *), be produced till they meet at* S, *and from the fulcrum* F *the line* FI *be drawn parallel to the direction of one force till it meets that of the other, then will* SI, IF, *repreſent the two forces, and* SF *the preſſure upon the fulcrum.*

For, if SF be radius, FM and FL (perpendiculars to SD and SP reſpectively) are the ſines of the angles FSM, FSL, or of the angles FSI, IFS: hence SI : IF : : ſin. SFI : ſin. ISF : : FL : FM; and therefore the forces W and P are by the propoſition repreſented in magnitude and direction by SI, IF, and (art. 32.) may be ſuppoſed to be applied at S. But the third force by which the equilibrium is ſuſtained is repreſented by SF, in the triangle SIF; and that third force is (by hyp.) no other than the re-action of the fulcrum, which is equal and oppoſite to the preſſure upon it; conſequently that preſſure is denoted by SF.

Cor. 2. *The magnitudes of* W, P, *and the preſſure, are each as the ſine of the angle formed by the directions of the other two; or, any two of them are inverſely as the perpendiculars let fall upon their direction from any point in the direction of the third.* So that theſe deductions coincide with what has already been demonſtrated in arts. 82, 83. &c. Ch. II. as they certainly ought to do.

Cor. 3. Since the energy of the powers is the ſame at whatever point in their reſpective directions they may be applied, a bended or an angular lever may be reduced to a ſtraight one, making an invariable angle with it, and the powers may therefore be always ſuppoſed to act upon different points of one ſtraight line paſſing through the centre of motion.

Cor. 4. If the directions of forces acting upon a ſtraight lever are parallel, and keep it in equilibrio, they will be inverſely as their diſtances from the fulcrum, either both meaſured along the lever, or both eſtimated perpendicular to the directions. This, therefore, agrees with what has previouſly been ſhown with reſpect to the ſtraight lever.

Cor. 4. If more than two forces act upon a lever, there will be an equilibrium when the ſum of the products ariſing from multiplying each into the neareſt diſtance of its direction from the fulcrum, on one ſide, is equal to the ſum of the products on the other.

Cor. 5. If the weight of the lever is taken into the account, that of each arm is to be regarded as a new force acting at its centre of gravity.

Cor. 6. If any body AB (fig. 11. pl. IV.) is moveable about its centre of gravity C, and two forces P, P′, act upon it at thoſe points in the figure in the directions PD, P D′, in the ſame plane: then joining CP, CP′, the body may be conſidered as a lever PCP′, and there will be an equilibrium when the powers are inverſely as the perpendiculars upon their reſpective directions; that is, when P : P′ : : CM′ : CM.

Cor. 7. If two weights are ſuſpended from the ends of an angular lever, whoſe fulcrum is at its angular point, the whole will be at reſt when the vertical line paſſing through the fulcrum divides the right line joining the extremities of the lever into two parts, which are inverſely as the weights ſuſpended from the contiguous arms of the lever.

For, let the lever WFP (fig. 1. pl. V.), having the weights W and P attached to its extremities, be in a ſtate of equilibrium; the directions WL, PM of the forces being in this caſe parallel, the perpendiculars FL, FM, form one ſtraight line, and (*Cor.* 3.) FL : FM : : P : W. But the vertical line FG is parallel to WL and PM: ſo that, if L′ EM be drawn through F, parallel to WP, we ſhall have WG (=L′ F) : GP (=FM′) : : LF : FM : : P : W.

Cor. 8. If the arms of the angular lever be priſms or cylinders of uniform matter and thickneſs, the weight of the arm WF being denoted by w, and that of PF by p, then in the caſe of the equilibrium WG : GP : : $P + \frac{1}{2}p : W + \frac{1}{2}w$. For this is only to ſuppoſe $\frac{1}{2}$ of the weight of either arm to act at double the diſtance of its centre of gravity from the fulcrum, which is manifeſtly the ſame in effect as the whole weight acting at the ſingle diſtance.

141. Prop. *In any combination of ſtraight levers* AB, A′B′, A″B″, &c. *whoſe centres of motion are* F, F′, F″, &c. *the whole lying in one direction*, the ratio of P to W acting in *the ſame plane at their extremities, is that of* B″F″ . B′F′ . BF : AF . A′F′ . A″F″.

For, let the forces Q, Q′, &c (fig. 2. pl. V.) acting at the points B, B′, &c. in directions parallel to thoſe of P and W, be thoſe which would keep each lever in equilibrio: then (art. 132.)

$$P : Q :: BF : AF.$$
$$Q : Q' :: B'F' : A'F'.$$
$$Q' : W :: B''F'' : A''F'', \&c.$$

whence, by compoſition of ratios $P : W :: BF : B'F' . B''F'' : AF . A'F' . A''F''$. Q. E. D.

Cor. 1. The preſſure upon the fulcrum $F = P + Q = \frac{P \times AB}{FB}$.

the preſſure upon $F' = Q + Q' = \frac{P . AF}{BF} + \frac{W . B''F''}{A''F''}$.

the preſſure upon $F'' = Q' + W = W + \frac{W.B''F''}{A''F''} + \frac{W.A''B''}{A''F''}$.

Cor. 2. If P and W act in different directions, as well as Q, Q′, &c. and the whole be in equilibrio, we muſt ſubſtitute perpendiculars from the points F, F′, &c. upon the ſeveral directions of the powers, for AF, FB, &c. in the original proportion.

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142. Before we cloſe our diſcuſſions reſpecting the lever, it may not be amiſs juſt to remark, that in every attempt to determine the advantage gained by this machine peculiar attention muſt be paid, not only to the directions in which the forces are exerted, but to the points on the lever to which their action is to be referred. Without a due regard to theſe particulars, the mechaniſt will often be involved in error, even in ſimple caſes where there might be ſuppoſed but little probability of miſtake. In fact, even the ſimple property of the ſtraight lever that equal weights acting at equal diſtances from the fulcrum on oppoſite ſides will be in equilibrio, while at unequal diſtances the one which acts moſt remotely from the fulcrum will preponderate, has more than once been a ſource of error in unſkilful hands; and in particular, it has lain at the foundation of moſt of thoſe ill-fated and uſeleſs contrivances which have been ſtruck out by ſuch as were in purſuit of the *perpetual motion*. In theſe contrivances the object of the projector has generally been to apply different weights to a rotatory machine in ſome ſuch manner, that, at ſucceſſive moments of time, firſt one and then another ſhould be brought to greater diſtances from the centre, and ſo, by being placed at the extremity of a longer lever, ſhould produce a conſtant motion. To prevent, therefore, the waſte of time and ingenuity, we ſhall here deſcribe an apparatus invented by Dr. Deſaguliers (See *Phil. Trans.* No. 419. Or, *Abridg. vol.* VI. *pa.* 311.) in which two equal weights may be placed at any unequal diſtances whatever from the centre of motion, and ſtill remain in equilibrio. In fig. 3. pl. V. AB re-

preſents a balance with equal arms, and EF another of equal dimenſions: they turn freely upon the centres C, D, and their extremities are connected by equal inflexible bars AE, BF; the whole being permitted to move freely at the joints A, B, C, D, ſo as to aſſume the forms of varying parallelograms, in conſequence of any motion upon the points C and D. Acroſs the bars AE, BF, are fixed others as WU, PQ, from any points of which equal weights P, W, may be ſuſpended. Now, on whatſoever part of the bar PQ the weight P is fixed, it is manifeſt that it will, on account of that bar being firmly connected with the vertical rod BF, *act* as though it were placed at F: and, in like manner, in whatever part of the bar WU the weight W be ſuſpended, it will *act* as though it were placed at E: ſo that, however great may be the difference of the diſtance of the bodies P and W from CD, they will ſtill, if equal in weight, balance each other in any poſition of the ſyſtem. Nor is this in any reſpect incompatible with the principle of the equal products of weight and velocity, which we have mentioned (130.) as a uſeful indication of an equilibrium: for, ſuppoſe this compound balance to be brought by motion on its centres into the poſition *abcd*, the weights being then at *w* and *p*; thoſe weights will have moved through the arcs W *w*, P *p*, while the extremities of the levers will have paſſed through the equal and reſpectively parallel arcs A *a*, E *e*, B *b*, F *f*; of conſequence the velocities of the two weights will have been equal, as they ought to be, in conformity with that principle. Thus, then, it appears from this ſimple contrivance, that weights do not preponderate in machines merely on account of their different diſtances from the centre of motion; and conſequently a mere increaſe of diſtance does not *univerſally* give a mechanical advantage.

II. *Of the Wheel and Axle.*

143. DEF. The *Wheel and Axle*, or, as it is often called, the *Axis in Peritrochio*, is a machine which conſiſts of a cylinder and a wheel having the ſame axis, at the two extremities of which are pivots on which the whole may turn. The power is applied at the circumference of the wheel, generally in the direction of a tangent, and a cord is wrapped about the cylinder, in order to overcome the reſiſtance, or elevate the weight.

This machine is ſometimes, without any impropriety, named the perpetual lever; it being, in reality, a lever on whoſe arms the power and weight may always act perpendicularly although the lever turns round its fulcrum. It is often conſtructed of the form repreſented fig. 4. pl. V. where CD is the cylinder, at the ends of which are the pivots FE, turning in the ſolid pieces of timber HF, AE; the weight W is raiſed by means of a rope coiled about the cylinder; the power being applied to the wheel SSB, either by the cord II, or by the handles S, S, S. Sometimes, inſtead of the wheel, we find this machine made up of levers, fixed into the cylinder, as ſpokes into the nave of a wheel; at others, a ſimple handle ſerves for the application of the power, as in fig. 5: but the effect is ſtill the ſame, except that the rotation is leſs uniform. In ſome caſes the cylinder is horizontal, as in the figures, and in ſome kinds of *cranes;* in others, it is vertical, as in the capſtan, &c. But whether the cylinder be horizontal or vertical, this machine has a manifeſt advantage over the ſimple lever in point of convenience: for, by the continual rotation of the wheel the weight may be raiſed to any height, or from any depth; while it could be elevated only a little way by any lever.

144. Prop. *In the wheel and axle if the power act perpendicularly to the radius of the former, and the weight perpendicularly to that of the latter, there will be an equilibrium when the weight and power are reciprocally as the radii of the circles at which they act.*

Conceive fig. 6. pl. V. to be a vertical ſection of the machine perpendicular to the axis: and ſince the effort of the weight to turn the axle round is the ſame at whatever point of that axle it be applied, ſuppoſe both P the power, and W the weight or reſiſtance, to be applied at A and B, in the ſame plane, perpendicular to the axis of rotation. Then, whether the power P act at A, or A′, ſince it acts perpendicularly to the radius CA, CA′ (by hyp.) while the weight acts perpendicularly to CB, it is manifeſt that either ACB, or A′CB may be conſidered as a lever, whoſe fulcrum is C; and conſequently (140.) P : W : : CB : CA. Q. E. D.

Cor. 1. If the power act in any other direction than the tangential one, as in A′ P″, for example; then, drawing from C the line CD perpendicular to the direction of the power, we have P : W : : CB : CD; whence, becauſe CD is always leſs than CA, it is obvious that the *tangential is the moſt advantageous direction in which the power can be applied.*

Cor. 2. Since we have always P : W : : CB : CA, or CA . P = CB . W, it follows, that when the power is variable, if the diameter of the wheel increaſe in the ſame proportion as the power diminiſhes, and *vice verſa*, the force with which the wheel will continue to be turned, will always be of the ſame magnitude.—This principle is ingeniouſly applied in the action of the main ſpring on the fuzee of watches, and of the main ſpring on the tumbler of gun-locks.

Cor. 3. When the moving force is applied by means of handles, as at S, S, S, (fig. 4.) it often happens that many forces act ſimultaneouſly, one at each handle: then, if they all act in directions perpendicularly to the reſpective radii, there will be an equilibrium, when the ſum of all the powers is to the weight, as the radius of the wheel, to the mean diſtance at which the powers act; or, when the ſum of the moments (31.) of the powers, is equal to the moment of the weight.

COR. 4. When the power and weight act by means of a rope; and it have a ſenſible thickneſs compared with that of the axle, there will be an equilibrium when the power is to the weight, as the ſum of the radii of the axle and rope, to the ſum of the radii of the wheel and rope.

For the action of both forces is tranſmitted by the axis of the rope, and conſequently its radius ought to be added both to that of the wheel, and that of the axle.

COR. 5. In the caſe of the laſt corollary the ratio of the power to the weight is greater than when the thickneſs of the rope is not taken into the account: if, therefore, the rope is ſo folded upon the axle as to cover the ſurface, and the weight acts by a ſecond ſpire of rope, the power muſt be augmented to maintain the equilibrium; and ſo on continually for every increaſed courſe of rope upon the axle.

COR. 6. The diſtance which the weight hangs from the axle will make no difference in the reſult, except that which may be occaſioned by the weight of rope, or that which might be cauſed by a varying force of gravity at different diſtances from the earth's centre.

145. PROP. *To determine the preſſure upon the pivots of the wheel and axle, when in equilibrio.*

Here we ſhall merely conſider the moſt common and uſeful caſe, in which the power and the weight act in parallel directions, and on oppoſite ſides of the horizontal axis of motion. The preſſures upon the pivots will ariſe from the action of the weight, that of the power, the weight of the wheel, and that of the axle. Call the weight of the axle A, that of the wheel V, the power and the reſiſtance to be overcome being denoted by P and W. Then (fig. 8. pl. V.) the weight A may be conſidered as acting at the middle of the axle, and the conſequent preſſure upon each pivot will be $\frac{1}{2}$ A. The weight V will act at C, the centre of the wheel; therefore (133. cor. 6.) preſſure upon A $= \frac{V \cdot BC}{AB}$, and preſſure upon B $= \frac{V \cdot AC}{AB}$. The forces P and W may be conceived to act together at their common centre; that is, if I be the centre of a ſection of the axle where W acts, and ID : DC : : P : W, D will be the point where P + W will act together: hence, the third preſſure upon A =

$(P+W)\frac{BD}{AB}$, correſponding preſſure upon $B=(P+W)\frac{AD}{AB}$. Conſequently, in the caſe of equilibrium, the aggregate preſſure upon $A=\frac{1}{2}A+\frac{V.BC+(P+W).BD}{AB}$, and the whole preſſure upon $B=\frac{1}{2}A+\frac{V.AC+(P+W).AD}{AB}$.

The preſſure upon the pivots when the machine is in motion will be inveſtigated in the ſecond book. (art. 321.)

146. Prop. *If a ſeries of wheels and axles be ſo connected by cords as to act upon one another, the power being applied tangentially to the firſt wheel, and the weight in like manner to the laſt axle, there will be an equilibrium when the power is to the weight, as the continual product of the radii of all the axles, to the continual product of the radii of all the wheels.*

Let ſuch a ſyſtem of wheels and axles be repreſented in fig. 9, and let the force Q in direction OF balance P acting at G, and the force Q′ in direction NE balance Q: then, by art. 144. we have

$$P : Q :: OD : DG.$$
$$Q : Q' :: CN : CF.$$
$$Q' : W :: AB : AE.$$

Whence, by compoſition of ratios, $P : W :: DO.CN.AB : DG.CF.AE$. Q. E. D.

Cor. 1. If the wheels and axles inſtead of being at a diſtance from each other, and connected by cords, are placed as in fig. 7. and the axle of one wheel made to act upon the circumference of the next by means of equal teeth, the proportion between the power and the weight will, nevertheleſs, be as ſtated in the propoſition: for each wheel and axle will act as a lever whoſe fulcrum is at the centre of the axle, and its arms reſpectively equal to the radii of the wheel and the axle; and the joint effect will be as above. (141.)

This is generally called *tooth and pinion work*, the axle when its ſurface is indented being called a *pinion*. And becauſe the number of teeth in the wheels and pinions are to each other as their peripheries, or as their radii, it follows that the power is to the weight, as the continual product of the teeth in the pinions, to the continual product of the teeth in the wheels. Or, if P acts on the wheel A by a cord, and W is ſuſpended from the axle *e* by another, we ſhall have P : W :: teeth in *a* × teeth in *b* × teeth in *c* × teeth in *d* × rad. *e* : rad. A × teeth in B × teeth in C × teeth in D × teeth in E. If the wheels and pinions, inſtead of being diſpoſed as in fig. 7. are diſpoſed in any other manner, as in fig. 10. for inſtance, the ratio of the power and weight will ſtill be determined by a ſimilar proceſs.

COR. 2. If the wheels be equal to each other, and the axles equal to each other, or if each wheel be to the axle on which it is fixed in a conſtant ratio, $r : s$; then n being the number of wheels, we ſhall have in the caſe of equilibrium, $P : W :: s^n : r^n$. Thus, if there be three wheels and axles, the radii of each being as 10 : 1; then will P : W : : 1 : 1000: if there be four ſuch wheels and axles P : W : : 1 : 10000, and ſo on.

COR. 3. Since, when a toothed pinion works a wheel the peripheries of both move with the ſame velocity; and the pinion of the ſecond wheel moves the third wheel with a leſs velocity than the ſecond moves, being at a leſs diſtance from the centre of motion; and the ſame thing obtains throughout the ſyſtem; hence, in a combination of wheels in motion the number of revolutions of the wheel where the power acts, is to the number of ſynchronal revolutions of the wheel where the weight acts, as the product of the radii (or teeth) of the wheels, to the product of thoſe of the axles.

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147. In forming the teeth it is of conſiderable importance to determine their proper curvature, ſo that the motion may be communicated equally, and with as little friction as poſſible. Two methods of accompliſhing this end have been recommended: of theſe the firſt was originally propoſed by M. de la Hire, who affirmed that the preſſure would be uniform *if the teeth were formed into epicycloids*; and M. Camus, in his Cours de Mathematiques, has purſued M. de la Hire's principle, and applied it to the various caſes which are likely to ariſe in practice. The conſtruction, however, is ſubject to a limitation; on which account a ſecond method has been propoſed, which ſecures the perfect uniformity of action without any ſuch limitation. This method conſiſts in making both teeth portions of involutes of circles. Thus, let IHF, KE *b*, (fig. 1. pl. VI.) be the wheels to which the teeth are to be accommodated: the acting face GCH of the tooth *a* muſt have the form of the curve traced by the extremity H of the flexible line F *a* H, as it is unwrapped from the circumference; and, in like manner, the acting face of the tooth *b* muſt be formed by the unwrapping of a thread from the circumference of the circle KE*b*. The line FCE drawn to touch both circles will cut the ſurfaces of the two teeth in C, the point where they touch each other; the faces of both teeth will always touch each other at a point in the common tangent to both circles, and the force ariſing from their mutual preſſure will always act in the direction of the circumferences of the wheels at E and F. This form, by allowing the teeth to act on each other through the whole extent of the line FCE, will admit of ſeveral teeth to be acting at the ſame time; and thus,

by dividing the preſſure among ſeveral teeth, will diminiſh its quantity upon any one of them, and therefore diminiſh the cauſe of the indentations they unavoidably make upon each other. Conſequently, a conſiderable number of teeth thus formed acting at once cauſe the communication of motion to be extremely ſmooth and regular. But this by the bye: the conſideration of teeth and pinions, &c. will be farther purſued in the ſecond volume.

III. *Of the Pulley.*

148. Def. The *pulley* is a wheel of wood, braſs, or iron, turning on an axis, and encloſed in a kind of frame or caſe called its *block*, which admits of a rope to paſs freely over the circumference of the pulley, in which there is uſually a groove to prevent the ſlipping out of the rope. The pulley is ſaid to be *fixed* or *moveable* according as its block is fixed or riſes and falls with the weight. An aſſemblage of ſeveral pulleys is called a ſyſtem of pulleys, a *muffle*, or *polyſpaſton*: of which ſome are in a fixed block, and others in a moveable one.

The nature and effects of the pulley have by ſome writers been explained by conſidering a fixed pulley as a lever of the firſt order, and a moveable pulley as one of the ſecond. But, Profeſſor Hamilton and others have been of opinion that "the "pulley cannot properly be conſidered as a lever of any kind; "for when any power ſuſtains a weight by means of a ſyſtem "of pullies that power will ſuſtain the ſame weight if the pullies "be removed, and the ropes be brought over the axles on which "the pullies turned. If the weight were to be raiſed up there "would in this caſe, be a very great reſiſtance from the friction "of the ropes on the axles; and it is merely to avoid this re"ſiſtance that pullies are uſed, which move round the axles "with but little friction." One of the moſt ſimple and natural methods of computing the power and explaining the effects of the pulley, is by conſidering that every moveable pulley hangs by two parts of the ſame rope equally ſtretched, which muſt ſuſtain equal parts of the weight; and therefore, when one and the ſame rope goes round ſeveral fixed and moveable pullies, ſince all its parts are equally ſtretched, the whole weight muſt be divided equally among all the ropes by which the moveable pullies hang. And conſequently, if the power which acts on one rope be equal to the weight divided by the number of ropes, that power muſt ſuſtain the weight. This principle may be applied to many of the caſes which occur, with great facility; particularly when the cords run in directions nearly parallel, as in the ſyſtems exhibited, fig. 2. pl. VI. But when the ropes are drawn in directions which are not parallel, this method may lead to error, on which account we ſhall give a general propoſition, depending on the pure principles of equilibrium, from which the application to particular caſes may be eaſily deduced.

149. Prop. *Each of two forces in equilibrio about a pulley is to that which retains the axis, as the radius of the pulley, to the chord of that arc with which the rope is in contact.*

Let DEC (fig. 3. pl. VI.) repreſent a pulley which is kept in equilibrium by three forces P, W, R; of which the two former act by a rope PEGDW, which touches the pulley throughout the arc EGD, the latter acting by a rope or a bar RG whoſe direction paſſes through C, the axis of the pulley. Now, ſince the whole is in equilibrium, the force R is equal and oppoſite to the reſultant of P and W; and ſince the moment of R with reſpect to the point C is zero, the moments of P and W muſt be equal (60. 61.): wherefore, ſince DC=CE, W muſt be equal to P. Produce, therefore WD, PE, till they meet CR, and ſetting off from the point of concourſe the equal diſtances RA, RB, on the directions of the powers to repreſent them, complete the rhombus RAIB; then, we ſhall have P : W : R :: RB : BI : IR. But the triangles RBI, ECD, having their ſides reſpectively perpendicular to each other, are ſimilar; and conſequently P : W : R :: CE : CD : DE. Q. E. D.

Cor. 1. When WD and PE are parallel, CD and CE would coincide with the ſemi-chords FD, FE, and we ſhould have P=W, or the power equal to the weight, as in the fixed pulley with ropes parallel. Or, ſuppoſing the figure inverted, the rope fixed at one end as W, and the power acting at the other end P, R would then be the weight and would be equal to 2 P.

Cor. 2. If the angle DRE be denoted by $2a$, then will $R = 2P\cos. a$. For, ſince P : R :: CE : DE, we have $R = P\frac{DE}{CE}$: and ſince moreover $DE = 2EF = 2CE. \cos. CEF = 2CE. \cos. a$; conſequently $R = P.\frac{2CE.\cos. a}{CE} = 2P\cos. a$.

Cor. 3. If the angle DRE be equal to 120°, then will $\cos. a = \frac{1}{2}$, and P=W=R.

150. Prop. *In a ſyſtem of moveable pullies, each of which has a ſeparate rope, the power is to the weight, as radius to the continual product of the coſines of the half angles made by the rope ſuſtaining each pulley, into that power of 2 whoſe exponent is the number of pullies.*

Let the weight W (fig. 5. pl. VI.) be kept in equilibrio by the power P through the medium of the pullies A, A′, A″, the angles $n\,AA'$, $n\,A'A''$, made at each being denoted by $2a$, $2a'$, $2a''$, &c. and let t, t', t'', &c. be the tenſions of the cords by which the pullies A, A′, A″, are ſupported. Then by Cor. 2. laſt prop. we have

$$W = 2t \cos. a.$$
$$t = 2t' \cos. a'.$$
$$t' = 2t'' \cos. a''.$$
$$\dots\dots\dots$$
$$t^{n-1} = 2t^{n} \cos. a^{n}.$$

Multiplying together the two firſt, three firſt, &c. of theſe we ſhall obtain the tenſions of the ſeveral cords in ſucceſſion; and if we multiply together the whole, we have for the relation between P and W, the following equation

$$W = 2^{n} P (\cos. a. \cos. a'. \cos. a'' \dots \cos. a^{n}.)$$

Whence the proportion ſtated in the propoſition is manifeſt.

Cor. 1. If the cords are parallel (as in fig. 4.) then will cos. $a = \cos. a' = \cos. a''$, &c. $= 1$, and the preceding equation becomes $W = 2^{n} P$, whence $P : W :: 1 : 2^{n}$; that is, *the power is to the weight, as unity, to that power of 2 which is denoted by the number of pullies.*

Cor. 2. If the angles at A, A′, A″, are equal to each other, then will $P : W :: 1 : 2^{n} \cos.^{n} a$, a denoting the angle at each pulley.

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151. A ſyſtem of pullies being tolerably portable and not very heavy, can be readily conveyed from place to place, and thus made uſe of in many inſtances where other machines cannot be reſorted to. In a judicious combination of them a conſiderable weight may be moved by a very ſmall power, though in this as in all other machines, the power muſt paſs over a proportionably greater ſpace. But the chief diſadvantages to which pullies are liable, ariſes from their great friction, and the ſtiffneſs of the ropes which paſſes over and under them. Some ingenious contrivances have been deviſed to remove in a great degree theſe and other inconveniences; which will be deſcribed under the article Pulley in the ſecond volume of this work.

IV. *Of Inclined Planes.*

152. Def. An *Inclined Plane*, as its name imports, is one which forms with an horizontal plane any angle whatever. It is reckoned among the mechanical powers becauſe when any body is laid upon it a part of its weight is ſupported, and it is therefore moved with greater eaſe. The inclination of the plane is meaſured by the angle formed by two lines drawn upon the ſloping plane and the horizontal plane, perpendicular to their

common interſection, and meeting at a point in that interſecting line.

153. Before we demonſtrate the general properties of this ſimple machine we ſhall premiſe a few obſervations on the manner in which *bodies* are ſupported upon any planes whatever.

I. If a body of any figure touches a plane in a ſingle point, and is ſolicited by one force only; two conditions are requiſite to its continuance at reſt on that plane. *Firſt*, that the direction of the ſingle force be perpendicular to that plane. *Secondly*, that its direction paſs through the point where the body is in contact with the plane. The neceſſity of the firſt of theſe conditions appears from art. 134. As to the ſecond, it is no leſs neceſſary, ſince if the direction of the power, though perpendicular to the plane, paſs not through the point of contact of the plane and the body, the reſiſtance of the plane which is exerted perpendicularly to the point of contact, will not be directly oppoſed to the force, and of conſequence they cannot deſtroy each other although they ſhould be equal.

II. If the body inſtead of touching the plane in one point only, touch it in many points, or throughout plane ſurfaces, then it is not indiſpenſable that the ſingle force which acts on the body ſhould paſs through any of the points of contact: but it muſt be perpendicular to the plane, and ſo ſituated that it may be decompoſed ſolely into as many forces perpendicular to the plane as there are points reſting on the plane, and the forces reſulting from the decompoſition muſt paſs through thoſe points.

III. Therefore, if a body which touches a plane in many points is ſolicited by various forces in different directions, it is neceſſary, *Firſt*, that theſe forces may be reduced to one only which is perpendicular to the plane: and, *Secondly*, that the latter in the caſe in which it does not paſs through one of the points of contact, may be decompoſed into as many forces parallel to it as there are points of contact, each of theſe new forces paſſing through a point of contact.

IV. If the ſingle force which acts upon a body be that of gravity, it follows that the plane muſt be horizontal; and, if the vertical line drawn through the centre of gravity of the body, does not paſs through one of the points of contact, it is farther neceſſary that all the points of contact ſhould not fall on one and the ſame ſide of the ſaid vertical line.

V. If, therefore, the body is ſolicited by two forces only, it is neceſſary, *Firſt*, that theſe two forces are in the ſame plane: *Secondly*, that this plane be perpendicular to that on which the body lies: *Thirdly*, that the reſultant, which ought always to be perpendicular to the latter mentioned plane, leave not all the points of contact on the ſame ſide of it. And, if one of the two forces be that of gravity, it is neceſſary moreover that the plane in which the reſultant falls be vertical, and paſs through the centre of gravity of the body.

VI. With regard to bodies which reſt on many planes at once, whether in virtue of a ſingle force, or of many forces, among which we comprehend that of gravity, the general laws of their equilibrium are, *Firſt*, that the reſultant of all theſe forces may be decompoſed into as many forces as there are points on which the body reſts: *Secondly*, that theſe latter are perpendicular to the reſpective planes at the ſeveral points of contact. Whence, we conclude that, in order that a body ſolicited only by gravity may remain in equilibrio between two inclined planes, it is neceſſary that there be in the vertical which paſſes through its centre of gravity, at leaſt one point from which we may let fall a perpendicular on each of theſe planes; and that each of theſe perpendiculars have the conditions ſtated above.

154. Prop. *When two forces acting on a body keep it in equilibrio upon a plane, if we conceive two other planes to which theſe two forces are reſpectively perpendicular; the two forces and the preſſure upon the plane, are each repreſented by the ſine of the angle compriſed between the planes to which the two other forces are perpendicular.*

This is nothing elſe than a manifeſt inference from art. 48. It may, however, be demonſtrated in a manner immediately applicable to the preſent caſe, thus: Let CP, CG, be the directions of the two forces, which ſuſtain the body whoſe centre of gravity is C at reſt upon the plane; and let AD (fig. 6. pl. VI.) be the interſection of the plane of the two forces with that on which the body lies: draw CF perpendicular to AD, and on this line as a diagonal deſcribe the parallelogram CEFG two of whoſe ſides ſhall fall on the directions of the given powers; then, it is evident from the preceding article, that if the reaction of the plane be repreſented by FC, the two other forces which ſuſtain the body will be repreſented by EC and GC reſpectively: conſequently if the preſſure on the plane which is equal to its reaction, be called R, and P and G be the powers acting in PC, GC, then will P : G : R : : EC : GC : FC : : FG : EF : FC : : ſin. GCF : ſin. FCE : ſin. GCE. Now from two points A and *i* taken arbitrarily upon AD, draw *i* B, and AB perpendicular to the directions of P and G; ſo ſhall the triangle AB *i* be ſimilar to CGF: whence, becauſe P : G : R : : ſin. GCF : ſin. FCE : ſin. GCE or ſin. GC*p*, we ſhall have P : G : R : : ſin. *i* AB : ſin. A *i* B : ſin. AB *i*. Q. E. D.

Cor. 1. Since the relations juſt eſtabliſhed have place, whatever the forces may be, we may conſider one of them, as G for inſtance, to be the force of gravity; then will AB be parallel to the horizon: *and the power, weight, and preſſure upon the plane,* *will be reſpectively, as the ſine of the plane's inclination, the coſine of the angle which the direction of the power makes with the plane, and the coſine of the angle which the direction of the power makes with the horizon:* for ſin. iAB, ſin. AiB, and ſin. ABi, are the ſame as ſin. iAB, cos. CEi, and cos. iBD.

COR. 2. Hence, whether the line of direction of the power be elevated above or depreſſed below, *if the angles which it makes with the plane are equal, equal powers will ſuſtain the weight.* Thus, if the lines PCp, P′Cp', be equally inclined to the plane, equal powers acting in thoſe directions will ſuſtain the weight, whether they draw in the directions from C to P or P′, or puſh from p or p' towards C. But the preſſures on the planes vary with thoſe directions, being leſs and leſs as the direction is more elevated.

COR. 3. *The power P'' or p'' is leaſt when its line of direction is parallel to the plane:* for ſince P : G or W : : ſin. iAB : cos. CEi, we have $P = W.\frac{\text{sin. } i\text{AB}}{\text{cos. CE}i}$, which is manifeſtly a *minimum*, when cos. CEi=radius, that is, when CEi vaniſhes.

The truth of this will alſo appear by conſidering the proportion, P : W : : Bi : BA, in which it is obvious Pi will be the leaſt poſſible when it is perpendicular to AD, a perpendicular being the ſhorteſt diſtance from a given point to a line; ſo that *the moſt advantageous direction in which the power can act is that which is parallel to the plane.*

COR. 4. In this latter caſe, becauſe of the ſimilar triangles ABi'', ADB, we have P : W : R : : Bi'' : BA : Ai'' : : BD : DA : AB : : ſin. A : rad. : cos. A. That is, *when the direction of the power is parallel to the plane, the power, weight, and preſſure on the plane, are reſpectively as the height, length, and baſe of the plane; or as the ſine of inclination, radius, and coſine of inclination.*

COR. 5. If two weights W, W′, ſuſtain each other upon two inclined planes AC, CB, which have a common altitude CD, by means of a cord which runs freely over a pulley and is parallel to both planes, then will W : W′ : : AC : CB (fig. 7. pl. VI.). For let the power P acting in direction CW, be that which ſuſtains the weight upon the plane CA, then by reaſon of the uniform tenſion of the ſtring, the ſame power ſuſtains the weight upon W′ upon the plane CB, acting in a direction parallel to it: hence, Cor. 4.

W : P : : AC : CD,
P : W′ : : CD : CB,
And, compo. W : W′ : : AC : CB.

COR. 6. Hence alſo, when one of the planes as CB, becomes vertical, we ſtill have W : W′ : : AC : CB. (fig. 8.)

COR. 7. If the pulley at C be ſo elevated that the two parts

CW, CW′, of the rope are not parallel to the plane, we ſhall have an equilibrium when W : W′ : : ſin. CBD × cos. CWC′ : ſin. CAD × cos. CW′C′. For, in this caſe

W : P : : cos. CWC′ : ſin. CAD,
P : W′ : : ſin. CBD : cos. CW′C′,

Whence, W : W′ : : ſin. CBD . cos. CWC′ : ſin. CAD . cos. CW′C′.

Cor. 8. *If the direction of the power be parallel to the horizon, then will the power, the weight, and the preſſure on the plane, be, reſpectively, as the height, the baſe, and the length of the plane ; or, as the ſine of inclination, its coſine, and radius.* For, if CP (fig. 6.) be parallel to AB, then will B *i* coincide with BD, and it will be P : W : R : : BD : AB : AD : : ſin. A : cos. A : rad.

Cor. 9. If inſtead of oppoſing a ſingle power to the action of the weight we oppoſe ſeveral, then all which we have ſtated in this propoſition and the corollaries as relating to the power P, muſt be extended to the reſultant or equivalent of thoſe powers. For example, if the body W (fig. 10. pl. VI.) is ſuſtained on the inclined plane by the combined action of a power P, and the reſiſtance of a fixed point D to which is attached the rope DKP which embraces the body : then, imagine a line HC to be drawn from the point of concourſe of DF and PK, biſecting the angle made by thoſe directions. If this line interſect the vertical drawn through the centre of gravity of the body in a point C, from which we may demit on the plane a perpendicular paſſing through the point of contact F, the equilibrium will be poſſible; and the relation between the weight W, and the effort according to HC, will be determined by what has preceded. As to the ratio of the force in direction HC, to the power P, it will be the ſame as in the moveable pulley. Thus, if the power P acts parallel to the plane, the weight will be to that power, as the length of the plane to half its height ; viz. *the power will be only half that which would have been requiſite had the weight been ſuſtained without the aid of the fixed point D.*

155. When two bodies whoſe weights are repreſented by w, w', are attached to a thread w C w' (fig. 9. pl. VI.) paſſing over a pulley at C, and are always in equilibrium upon two curves FA, EB, whatever are their reſpective poſitions, it will be eaſy to determine the neceſſary conditions of thoſe two curves. In order to this draw the vertical CM through the point C, and ſuppoſe that the equations of the two curves with reſpect to that line as a common axis are $y = fx$, $u = \mathrm{F}t$, f and F being any functions of the axes x, and t, and C their common origin. The weight w is a force acting in the vertical direction wb ; and if that force be repreſented by wb, it may be reſolved into the two wd acting in the direction of the thread, and wc acting in

that of the normal w M; the parallelogram of forces being $c\,d$. Here the ſimilar triangles $w\,d\,b$, C w M, give

$$\text{CM} : \text{C}\,w :: w\,b : w\,d.$$

Likewiſe, putting C $w=z$, we have

$$\text{CM}=x+\text{ſubnorm.}=\frac{x\dot{x}+y\dot{y}}{\dot{x}}=\frac{1}{2}\,\frac{\dot{\overline{x^2+y^2}}}{\dot{x}}=\frac{z\dot{z}}{\dot{x}}.$$

Therefore, $w\,d=\frac{w\dot{x}}{\dot{z}}$. In like manner putting C $w'=z'$, and operating for the weight w', we have $\frac{w'\dot{t}}{\dot{z}'}$ for the component of that weight acting in the direction C w'. In the caſe of an equilibrium theſe two components muſt be equal. But the length of the thread w C w' being conſtant and known, we have $z+z'=\text{C}$, whence $\dot{z}=-\dot{z}'$, and conſequently

$$w\dot{x}+w'\dot{t}=0,\ \text{or}\ wx+w't=\text{A}.$$

From this equation it appears that whatever poſition we give to the bodies w, w', the centre of gravity of their ſyſtem will be always on the ſame horizontal line; becauſe the co-ordinate of that centre with reſpect to GH is conſtant: for, from the equation (108. I.) that co-ordinate is

$$\frac{wx+w't}{w+w'}=\frac{\text{A}}{w+w'}=\text{a conſtant quantity.}$$

The curve FA being given, if we demand what the curve EB ought to be, that the equilibrium ſhould obtain in all its points. 1ſt. We ſubſtitute in the equation for the firſt $\sqrt{z^2-x^2}$ for y, and $\text{C}-z'$ for z, that is to ſay, we make $y=\sqrt{(\text{C}-z')^2-x^2}$. 2dly, we ſubſtitute $\frac{\text{A}-w't}{w}$ for x, and thence have the equation of the curve BE in functions of t and z. 3dly, we put $\sqrt{t^2+u^2}$ for z', and thence determine the conſtant quantity A from the poſition of the arbitrary point E where the curve BE muſt interſect the vertical line CM.

If, for example, the line AF is a right line and the pulley C is placed at its point of interſection F with the vertical, that line has for its equation $y=\frac{s}{c}x$, s and c being the ſine and coſine he angle ACD (fig. 7. pl. VI.): and ſubſtituting $(\text{C}-z')^2-x^2$ for y^2, we have $(\text{C}-z')^2=\frac{s^2}{c^2}x^2+x^2$, or $z'=\text{C}-\frac{x}{c}$

Subſtituting $\frac{\text{A}-w't}{w}$ for x, and thence will ariſe

$$z'=\text{C}-\frac{\text{A}+w't}{wc}.$$

This equation is manifeſtly that of a right line: if we wiſh it to paſs through C likewiſe, its equation muſt give at the ſame time $z' = 0$, and $t = 0$; whence $C - \frac{A}{wc} = 0$. Put therefore $u^2 + t^2$ for z'^2, and there will ariſe

$$u^2 + t^2 = \frac{w'^2 t^2}{w^2 c^2}, \text{ and } u = t\sqrt{\frac{w'^2}{w^2 c^2} - 1}.$$

The laſt equation leads to the concluſion already announced (154. cor. 5.). For, the angle DCB being ſuppoſed to have s' and c' for its ſine and coſine, the equation of the right line BC is $u = \frac{s'}{c'} t$; therefore, that the equilibrium may obtain we muſt have $\frac{s'^2}{c'^2} = \frac{w'^2 - w^2 c^2}{w^2 c^2}$, from which reſults $wc = w'c'$.

156. Prop. *When a heavy body is ſupported by two planes to determine the relation between the weight of the body and the preſſure upon each plane.*

Here we apply the obſervations laid down in art. 153. VI. in this manner: Let G be the centre of gravity of the body ſupported by the planes (fig. 11. pl. VI.) through which draw the vertical GF; then if from any point F in that vertical perpendiculars FC, FE, be demitted upon the two planes, thoſe perpendiculars muſt paſs through points of contact of the body and planes: otherwiſe a new force will be required to ſupport the body, contrary to the hypotheſis. Let the ſpace F*b* on the vertical line repreſent the weight of the body, on which as a diagonal complete the parallelogram *c*F*be*, two of whoſe ſides fall upon FC, and FE; then ſhall F*c* or *eb*, repreſent the preſſure P upon the plane AB, and F*e* or *cb*, the preſſure P′ upon the plane BD. Hence if KL be drawn parallel to HI, the ſides of the triangle BKL will be reſpectively perpendicular to the ſides of the triangle *c*F*b*, and we ſhall have $W : P : P' :: Fb : Fc : bc :: KL : BL : BK ::$ ſin. ABD : ſin. ABH : ſin. DBI. That is, *if the weight of the body be repreſented by the ſine of the angle comprehended between the two planes, the preſſures upon them are reciprocally proportional to the ſines of the inclinations of thoſe planes with the horizon.*

Cor. 1. If the angles of inclination of the planes be each 60 degrees, the ſum of the preſſures of the body upon both planes will be equal to twice its weight.

Cor. 2. If the angle ABD be a right angle, the ſum of the preſſures upon two planes ſhall be as two ſides of the right-angled triangle KBL to its hypothenuſe.

Cor. 3. The more inclined the two planes are, the greater will be the preſſure upon them; and *vice verſa*.

Cor. 4. If one of the planes is horizontal the body cannot remain in equilibrio, independent of friction, except in the caſe where the vertical drawn through its centre of gravity paſſes through one of the points with which the body touches the horizontal plane.

COR. 5. When a body is ſuſtained by three planes, we muſt in a ſimilar manner demit perpendiculars from ſome point in its line of direction (106.) upon thoſe planes, and upon them conſtruct a parallelopipedon whoſe vertical diagonal ſhall repreſent the weight of the body, and its three contiguous edges the preſſures upon the planes to which they are perpendicular.

157. PROP. *Given the weight, magnitude, and poſition of a beam or other heavy body, to find the relative poſition of two props that may ſupport it in equilibrio, one prop and the lower end of the other having fixed ſituations.*

Let the body, AD (whether ſimple or compound, as a beam, or a beam with a load), have its centre of gravity at G; it is required to find the relative poſition of two props which will ſupport it, their feet ſtanding at C and I (fig. 1. pl. VII.), and the poſition of one of them as CA being fixed. Produce CA until it meet the vertical paſſing through G in F; join IF, and it will cut AD in D, the place where the top of the prop ID muſt be applied. For, drawing AB, BD, perpendicular to CA, DI, they would be ſections of inclined planes which would ſupport the body in equilibrio, by the laſt propoſition: and ſince the action of the body upon thoſe planes would be perpendicular to them, that action would be completely reſiſted by the props CA, ID, their feet being ſuppoſed immoveable.

COR. 1. *The weight of the body, the preſſure upon* CA, *and the preſſure upon* ID, *are reſpectively, as ſine of* C+I, *coſine of* I, *and coſine of* C. For, ſine of C+I=ſine ABD, cos. I=ſin. DBI, and cos. C=ſin. ABC: conſequently this agrees with art. 156. as it ought to do.

COR. 2. The equilibrium will be equally preſerved whether the body be ſuſtained by the two props CA, ID; the two inclined planes BA, BD; or by two ropes FA, FD, fixed to a pin or hook at F. For in either caſe the forces, and directions are the ſame.

The ſame principles are aſſumed as the foundation of the more complex inveſtigations relative to the equilibrium of vaults, arches, domes, &c. as will be ſhewn in their proper place. (Chap. VI.)

V. *Of the Screw.*

158. DEF. The *Screw* is a mechanical power chiefly uſed in preſſing or ſqueezing bodies cloſe, and ſometimes in raiſing

weights. It is a very ſtrong machine, though it cannot be accounted a ſimple one, as no ſcrew can be made uſe of without a lever or winch to aſſiſt in turning it. The ſcrew is chiefly diſtinguiſhed by its ſpiral thread, of which a tolerable conception may be obtained by cutting a piece of paper into the form of an inclined plane, and then wrapping it round a cylinder, as in fig. 2. pl. VII.

159. The ſcrew may be conſidered as compoſed of the lever and the inclined plane; as will be evident from a more minute account of the manner in which it may be conceived to be generated. If an iſoſceles triangle BFG turn about the axe AZ (fig. 5. pl. VII.) there will be generated by that revolution two conic fruſtums united by their greater ends: conceive now, that beſides the motion of rotation, this triangle has alſo a motion of tranſlation in the direction of the axe AZ, ſo regulated that while the triangle makes a complete revolution, the point B is moved to G, and the whole triangle is found in the poſition GF′G′, and ſo on: the ſolid thus generated is called the *interior ſcrew;* and the height GB is called *the diſtance of the threads.* The *exterior ſcrew* is ſo adapted to the other, as if it were its mould; and is nothing elſe than the ſolid generated by the polygon HGFBC, ſuppoſing it to partake of the ſame motions as the triangle BGF. For the ſake of diſtinction we ſhall apply the name the *ſpindle* to the interior ſcrew, calling the exterior one only, the *ſcrew*. The ſpindle then is a cylinder inveſted with a ſpiral band of uniform thickneſs, and of which the inclination with reſpect to the axe of the cylinder is conſtant: the ſcrew, on the contrary, is a ſolid having a correſponding ſpiral hollow. In ſome caſes the ſpindle is fixed in a ſolid block, as AB, fig. 3. while the ſcrew E is moved upon it by means of a lever DC. At other times the ſcrew is fixed and the ſpindle moveable: but this cauſes no difference in the theory.

The curve which any one of the points of the generating polygon, as N for inſtance, deſcribes about AZ, is obviouſly traced on the ſurface of a right cylinder whoſe axis is AZ, and radius of its base EN (fig. 5.) If we develope this, then *dc* (fig. 6.) being the circumference which has EN for its radius, and taking the perpendicular *bc* equal the diſtance between two contiguous threads, the hypothenuſal line *db* will be the developement of an entire revolution of the point N. In effect, the helix being throughout of conſtant inclination with reſpect to any poſition whatever of the generating line of the cylinder, every parallel to AD will make with the developement of that curve the ſame angle: thus, the developement will be a right line, as *db*; and in like manner the right line *af* will be the developement of a ſecond revolution. This being granted, we may demonſtrate in a very ſatisfactory manner, the truth of the following propoſition.

160. PROP. *There will be an equilibrium in the ſcrew when the power is to the reſiſtance, as the diſtance between two contiguous threads in a direction parallel to the axis, to the circumference deſcribed by the power.*

Let us ſuppoſe the ſpindle AB to be fixed (fig. 3.), and that the ſcrew is moveable by the aid of a power P applied to the extremity C of a lever CE=R, acting horizontally perpendicular to the lever. Let W be the weight of the ſcrew, or that which the ſcrew ſupports, or the reſiſtance oppoſed by the ſcrew to the power P. If the ſcrew preſſed only on one of the points of the ſpindle, ſuppoſe it to be at the diſtance r from the axis, and that its poſition on the developement db of the ſpiral be at n; then will the preſſure on the ſpindle be exactly the ſame as on the inclined plane db. From the theory (154. cor. 8.) the power M which we ſuppoſe applied horizontally in direction Mn, muſt to retain the equilibrium, ſatisfy this proportion:

$$M : W :: cb : cd;\ \text{whence}\ M = W.\frac{cb}{cd} = W.\frac{h}{2\pi r}$$

where $h = bc$, and $\pi = 3{\cdot}141593$, as heretofore. The force M which is ſuppoſed applied in n, when the helix is not developed is perpendicular to the edge of the cylinder, or acts in a tangential direction to the cylinder, and of conſequence always parallel to the power P.

Now, ſubſtituting for this ſubſidiary power M, the power P, acting at the diſtance R, we have from the principles of the lever $PR = Mr$, the lengths of the arms being R and r. For M in this equation, ſubſtitute its value in the former one, and there ariſes $2\pi RP = Wh$. This equation, not containing r, is entirely independent of the diſtance at which the point n is ſuppoſed from the axis; it will therefore be the ſame if we ſuppoſe that point any where elſe on the ſpindle. Hence we deduce a general reſult: for this equation will even be true, if the ſcrew inſtead of touching the ſpindle in a ſingle point, as we have hitherto ſuppoſed, touch it in any number of points whatever. In this latter caſe every point on the thread of the ſpindle bears a portion of the weight W; theſe portions being denoted by W', W'', W''', &c. give $W' + W'' + W'''$, &c.$= W$. But, on the other hand, the force P which ſupports the weight of the ſcrew, may be conſidered as the ſum of as many forces P', P'', P''', &c. as there are points of contact, each of which is employed in ſupporting the weights W', W'', W''', &c. To each of theſe the laſt equation applies, we have, therefore, $W'h = 2\pi RP'$, $W''h = 2\pi RP''$, $W'''h = 2\pi RP'''$, &c. their ſum manifeſtly producing the equation $Wh = 2\pi RP$; whence $P : W :: h : 2\pi R$. Q. E. D.

Cor. 1. If the ſcrew had a ſquare or rectangular fillet inſtead of a triangular one, the concluſion would be the ſame, for it is independent of the form of the generating polygon.

Cor. 2. In the ſame ſcrew the effect is always the greater, as the power is applied farther from the axis.

Cor. 3. In two different ſcrews, a force acting with the ſame diſtance of lever, produces a greater effect in proportion as the threads of the ſcrew are nearer together.

Cor. 4. In the endleſs or *perpetual ſcrew* BC (fig. 7.) which drives the teeth of the wheel FD, we ſhall, in the caſe of an equilibrium, have P × AB × Rad. of FD = W × diſt. of threads × rad. of axle. For the perpetual ſcrew is a combination of the axis in peritrochio and the ſcrew.

SCHOLIUM.

161. The ſcrew is of very extenſive uſe in Mechanics, its great power rendering it more eligible for compreſſing bodies together than moſt other machines, and the great diſparity betwixt the velocity of the handle and that of the threads of the ſcrew rendering it proper for dividing ſpace into an almoſt infinite number of parts. Hence, in the conſtruction of many mathematical inſtruments, ſuch as teleſcopes, where it is neceſſary to adjuſt the focus to the eyes of different people, the ſcrew is always made uſe of in order to move the eye-glaſs a very little farther from or nearer to the object glaſs. In the Philoſophical Tranſactions, vol. 71. a new method of applying the ſcrew, ſo as to make it act with the greateſt accuracy, is deſcribed by Mr. Hunter. This method depends upon theſe general principles, applicable to moſt machines.

1. That the ſtrength of the ſeveral parts of the engine be adjuſted in ſuch a manner to the force they are intended to exert, that they ſhall not break under the weight they ought to counteract, nor yet encumber the motion by a greater quantity of matter than is neceſſary to give them a ſuitable degree of ſtrength.

2. That the increaſe of power by means of the machine be ſo regulated, that while the force is thereby rendered adequate to the effect, it may not be retarded in procuring it more than is abſolutely neceſſary.

3. That the machine be as ſimple as is conſiſtent with other conditions.

4. That it be as portable, and as little troubleſome as poſſible in its application.

5. That the moving power be applied in ſuch a manner as to act to the greateſt advantage; and that the motion ultimately

produced may have that direction and velocity which is moſt adapted to the execution of the ultimate deſign of the machine.

6. Of two machines, equal in other reſpects, that deſerves the preference in which the friction leaſt diminiſhes the effect propoſed by the whole.

To attain all theſe advantages in any one machine is perhaps impoſſible; but Mr. Hunter's method of applying the ſcrew certainly combines a great portion of them. Let AB (fig. 8. pl. VII.) be a plate of metal in which the ſcrew CD plays, having a certain number of threads in an inch, ſuppoſe 10. Within the ſcrew CD there is an exterior ſcrew which receives the ſmaller ſcrew DE of 11 threads in an inch. This ſcrew is kept from moving about with the former by means of the apparatus at AFGB. If the handle CKL be turned 10 times round, the ſcrew CD will advance an inch upwards; and if we ſuppoſe the ſcrew DE to move round along with CD, the point E will advance an inch. If we now turn the ſcrew DE ten times backward, the point E will move downwards $\frac{10}{11}$ths of an inch, and the reſult of both motions will be to lift the point E an eleventh of an inch upwards. But, if, while the ſcrew CD is turned 10 times round, DE be kept from moving, the effect will be the ſame as if it had moved 10 times round with CD, and been turned back again 10 times; that is, it will advance $\frac{1}{11}$ of an inch. At one turn, therefore, it will advance $\frac{1}{10}$ of $\frac{1}{11} = \frac{1}{110}$ of an inch. If now the handle be ſix inches long, the power to produce an equilibrium muſt be to the weight as 1 to $110 \times 6 \times 2\pi = 4146{\cdot}912$. Thus, the force of Mr. Hunter's ſcrew is greatly ſuperior to the common one; for a common one with a ſix inch handle muſt have 110 threads in an inch to produce the ſame effect, and this great number of threads would render it too weak to reſiſt any conſiderable violence.

With regard to the ſecond general maxim, both kinds of ſcrews are equally applicable; only that the more complicated ſtructure, and conſequently greater expence of Mr. Hunter's ſcrew, renders it convenient to uſe the common ſcrew where only a ſmall increaſe of power is requiſite, and the improved one where a great power is wanted. The handle being ſhort makes this machine accord with the fourth maxim.

To anſwer the fifth both ſeem equally proper; but for the ſixth the preference muſt be given to ſuch as beſt anſwer the ſpecific purpoſe propoſed. Thus, if the ſcrew DE be deſigned to carry an index which muſt turn round at the ſame time that it riſes upward, the common ſcrew ſeems preferable; though Mr. Hunter propoſes a method by which his may anſwer the ſame purpoſe: with this view a ſtill ſmaller ſcrew ought to play within DE, and be connected with CD, ſo as to move round along with it. It muſt have, according to the foregoing proportions, 111 threads in an inch: and they muſt lie in a contrary direction to thoſe of CD; ſo that when they are both turned together, and CD moves upwards, this other may move downwards. At one turn this will move upwards $\frac{1}{12210}$th part of an inch, and at the ſame time will move in a circular direction. Similar methods may be applied in many other caſes: indeed they have lately been applied very frequently; though few of thoſe who have adopted them have acknowledged by whom they were firſt propoſed; on which account we have given this brief decription of Mr. Hunter's contrivance, and of his judicious practical maxims.

VI. *Of the Wedge.*

162. Def. A *Wedge* is a triangular priſm, or a ſolid conceived to be generated by the motion of a plane triangle parallel to itſelf upon a ſtraight line which paſſes through one of its angular points. The wedge is called *iſoſceles*, *rectangular*, or *ſcalene*, according as the generating triangle is iſoſceles, right-angled, or ſcalene.

It is very frequently uſed in cleaving wood, as repreſented fig. 12. pl. VII. and often in raiſing great weights. The theory of knives, ſwords, coulters, nails, &c. is generally reduced to that of the wedge. The doctrine of the wedge is very imperfect, and can only be exhibited at all by making gratuitous aſſumptions: ſuch of thoſe as are moſt likely to obtain in practice are made the baſis of the three ſucceeding propoſitions.

163. Prop. *When a reſiſting body is ſuſtained againſt the face of a wedge, by a force acting at right angles to its direction, in the caſe of equilibrium the power is to the reſiſtance, as the ſine of the ſemi-angle of the wedge, to the ſine of the angle which the direction of the reſiſtance makes with the face of the wedge; and the ſuſtaining force will be as the coſine of the latter angle.*

Let ABC (fig. 9. pl. VII.) be a rectangular wedge, whoſe edge is C, face BC, and back AB. Let this wedge ſlide freely along the plane LN; let a body E be drawn or urged in the direction KE againſt the face of the wedge, and let it be kept in that direction by a force acting in the direction DE, at right angles to KE. There are now three forces acting on the body E, viz. the reſiſting force KE, the ſuſtaining force DE, and the re-action of the wedge in the direction AE, perpendicular to the ſurface BC. On ED demit the perpendicular AG; and ſince the three forces are in equilibrio, they will be to each other as the ſides of the triangle AEG drawn parallel to their directions. Draw EF perpendicular to AC, and the force AE will be reſolved into two, one of which EF preſſes the wedge perpendicularly againſt the plane LN, and is balanced by the reaction of the plane; the other FA endeavours to move the wedge upwards along the plane LN, and is balanced by the power on the back of the wedge. If, therefore, AG repreſent the force KE, EG will be the ſuſtaining force, and AF the power applied on the back of the wedge, when theſe forces balance each other. Hence, making AE radius, AF is the ſine of the angle AEF or ACB; and AG is the ſine of the angle AEG or KEC, theſe two angles being the complements of AEK.

If the wedge be iſoſceles, or compoſed of two rectangular wedges, the force EF, which in the former caſe was counteracted by the plane, will now be counteracted by the other half of the wedge; and the power, reſiſtance, and ſuſtaining force, will remain in the ſame ratio as before. Q. E. D.

Cor. I. When EK is parallel to BA, AG becomes equal and parallel to EF; and EG equal and parallel to AF; and the power is to the reſiſtance as AF to EF, or AB to AC, and equal to the ſuſtaining force.

Cor. 2. If EK be perpendicular to BA, the direction of the reſiſting force will be parallel to AB; therefore the reſiſting and ſuſtaining forces changing denominations, this will be a caſe correſponding with the former.

Cor. 3. When KE is perpendicular to BC, the ſine of the angle KEC is radius; and its coſine, which repreſents the ſuſtaining force, vaniſhes: therefore, the power is to the reſiſtance, as the ſine of the ſemiangle of the wedge, to radius.—See Ludlam's 2d Eſſay.

164. Prop. *When the reſiſtance is made againſt the face of a wedge by a body which is not ſuſtained, but will adhere to the place to which it is applied without ſliding, the power is to the reſiſtance, in the caſe of equilibrium, as the coſine of the difference between the ſemiangle of the wedge and the angle which the direction of the reſiſtance makes with the face of the wedge, to radius.*

From any point K (fig. 10. pl. VII.) draw the line KE through the middle point of the back, meeting the face of the wedge in E; let E be the unſliding body, which acts in the direction EK, and let the magnitude of the force with which it is urged be repreſented by AE: from E let fall the perpendicular EF upon AC; and the force AE will be reſolved into two, one of which EF will be balanced by the oppoſite half of the wedge, and the other AF will be counteracted by the power; therefore the power is to the reſiſtance as AF to AE, that is, making AE radius, as the coſine of the angle EAF to radius.

Cor. 1. When KE is perpendicular to BC, the power is to the reſiſtance as AF to AE, that is, as the ſine of the ſemiangle of the wedge to radius.

Cor. 2. When KE is parallel to AB, AF vaniſhes, that is, the power is indefinitely leſs than the weight.

Cor. 3. When KE is perpendicular to AB, EF vaniſhes, and AF and AE, which repreſent the power and reſiſtance, become equal.

165. Prop. *When the reſiſting body is neither ſuſtained nor adheres to the point to which it is applied, but ſlides freely along the face of the wedge, the power is to the reſiſtance, as the product of the ſines of the ſemiangle of the wedge and the angle in which the reſiſtance is inclined to its face, to the ſquare of radius.*

Let AE (fig. 11. pl. VII.) be perpendicular to BC, and let the body E be urged againſt the face of the wedge in the direction KE; and let KE repreſent the magnitude as well as the direction of that force. On AE produced let fall the perpendicular KO, which will be parallel to BC; thus will the force KE be reſolved into two, one of which KO will carry the body down along the face of the wedge, and the other OE will propel it perpendicularly againſt it. Now in the caſe of equilibrium, the power is to OE, that part of the reſiſtance which acts perpendicularly againſt the face of the wedge, as the ſine of the angle ACB to radius; and OE is to the whole reſiſtance, as OE to KE: that is, making KE radius, as the ſine of the angle OKE, or its alternate KEB, to radius. Therefore, *ex æquo & componendo*, the power is to the reſiſtance, as *ſine* ACB × *ſine* KEB, to the ſquare of radius.

Cor. 1. When KE is perpendicular to BC, the ſine of the angle in which the reſiſtance is applied, is radius; therefore the power is to the reſiſtance, as the ſine of the ſemiangle of the wedge to radius.

Cor. 2. When KE is parallel to AB, the angle of inclination is the complement of the ſemiangle of the wedge; and therefore, the power is to the reſiſtance, as the product of the ſine and coſine of the ſemiangle of the wedge to the ſquare of radius.

Cor. 3. When KE is perpendicular to AB, the angle of inclination is equal to the ſemiangle of the wedge, and the power is to the reſiſtance in a duplicate ratio of the ſine of the ſemiangle of the wedge to radius. See Dr. M. Young's Lectures.

SCHOLIUM.

166. The theory of the equilibrium of the wedge has greatly engaged the attention of many philoſophers, as Mr. *Ludlam*, &c. but it is not of very great uſe in practical mechanics, becauſe the wedge is ſcarcely ever otherwiſe urged than by percuſſion. In cleaving of wood, the reſiſtance oppoſing the force of the mallet (ſuppoſing the ſides of the wedge perfectly poliſhed, and its edge a line without breadth), is the coheſion of the particles of

the wood which are about to be ſeparated; and this being a kind of preſſive force acting againſt the ſides of the wedge, it is by many philoſophers thought abſurd to attempt to compare it with the percuſſive force of the mallet. For the greateſt finite preſſive force muſt, in their opinion, give way to the leaſt percuſſive one, and there cannot be an equilibrium between two ſuch different forces. "Any percuſſive force (ſay they) acting on a moveable body, generates a finite quantity of motion in an indefinitely ſmall portion of time; but the time will be finite in which any given preſſive force whatever, acting on the ſame body can generate or deſtroy the ſame quantity of motion. Therefore, a body being urged in a certain direction by any preſſive force whatever, and in the contrary direction by any percuſſive one, the preſſive force will be ſome finite time in deſtroying the quantity of motion which the percuſſive one generated in an inſtant. Conſequently, how great ſoever the preſſive force may be, and how ſmall ſoever the percuſſive one, the body will be moved (at leaſt for ſome ſhort time) by this laſt force." Indeed, after the ſtroke is given, the preſſive force may quickly prevail and force back the body, which the impulſe of the other force had driven forward. And ſo it would frequently be in the operation of cleaving wood, if the ſides of the wedge were perfectly ſmooth. For, after the ſtroke of the mallet, the wedge, unleſs its weight were equivalent to the attraction of the parts of the wood about to be ſeparated, would preſently be forced back from the place to which it had been driven by the mallet. And it is chiefly the roughneſs of the ſides of the wedge, and of the parts of the wood in contact with it, which, in that operation, keeps the wedge from receding. It is that roughneſs too, and the bluntneſs of the edge, which ſometimes prevent the wedge from being moved by the ſtroke of the mallet. For were it not obſtructed by ſuch roughneſs and bluntneſs, it would, according to what we juſt now obſerved, be always driven forward, even by the leaſt percuſſive force.

Several of theſe remarks, it will be ſeen, reſt upon the commonly received doctrine of percuſſion: a doctrine, however, which, in our humble opinion, has no very durable baſis. It is manifeſt that a blow may perform many things (particularly in effecting fractures or breaches) which a conſiderable preſſure cannot accompliſh: but this is probably owing to the circumſtance that the rapidity with which the excited preſſure increaſes to its maximum does not leave ſufficient time for the forces which connect the particles of the body ſtruck to be excited throughout to its more remote parts.—We hope to place this matter in a rather better light when the ſubjects of Colliſion and Percuſſion come under conſideration. See Book II. Chap. 5.

CHAPTER V.

On the Strength and Streſs of Materials.

167. THE *Reſiſtance* of ſolids, or that force with which the quieſcent parts of ſolid bodies oppoſe the motion of others contiguous to them, is generally conſidered as of two kinds, in one of which the reſiſting and reſiſted parts, though contiguous, conſtitute ſeparate maſſes; this will be conſidered in another place, under the title of *Friction*: in the other kind the reſiſting and reſiſted parts are not only contiguous but cohere, being parts of the ſame body or maſs; and it is this which we now propoſe to conſider.

This kind of Reſiſtance has exerciſed the ſagacity of ſome of the moſt eminent philoſophers from the time of Galileo down to the preſent period; and different theories have been propoſed by Mariotte, Leibnitz, Varignon, Euler, Lagrange, and Girard (perhaps others which have not come to our knowledge), but none of them are ſo free from objection and from error as might be wiſhed. Indeed, the figure and conſtitution of bodies are ſo variable and irregular, that we cannot with the deſireable preciſion determine thoſe elements which ſhould precede and regulate this diſcuſſion. Of the theories above adverted to ſome are certainly very ingenious; but at the ſame time they are very complex and intricate, and cannot by any means be relied upon independent of experiment: we therefore prefer the comparatively ſimple theory of Galileo, originally laid down in his dialogue "*On the Cauſe of the Coherence of Solids*," with which the other hypotheſes agree in the moſt eſſential particulars, and which, when aided by proper experiments, may ſerve as a ſafe approximation to the ſtrength and ſtreſs of the different parts of machines.

"That the reſiſtance of ſolids might be ſubjected to calculation, Galileo ſuppoſed firſt that bodies were compoſed of ſolid fibres, parallel to one another; he then enquired what was the force with which they reſiſt the action of a power ſtretching them in a direction parallel to their length, and found that it was proportional to the number of integral fibres: next, conſidering the fibres as ſubjected to an effort perpendicular to their length, he found that the reſiſtance of the integral fibres was proportional to their ſum multiplied by an arm of a lever, which is always at a certain part of the vertical dimenſions of a ſolid in the plane of its rupture." The length of this arm of lever was regulated, according to Galileo, by the poſition of the centre of gravity of the plane of rupture, according to others, by the centre of percuſſion, &c. But "the diſtinctive character of Galileo's hypotheſis conſiſts in this, that the reſiſtance of each of the fibres is independent of their quantity of extenſion at the inſtant of their rupture." Galileo's reaſoning on this intereſting topic is compriſed in 17 propoſitions: his *mode* of diſcuſſion will not be entirely adopted; but that the reader may know how far this philoſopher advanced the ſubject, we ſhall diſtinguiſh by a [G] ſuch of the following propoſitions and corollaries as are found in his Treatiſe.

168. DEF. *Strength*, and *Streſs* or *Strain*, are terms, the former of which is uſed to denote the force or power with which any maſs or body reſiſts a breach or change in its ſtate, which a preſſure or ſtroke upon it has a tendency to produce; and the latter are uſed indiſcriminately to expreſs the force which is excited in any ſuch maſs and tending to break it. Thus, every part of a pillar is equally *ſtrained* by the load which it ſupports. Hence, it is evident that we cannot make any ſtructure fit for its purpoſe, unleſs the *ſtrength* in every part be at leaſt equal to the *ſtreſs* laid on, or the *ſtrain* excited in that part: and hence the neceſſity of an acquaintance with the nature of the reſiſtance of bodies, ſo that there ſhall be neither a ſurplus nor a deficiency of materials in any machine.

169. PROP. *The ſtrength of a beam or bar to reſiſt a fracture by a force acting laterally, is as the ſolid, made by a ſection of the beam in the place where the force is applied, into the diſtance of its centre of gravity from the point or line where the breach will end.* [G.]

Suppoſe AB (fig. 13. pl. VII.) to be the beam (of uniform matter throughout) fixed firmly at its two ends A, B, at the middle of which is laid the weight W. In the caſe of a rupture we conceive the beam will be ſeparated firſt in the line *c d* oppoſite to W and fartheſt from it, and the ſeparation be gradually continued till it arrives at *a b*, which may, therefore, be conſidered as a fixed line till the termination of the fracture. Now the area *a b c d* repreſents the ſum of all the fibres to be broken or torn; and as they are equal to each other both in magnitude and ſtrength (by hypotheſis), this area will likewiſe expreſs the aggregate of the ſtrength of the fibres in the longitudinal direction. But with reſpect to lateral ſtrength, we muſt conſider each fibre as acting at the extremity of a lever whoſe centre of motion is on the line *ab*: thus, each fibre in the line *c d*, will reſiſt the breach by a force proportional to the product of its individual ſtrength into its diſtance db from the centre of motion; and conſequently the reſiſtance of all the fibres in cd will be repreſented by $cd \times bd$. In like manner, the aggregate reſiſtance of another courſe of fibres parallel to ab, as oo, will be repreſented by $oo \times bo$: and of a third, as ii, by the rectangle $ii \times bi$, and ſo throughout. Therefore the ſum of all theſe products will expreſs the total ſtrength or reſiſtance of the beam in that part. But (108.) the ſum of all theſe products is equal to the product of the area $abcd$, into the diſtance of its centre of gravity from ab: whence the propoſition is manifeſt.

Cor. 1. *In ſquare beams the lateral ſtrengths are as the cubes of the breadths or depths.* [G.]

Cor. 2. *In cylindric beams the lateral ſtrengths are as the cubes of the diameters.* [G.]

Cor. 3. *The lateral ſtrengths of any beams whoſe ſections are ſimilar figures, are as the cubes of correſponding dimenſions of the ſections.*

Cor. 4. *In rectangular beams the lateral ſtrengths are conjointly as the breadths and ſquares of the depths.* [G.]

For the areas are $\propto$ breadth × depth, and the diſtances of the centre of gravity are $\propto$ depth: conſequently, ſtrength $\propto$ breadth × depth.^{2}

Cor. 5. *The lateral ſtrength of a beam with its narrower face upwards, is to its ſtrength with the broader face upwards, as the breadth of the broader face, to the breadth of the narrower.* [G.] For $bd^2 : db^2 :: d : b$.

Cor. 6. If a beam were fixed firmly at one end into a wall and the fracture were cauſed by a weight ſuſpended at the other end, the proceſs of nature would be ſimilar, only that the breach would terminate at the lower part of the beam; and the propoſition and five firſt corollaries would ſtill obtain.

170. Prop. *The lateral ſtrengths of priſmatic beams of the ſame materials are as the areas of the ſections and the diſtances of their centres of gravity, directly, and as their lengths and weights, inverſely.*

Let BC, GH (fig. 14. pl. VII.), be two beams of like materials fixed in horizontal poſitions to the upright wall AB, by their ends B, G. Let A be the area of the end G of the beam GH, G the diſtance of the centre of gravity of that end from its loweſt point, L its length, W its weight, and S its ſtrength: and let a, g, l, w, and s, be correſponding particulars in the beam CB. Then $S : s :: \frac{A \,.\, G}{L \,.\, W} : \frac{a \,.\, g}{l \,.\, w} : A \,.\, G \,.\, l \,.\, w : a \,.\, g \,.\, L \,.\, W$.

For, the direct ſtrength, or effort tending to preſerve the adheſion of the fibres, varies as the product of A . G, $a \,.\, g$, by the laſt prop. while the efforts tending to deſtroy their adheſion, and which are therefore in the inverſe ratio of the ſtrengths, vary both in proportion to the weights of the beams, and the diſtances at which thoſe weights act: but the weights of the beam may evidently be conſidered as acting at their centres of gravity, the diſtances of which from the end ſupported vary as the length of the beams; and conſequently the efforts tending to deſtroy the adheſion of the beams are as $L . W$, $l . w$. Whence, by incorporating the direct and inverſe ratios, we obtain that ſtated in the propoſition.

COR. 1. Had the beams been conſidered as fixed at both ends, the ſame thing would follow, *with this difference only, that a beam when fixed at both ends is as ſtrong as one of equal breadth and depth, and but half the length, which is fixed only at one end.* For, if the longer beam were biſected, each of its halves would be ſituated with reſpect to its fixed end, in the ſame manner as the ſhorter beam with reſpect to its fixed end.

COR. 2. When the ſtrength of a beam is very conſiderable in relation to its weight, we may, inſtead of the propoſition, take $S : s :: \frac{A . G}{L} : \frac{a . g}{l}$.

COR. 3. *Cylinders and ſquare priſms have their lateral ſtrengths proportional to the cubes of the diameters, or depths, directly, and their lengths and weights, inverſely.* [G.]

COR. 4. *Similar priſms and cylinders have their ſtrengths inverſely proportional to their like linear dimenſions.* [G.] For the cubes of the diameters or depths, vary as the cubes of the lengths, and the weights and lengths are as the cubes of the lengths and the lengths conjointly, or in the quadruplicate ratio of the lengths: therefore, the ſtrengths are as L^3 to l^3 directly, and L^4 to l^4 inverſely, or inverſely as the lengths L, and l.

SCHOLIUM.

171. From the preceding deductions it follows that greater beams and bars muſt be in greater danger of breaking than the leſs ſimilar ones; and that, though a leſs beam may be firm and ſecure, yet a greater ſimilar one may be made ſo long, as neceſſarily to break by its own weight. Hence Galileo juſtly concludes that what appears very firm, and ſucceeds well in models, may be very weak and unſtable, or even fall to pieces by its weight, when it comes to be executed in large dimenſions, according to the model. From the ſame principles he argues that there are neceſſarily limits in the works of nature and art, which they cannot ſurpaſs in magnitude: that immenſely great ſhips, palaces, temples, &c. cannot be erected, their yards, beams, bolts, &c. falling aſunder by reaſon of their weight. Were trees of a very enormous magnitude, their branches would, in like manner, fall off. Large animals have not ſtrength in proportion to their ſize; and if there were any land animals much larger than thoſe we know, they could hardly move, and

would be perpetually ſubjected to moſt dangerous accidents. As to the animals of the ſea, indeed, the caſe is different, as the gravity of the water ſuſtains thoſe animals in great meaſure, and in fact theſe are known to be ſometimes vaſtly larger than the greateſt land animals. It is, ſays Galileo, impoſſible for Nature to give bones for men, horſes, or other animals, ſo formed as to ſubſiſt, and proportionally to perform their offices, when ſuch animals ſhould be enlarged to immenſe heights, unleſs ſhe uſes matter much firmer and more reſiſting than ſhe commonly does; or ſhould make bones of a thickneſs out of all proportion; whence the figure and appearance of the animal muſt be monſtrous. This he ſuppoſes the Italian poet hinted at, when he ſaid,

Non ſi può compartir quanto ſia lungo,
Si ſmiſuratamente è tutto groſſo.

Whatever height we to the giant give,
He cannot without equal thickneſs live.

And this ſentiment being ſuggeſted to us by perpetual experience, we naturally join the idea of greater ſtrength and force with the groſſer proportions, and that of agility with the more delicate ones. The ſame admirable philoſopher likewiſe remarks, in connection with this ſubject, that a greater column is in much more danger of being broke by a fall than a ſimilar ſmall one: that a man is in greater danger from accidents than a child: that an inſect can ſuſtain a weight many times greater than itſelf; whereas a much larger animal, as a horſe, could ſcarcely carry another horſe of his own ſize. The ingenious ſtudent may eaſily extend theſe practical remarks, to any caſes which may come before him.

172. Prop. *The lateral ſtrengths of two cylinders (of the ſame matter) of equal weight and length, one of which is hollow and the other ſolid, are to each other as the diameters of their ends.* [G.]

Let ABE, HIK (figs. 1 and 2. pl. VIII.), be the ends of two cylinders of equal length, and containing equal quantities of matter, the former being the ſection of a tube conſtituted of cylinders having a common axis; then is the ſtrength of the tube to that of the ſolid cylinder as AB to HI. For the lateral ſtrengths (169.) are conjointly as the areas and the diſtances of the centres of gravity of the ſections from A or from B, according as the fractures terminate at the one or the other point: but the areas of the annulus in the firſt fig. and of the circle in the ſecond, are here equal, and the centres of gravity of both are at their centres of magnitude; wherefore, ſince the radii vary as the diameters, the ſtrengths in this caſe vary in the ſame ratio.

COR. 1. Since, when the area of a circular ſection is given, its diameter is greater when the ſection is an annulus, than when it is a circle without any cavity; and ſince the power with which the parts of the cylinder reſiſt extraneous force is greater in the ſame proportion, it follows according to the theory thus ſtated, that the ſtrength may be increaſed indefinitely without augmenting the quantity of matter.

This concluſion is, however, manifeſtly erroneous: becauſe after the diameter of the tube exceeds a certain magnitude (which can only be aſcertained experimentally) it will become flaccid, and bend under the ſmalleſt additional weight. The fact is, the reaſoning in the propoſition is founded upon the preſumption that the figure of the ſection will be *conſtantly circular;* and will, therefore, only hold true under thoſe limits in which the preſſure or ſtroke upon the tube will not cauſe its ſection to degenerate from the circle to an ellipſis or any other form.

COR. 2. When the two diameters of the end of a tube are given, the diameter of a ſolid cylinder of equal weight may be eaſily found (the lengths being ſuppoſed the ſame), by taking the ſquare root of the difference of the ſquares of the diameters of the tube; or the ſquare root of the product of their ſum and difference.

Or, the ſame may be effected geometrically by this ſimple proceſs: from one end of the exterior diameter AB ſet off AE equal to the interior diameter CD, and join EB, which will be the diameter of the ſection ſought. For EB^2 muſt be equal to $AE^2 - CD^2$, which it is by this conſtruction and Euc. I. 47. and III. 31.

Or, if BE be drawn a tangent to the inner circle till it cuts the exterior one in two points E and B, it will be the diameter ſought. For the triangles BEA, BFG, are then ſimilar, the angles at E and F being right angles; conſequently $EA = 2\,FG = CD$; as in the preceding conſtruction.

COR. 3. The lateral ſtrengths of tubes and ſolid cylinders of equal length and ſimilar materials, are as the areas of their ends and their diameters conjointly.

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173. From this propoſition Galileo juſtly concludes, that Nature in a thouſand operations greatly augments the ſtrength of ſubſtances without increaſing their weight: as is manifeſted in the bones of animals and the feathers of birds, as well as in moſt tubes, or hollow trunks, which though light, greatly reſiſt any effort to bend or break them. "Thus (ſays he) if a wheat "ſtraw which ſupports an ear that is heavier than the whole "ſtalk were made of the ſame quantity of matter but ſolid, it

" would bend or break with far greater eaſe than it now does. " And with the ſame reaſon Art has obſerved and Experience " confirmed that a hollow cane, or tube of wood or metal, is " much ſtronger and more firm than if, while it continued of " the ſame weight and length, it were ſolid, as it would then, " of conſequence, be not ſo thick: and therefore Art has con" trived a method to make lances hollow within, when they are " required to be both light and ſtrong," in this inſtance, as in many others, imitating the wiſdom of Nature.

In all ſuch inſtances, however, there is an obvious diſtinction between the works of Nature and thoſe of Art; " in the former (as M. *Girard* remarks, when treating the ſame ſubject), the cauſe and the effect eſſentially agree; the one cannot undergo any modification without the other experiencing a correſpondent change; or, to ſpeak more preciſely, a new effect always reſults from a new cauſe:—in the productions of human induſtry, on the contrary, there is no neceſſary proportion between the effect and cauſe: if, for example, a determinate weight is to be raiſed, it is indifferent whether we uſe the thread which has preciſely the adequate force, or the cable which has a ſuperabundant one; while, if the ſame weight had reſted naturally ſuſpended, it would have done ſo by means of fibres peculiarly appropriated in their organization to the object, and whoſe diſpoſition would have preſented the moſt advantageous form. Perfection reſides in a ſingle point, at which Nature arrives without effort; while man is obliged, by repeated trials, to paſs over an immenſe ſpace which ſeparates him from it."

174. Prop. *Of all hollow cylinders whoſe lengths, and the diameters of the exterior and interior circles, continue the ſame, thoſe have the greateſt lateral ſtrength, in which the interior touches the exterior circle, in the higheſt part, provided the cylinders are fixed at both ends and in a horizontal poſition; or when they touch in the loweſt part, if the cylinders are fixed only at one end: the cylinders in both caſes being conceived to exert their ſtrength againſt weights acting vertically.*

Here, ſince the diameters of the exterior and interior circles are ſuppoſed invariable, the area of the ſpace they include will be likewiſe invariable, ſo that the ſtrengths of the cylinders will be proportional to the diſtances of the centres of gravity of their ſections from the point where the fracture would end on the ſuppoſition the cylinders were broken. Now, when a beam is fixed at both ends, and broken by a weight laid between thoſe ends, the breach would manifeſtly commence at the lower part and terminate at the upper; and, when it is fixed at one end only, and the weight acts at the other, the breach would commence on the upper ſide, and terminate at the lower: conſequently, in the firſt inſtance, the centre of gravity of the ſection muſt be at the greateſt diſtance from its higheſt point, and in the ſecond inſtance at the greateſt diſtance from the loweſt point, to enſure a maximum of ſtrength. Let figs. 3 and 4. repreſent the ſections of the tubes, BC and AD being the diameters of the hollow part of both: let the area of the circle whoſe diameter is AB be M, the diſtance of its centre of gravity from A being called G, the area of the ſmaller circle m, the diſtance of its centre of gravity from A, g, the area of the ſpace between the two circles $(=M-m)$ μ, the diſtance of its centre of gravity from A, γ; then, by the nature of the centre of gravity, will $MG=mg+\mu\gamma$. In this equation M, G, m, and μ, are invariable; therefore γ muſt vary inverſely as g: and conſequently, when γ is a maximum g is a minimum, and *vicè verſa*. But g is a minimum when the inner circle touches the outer one in A, as in fig. 4.; and it is a maximum when the interior and exterior circles touch in B, as in fig. 3. Therefore the maximum and the minimum values of γ obtain when the circles touch in A and B reſpectively; and the comparative ſtrengths of the tubes are as expreſſed in the propoſition.

175. In caſes actually ariſing in practice, this propoſition, as well as that laid down in art. 172. will require ſome modification, and for the ſame reaſon as is there ſtated. The ſtrengths of the tubes, however, will increaſe as the circles approach nearer to each other, until they reach a certain limit, which can only be determined by experiment for each different kind of reſiſting body, and for various proportions of the exterior and interior circles.

176. PROP. *The ſtrongeſt rectangular beam which can be cut out of a given cylinder, is that of which the ſquares of the breadth, and depth, and the ſquare of the cylinder's diameter, are reſpectively as the numbers* 1, 2 *and* 3.

In fig. 6. pl. VIII. let BC the breadth of the beam be denoted by b, AC the depth by d, the diameter AB being denoted by D; then, when BC is horizontal, the lateral ſtrength will be repreſented by bd^2 (art. 169. cor. 4.), which is to be a maximum. But $AC^2 = AB^2 - BC^2$, or $d^2 = D^2 - b^2$; therefore $(D^2 - b^2) \times b = bD^2 - b^3 =$ a max. In fluxions $D^2\dot{b} = 3b^2\dot{b}$: whence $D^2 = 3b^2$; and $d^2 = D^2 - b^2 = 3b^2 - b^2 = 2b^2$. Conſequently, $b^2 : d^2 : D^2 :: 1 : 2 : 3$. Q. E. D.

COR. 1. Hence ariſes this eaſy practical conſtruction: divide the diameter AB into three equal parts in E, F; erect the perpendiculars ED, FC; and join the points C, D, to the extremities of the diameter: ſo ſhall ACBD, be the rectangular end of the beam required. For, becauſe AE, AD, AB, are in continued proportion, we have $AE : AB :: AD^2 : AB^2$: and, in like manner $AF : AB :: AC^2 : AB^2$. Hence $AE : AF : AB :: AD^2 : AC^2 : AB^2 :: 1 : 2 : 3$.

Cor. 2. The ratio of b to d is nearly that of 5 to 7, or more nearly that of 12 to 17. For $5^2 : 7^2 :: 25 : 49 :: 1 : 2$ nearly; and $12^2 : 17^2 :: 144 : 289 :: 1 : 2$ more nearly.

Cor. 3. A ſquare beam from the ſame cylinder, would have its ſide $= D\sqrt{\frac{1}{2}} = \frac{1}{2}D\sqrt{2}$. Its ſolidity would be to that of the ſtrongeſt beam, as $\frac{1}{2}D^2$ to $\frac{1}{3}D^2\sqrt{2}$, or as $\frac{1}{2}$ to $\frac{1}{3}\sqrt{2}$, or as 5 to ·4714; while its ſtrength would be to that of the ſtrongeſt beam, as $(D\sqrt{\frac{1}{2}})^3$ to $D\sqrt{\frac{1}{3}} \times \frac{2}{3}D^2$, or as $\frac{1}{4}\sqrt{2}$ to $\frac{2}{9}\sqrt{3}$, or as 3560 to 3849.

Cor. 4. Either of theſe beams will exert the greateſt lateral ſtrength, when the diagonal of its end is placed vertically. For the area being the ſame in both poſitions, the ſtrength will vary as the diſtance of the centre of gravity from the baſe of fracture: but when one of the ſides is vertical, as in fig. 7. the diſtance of the centre of gravity of the end will be $=$ AI, or ID, equal to half the ſide; whereas, when the diagonal is vertical as in fig. 5. that diſtance will be CE or ED, half the diagonal.

Cor. 5. The ſtrength of the whole cylinder will be to that of the ſquare beam when placed with its diagonal vertically, as the area of the circle to that of its inſcribed ſquare. For CE or ED (fig. 5.) is the diſtance of the centre of gravity in both caſes; therefore the ſtrengths vary as the areas.

177. Prop. *When a triangular beam is ſupported at both ends, its ſtrength when the edge of the beam is uppermoſt is to the ſtrength when the oppoſite ſide is uppermoſt, as 2 to 1.*

For in this caſe the fracture would terminate at the top of the beam; and as the area of the end continues the ſame whether ſide or edge be uppermoſt, the ſtrength will vary as the diſtance of the centre of gravity from the uppermoſt point. Now, in the triangle ABC, the diſtance CG of the centre of gravity from the vertex, is double the diſtance GD from the baſe (fig. 8. pl. VIII.): therefore when the triangle is inverted as $a\,b\,c$, the diſtance of the centre of gravity from d the higheſt point, is only half its diſtance from the higheſt point C of the triangle before it is inverted. Whence the propoſition is manifeſt.

Cor. If the beam be ſupported at one end only, it would if broken have the breach commence at the upper part and terminate at the lower: and in that caſe the beam will be ſtrongeſt when the edge is downward, and only half that ſtrength when its oppoſite face is downward and horizontal.

178. PROP. *If a weight be placed upon any part of a horizontal beam fixed at both ends, the ſtreſs of the beam at that part will be as the rectangle of its diſtances from the ſupported ends.* [G.]

Let the weight W preſs upon the beam at C (fig. 5. pl. IV.), then is the weight equal to the preſſures upon A and B, and preſſure upon A $= \frac{W . BC}{AB}$, while preſſure upon B $= \frac{W . AC}{AB}$. (See Cor. 6. art. 133.) But the reaction of either point of ſupport is equal to the preſſure upon it (ax. 3.); and this may be conſidered as a force acting at the point C as upon the arm of a lever: ſo that the ſtreſs at C is as the preſſure at either point of ſupport into its diſtance from C. That is, the preſſure is as $\frac{W . BC}{AB} \times AC$, or as $\frac{W . AC}{AB} . BC$, which are manifeſtly equal, the one to the other. But W and AB are given; therefore the ſtreſs varies as the rectangle AC . CB.

COR. 1. The ſame thing will obtain if the weight be equally diffuſed through the whole of the beam. For, in this caſe as in the former the ſum of the preſſures upon A and B will be equal to the whole weight: and if w be the weight of the part BC, its preſſure upon A, will be $\frac{w \times \frac{1}{2} BC}{AB}$; and this referred back to the point C will give $\frac{w \times \frac{1}{2} BC}{AB} \times AC$, for the ſtreſs; which varies as the rectangle AC . BC, as before.

COR. 2. The greateſt ſtreſs of a beam is in the middle between its ſupports, whether the weight be applied there or equally diffuſed over its whole length. [G.]

For the rectangle of the two parts of a line is a maximum, when thoſe parts are equal to each other.

COR. 3. Hence in all ſtructures we ſhould as far as poſſible avoid placing weights in the midde of beams: thus, in roofing, it is better to uſe prick-poſts than king-poſts, unleſs where there is a pillar or a partition wall to ſupport the beam in the middle.

COR. 4. If w be the greateſt weight a beam will ſuſtain at its middle point, and it be required to ſupport by that beam a greater weight W; the point C may be found by making $W : w :: \frac{1}{2} AB \times \frac{1}{2} AB : AC . CB$.

To perform this geometrically Galileo proceeds thus: let W and w be repreſented by lines, their mean proportional being a line M: find a fourth proportional S to W, M and AD (fig. 9. pl. viii). Let AD be the diameter of the ſemicircle AHD, in which draw the line AH=S; join H, D, and make DC=DH, then is C the point ſought. For, on BA deſcribe the ſemicircle ANB, raiſe the perpendicular NC and join N, D: then, becauſe

$NC^2 + CD^2 = ND^2 = AD^2 = AH^2 + HD^2$, and $HD^2 = DC^2$, we have $NC^2 = AC . CB = AH^2$, that is $= S^2$. But $S^2 : AD^2 :: w : W$; whence the truth of the conſtruction is obvious.

Cor. 5. Since the rectangle AC . CB diminiſhes as C approaches to A or B, and the ſtreſs varies as that rectangle, it follows that much may be taken from the thickneſs of beams towards each end without rendering them too weak for the load. [G.]

Cor. 6. The ſtrains at any two points C, and D, in a beam, are as the rectangles AC . CB, AD . DB; and, therefore when weights are laid on at C, and D, in the proportion of thoſe rectangles the beam will be no more liable to break at one point than at the other. [G.]

Cor. 7. The ſtrain at any point D cauſed by a weight at C, is equal to the ſtrain at C occaſioned by the ſame weight at D.

For, when the weight W is at C, the ſtreſs there is as AC . CB, and the ſtreſs at D $= \frac{BD}{BC} \times$ ſtrain at C $= \frac{BD}{BC} \times AC . CB = BD . AC$. And, when the weight is at D, the ſtrain there is as AD . DB; while the ſtrain at C is as $\frac{AC}{AD} \times$ ſtreſs at D $= \frac{AC}{AD} \times AD . DB = AC . BD$, the ſame as before.

179. Prop. *If a beam in form of an iſoſceles wedge be fixed by its baſe to a vertical wall, its faces being in a vertical poſition, and a weight be attached to its vertex, ſuch a beam will be equally ſtrong throughout.*

Let the wedge BA (fig. 10. Pl. VIII.) be fixed to the vertical wall BE, and a weight W be placed at A. The effort of the weight W upon any point D of the beam will, by the nature of the lever, be as the rectangle W × AD, or as AD, becauſe W is conſtant. And the ſtrength at any point D, will be as the breadth into the ſquare of the depth at that place (all the vertical ſections being rectangles), or as the breadth CD, the depth being conſtant. Therefore, ſince when the beam is equally ſtrong throughout the ſtrength and ſtreſs are in an invariable ratio, we ſhall have CD conſtantly as AC; and, of conſequence, ACD muſt be a rectilinear triangle, and the beam a wedge.

180. Prop. *If a beam is placed horizontally with one end fixed to a wall, and a weight hung at the other, then if its breadth be the ſame from one end to the other, it will be equally ſtrong throughout when the vertical ſides are in form of a parabola.* [G.]

For the ſtreſs is as AD (fig. 11.) as in the laſt prop. and the ſtrength is as the breadth into the ſquare of the depth, or, becauſe the breadth is conſtant, the ſtrength is as CD^2. But the ſtreſs and ſtrength muſt remain in a conſtant ratio; therefore AD muſt vary as CD^2 throughout the figure; which is the well-known property of a parabola whoſe vertex is A.

COR. 1. Since the parabola is $\frac{2}{3}$ of its circumſcribing parallelogram, parabolic beams require $\frac{1}{3}$ leſs matter than priſmatic ones; a circumſtance which may be beneficially attended to, particularly when iron is uſed.

COR. 2. In the beams of balances for very great weights, each arm may be conſtructed of a parabolic ſhape as in fig. 13. P being the centre of motion: which will cauſe a ſaving of materials without any diminution of *uſeful* ſtrength.

181. PROP. *If a beam has one end fixed to a wall, and gradually diminiſhes towards the other end where a weight is placed, ſo that all its vertical ſections are ſquares, then, in order that it may be equally ſtrong throughout, the bounding curve muſt be in the form of a cubic parabola.*

In fig. 12. Pl. VIII. the ſtreſs or effort of the weight upon any point F will be as AF: and, becauſe the ſections are all ſimilar the ſtrengths will vary as the cubes of the depths. Hence, in this caſe $AF \propto DC^3$, which is a well-known property of a cubic parabola.

182. PROP. *When a beam whoſe vertical ſides are parallel planes is fixed at both ends, it will be equally ſtrong throughout, if either the tops of thoſe ſides or the bottoms, or both, be terminated by ellipſes.*

For, the ſides being parallel planes, the beam will be of equal thickneſs throughout, and conſequently the ſtrength at any point DC, will be as CD^2, or as Cc^2, according as ADB, or AcB, is the bottom of the beam (fig. 13. Pl. VIII.). Now, the ſtreſs at the point D, is as the rectangle $AD \cdot DB$ (art. 178.); therefore CD^2, or Cc^2 muſt vary as $AD \cdot DB$, to enſure equal ſtrength throughout. And this is the fundamental property of the ellipſe whoſe vertices are A and B.

183. PROP. *Given the length and weight of a cylinder or priſm, which is placed horizontally with one end fixed firmly, and will juſt ſupport a given weight at the other end without breaking, to find the length of a ſimilar priſm or cylinder, which when ſupported in like manner at one end, ſhall juſt bear without breaking a ſecond given weight at the unſupported end.*

Let l denote the length of the given cylinder or priſm, d the diameter or depth of its end, w its weight, and u the weight hanging at the unſupported end: the capitals L, D, W, and U, repreſenting correſponding particulars with reſpect to the other priſm. Then the weights of ſimilar ſolids of the ſame matter being as the cubes of their lengths, we ſhall have l^3 :

$L^3 :: w : \frac{L^3}{l^3} w$, the weight of the priſm whoſe length is L. We may either ſuppoſe the whole weight of each beam to act at its centre of gravity, or half its weight at the unſupported end; for, on both theſe ſuppoſitions the ſtreſs upon the ſupported or fixed end ariſing from the weight of the beam will be the ſame: hence then, the ſtreſs upon the fixed end of the beam whoſe length is l, ariſing both from its own weight and the weight u attached to it, will be $(\frac{1}{2} w + u) \times l$; and the ſtreſs upon the other beam will be $\left(\frac{1}{2} w \frac{L^3}{l^3} + U\right) \times L$, or $\left(\frac{L^3}{2 l^3} w + U\right) \times L$. But the lateral ſtrength of the firſt beam is to that of the ſecond, as d^3 to D^3 (art. 169. cor. 2.) or as l^3 to L^3. Wherefore ſince the ſtrengths and ſtreſſes of the two beams in their reſpective circumſtances muſt be in the ſame ratio, to anſwer the conditions of the propoſition, we have $(\frac{1}{2} w + u)\, l : \left(\frac{L^3}{2 l^3} w + U\right) L :: l^3 : L^3$. This analogy converted into an equation, gives, after a little reduction, $L^3 - \frac{w + 2u}{w} l L^2 + \frac{2}{w} U l^3 = 0$: a cubic equation from which the numeral value of L may be determined, when thoſe of the other quantities are ſpecified.

Cor. 1. When U vaniſhes the equation becomes $L^3 = \frac{w + 2u}{w} l L^2$; whence $L = \frac{w + 2u}{w} l$, or $w : w + 2u :: l : L$. From which the length of the beam which will juſt break by its own weight, may be readily found.

Cor. 2. Hence of all beams of ſimilar ſhape and materials, there is one, and only one, that will merely ſuſtain itſelf when fixed at one end, being juſt on the point of breaking. [G.]

Cor. 3. If a beam break by its own weight, when fixed at one end, a beam of twice its length fixed in a ſimilar manner at both ends will alſo break by its own weight: or if one ſuſtain itſelf the other will.

For the ſtrain or ſtreſs is the ſame in both of them; each being equal to the ſtreſs of a beam of the double length, and ſupported at its middle point.

184. Prop. *Given the length and weight of a cylinder or priſm, which is fixed horizontally as in the foregoing propoſition, and a weight which when hung at a given point breaks the priſm, to find how much longer a priſm of equal diameter or of equal breadth and depth, may be extended before it break either by its own weight, or by the addition of any other adventitious weight.*

Here let l denote the length of the given priſm, w its weight, u a weight attached to it at the diſtance d from the fixed end; L the length of the required priſm, and U the weight attached to it at the diſtance D. Then $l : L :: w : \frac{Lw}{l}$, the weight of the priſm whoſe length is L: therefore the ſtrain it will occaſion upon the fixed end will be $\frac{Lw}{l} \times \frac{1}{2} L = \frac{wL^2}{2l}$; and the ſtrain ariſing from the weight U acting at the diſtance D will be DU; ſo that the whole ſtrain occaſioned by the longer beam and its weight will be $\frac{wL^2}{2l} + DU$. The ſtrain occaſioned, in like manner, by the weight of the original beam will be $\frac{1}{2} w l$, and by its weight u at the diſtance d, will be du, their ſum being $\frac{1}{2} wl + du$. Now the ſtrength of the beam which is juſt ſufficient to reſiſt theſe ſtrains is the ſame in both inſtances; conſequently $\frac{wL^2}{2l} + DU = \frac{1}{2} wl + du$; and by reduction $L = \sqrt{\frac{2l}{w} \times (\frac{1}{2} wl + du - DU)}$.

COR. 1. If the lengthened beam juſt breaks with its own weight, then U vaniſhes, and we have $L = \sqrt{l^2 + \frac{2dul}{w}} = \sqrt{l \times (l + \frac{2du}{w})}$.

COR. 2. If when U vaniſhes d becomes equal to l, we have $L = \sqrt{l \times \frac{wl + 2lu}{w}} = \sqrt{l \times \frac{l \times (w + 2u)}{w}}$. This may be expreſſed in words thus: make a line S in the ſame proportion to l, as the compound of the weight of l and double the weight attached to its end, is to the weight w of l; then ſhall the geometrical mean proportional between S and l be the length L required. This is the rule given by Galileo.

185. PROP. *The ſtrength of a rectangular beam in an inclined poſition is to the ſtrength of the ſame beam in an horizontal poſition, to reſiſt a vertical preſſure, as the ſquare of the radius to the ſquare of the coſine of elevation.*

For a tranſverſe vertical ſection of the beam will be rectangular, whether the beam be in a horizontal or oblique poſition: and conſequently, on the principles we have aſſumed, the ſtrengths in both caſes will be as the ſquares of the depths. Now AB (fig. 14. Pl. VIII.) being a longitudinal vertical ſection of the beam, the depth of the beam when inclined will be CD, and C d when it is horizontal. Hence the ſtrength in the one caſe to that in the other is as CD^2 to Cd^2, or becauſe of the ſimilar triangles CDd, ADO, as AD^2 to AO^2, that is, making AD radius, as radius^2 to cos.^{2} DAO. Q. E. D.

COR. 1. The ſtrength of any beam reſiſting a vertical preſſure is greateſt when it is in a vertical poſition; for then the coſine of inclination vaniſhes.

Cor. 2. The ſame prop. and corol. will hold, with reſpect to a preſſure in any other direction, provided AH, inſtead of being horizontal, be perpendicular to the preſſure.

186. Prop. *When two beams ſtanding obliquely bear two weights upon them, either at the middle points or in any ſimilar ſituations, or equally diffuſed over their whole lengths; the ſtrains upon them will be directly as the weights, the lengths, and the coſines of elevation, conjointly.*

For (art. 154. cor. 4.) the weight is to the preſſure upon the plane, as radius to the coſine of elevation: therefore the preſſure is as the rectangle of the weight and coſine of elevation; and this is the force acting againſt the beam perpendicularly. Whence the ſtreſs will univerſally be as the length of the beam and this force, that is, as the length × weight × coſine of elevation.

Cor. 1. If the lengths of the beams, and their weights, be the ſame, the ſtreſs will be as the coſine of elevation; and conſequently it will be greateſt when the beam is horizontal.

Cor. 2. If the beams are horizontal, or at any equal inclinations, and the weights vary as the lengths, or the beams are uniform, then will the ſtreſs vary as the ſquares of the lengths.

Cor. 3. If the weights are equal on the horizontal beam AC, and the inclined one AB (fig. 15.) and BC be vertical, the ſtreſs upon both beams will be equal.

For the length into the coſine of elevation is the ſame in both.

Cor. 4. But if the weights on the beams vary as their lengths, the ſtrength will alſo vary in the ſame ratio.

Cor. 5. And univerſally the ſtreſs upon any point of an oblique beam, is as the rectangle of the ſegments of the beam, the weight, and cos. inclination directly, and the length of the beam reciprocally.

187. Prop. *Let* AC *(fig* 1. *Pl.* IX.*) repreſent a beam moveable about a centre* C, *ſo as to make any angle* ACB *with the plane of the horizon* CB; *to determine the poſition of a prop or ſupporter* OS *of a given length, which ſhall ſuſtain it with the greateſt eaſe in any given poſition; alſo to aſcertain the inclination of* AC *to the horizon when the leaſt force that can ſuſtain it is greater than the leaſt force in any other poſition.*

Let G be the centre of gravity of the beam AC, and draw Gn, Gm, and CD, perpendicular to AC, CB, and OS reſpectively. Put SO$=p$, CG$=r$, C$m=x$, and the weight of the beam $=w$. Then, by the nature of the parallelogram of forces, we ſhall have Gm : Gn, or ſim. triangles, as CG

($=r$) : Cm ($=x$) : : w : $\frac{wx}{r}$, the force which acting at G in the direction Gn, is ſufficient to ſuſtain the beam: and, by the nature of the lever, CO : CG ($=r$) : : $\frac{wx}{r}$ (the requiſite force at G) : $\frac{wx}{\text{CO}}$, the force capable of ſupporting it at O in a direction perp. to AC or parallel to Gn: and, again, as CD : CO : : $\frac{wx}{\text{CO}}$ (the force acting at O in direction $\perp$ to CO) : $\frac{wx}{\text{CD}}$, the force or weight actually ſuſtained by the given prop SO in a direction $\perp$ to CD. This latter force will manifeſtly be the leaſt poſſible when the perpendicular CD upon OS is the greateſt poſſible, let the angle ACB be what it may. But of all triangles, having the ſame baſe OS and vertical angle SCO that which is iſoſceles is known to have the greateſt perpendicular [being an obvious corol. from Simpſon's 6th Theor. on the Max. and Min. of Geometrical quantities]: therefore the triangle OSC will be iſoſceles, and the angle S = angle o, when the weight ſuſtained by the prop OS is a *minimum*.

Secondly, in order to give a ſolution to the latter part of the problem, we have to find when $\frac{wx}{\text{CD}}$ is a maximum, the angles S and O being always = each other, while they vary in magnitude in conſequence of the change of the inclination ACB. Let CD produced meet Gm in I; then, becauſe of the ſimilar triangles CDS, CmI, we ſhall have CD : Cm ($=x$) : : SD ($=\frac{1}{2}p$) : mI, whence $\frac{x}{\text{CD}}=\frac{m\text{I}}{\frac{1}{2}p}$; and conſequently $\frac{x}{\text{CD}}\times w=\frac{m\text{I}}{\frac{1}{2}p}\times w$. But ſince CI biſects the angle mCG, we alſo have CG + Cm ($=r+x$) : Cm ($=x$) : : Gm ($=\sqrt{r^2-x^2}$) : Im = $\frac{x}{r+x}\sqrt{r^2-x^2}=x\sqrt{\frac{r-x}{r+x}}$: conſequently the force $\frac{m\text{I}}{\frac{1}{2}p}\times w$, acting upon the prop, is likewiſe truly expreſſed by $\frac{wx}{\frac{1}{2}p}\sqrt{\frac{r-x}{r+x}}$. The fluxion of this expreſſion being taken and put equal to *zero*, we obtain $x=\frac{r\sqrt{5}-r}{2}$; therefore CG : Cm : : 1 : $\frac{1}{2}\sqrt{5}-\frac{1}{2}$: : radius : coſine of GCB = 51° 50′, the inclination required.

188. PROP. *Suppoſe the beam* AC *inſtead of being moveable about the centre* C, *to be ſupported in a given poſition by means of the given prop* OS; *it is required to determine the poſition of that prop ſo that the priſmatic beam* CB *on which it ſtands may be the leaſt liable to breaking, this latter beam being only ſupported at its ends* C *and* B.

Let CB $=b$, OS $=p$, CG $=r$, weight of CA $=w$, CO $=z$, ſine and coſine of angle C $=s$ and c reſpectively, ſine $\angle$O $=x$, ſine $\angle$S $=y$. Then, by trig. $z : y :: p : s$, or $\frac{y}{z}=\frac{s}{p}$, and CS $=\frac{px}{s}$: alſo the force of the beam at G in direction G$n=cw$. Let F denote the force ſuſtaining the beam at O in the direction SO; then, becauſe action and reaction are equal and oppoſite, the ſame force will be exerted at S in the direction OS: therefore CG $\times cw=$ Fx, and F $=\frac{rcw}{zx}$. Again, the vertical ſtreſs at S, will $\propto$ F $\times$ ſin. S $\times$ CS . SB $=$ Fy . CS . SB $=\frac{rcwy}{zx}\times\left(b-\frac{px}{s}\right)\times\frac{px}{s}=$ (ſubſtituting $\frac{s}{p}$ for its equal $\frac{y}{z}$) $\frac{rcws}{px}\times\frac{px}{s}\times\frac{bs-px}{s}=rcw\times\frac{bs-px}{s}=\frac{rcwp}{s}\times\left(\frac{bs}{p}-x\right)=$ a min. by the prop. Conſequently $\frac{bs}{p}-x=$ a min. or x a max.: that is, $x=1$, and the angle COS a right angle. Hence the point O is readily found by this proportion, ſin. C : cos. C :: OS : OC.

GENERAL SCHOLIA.

189. We have already adverted to a general maxim, which, on account of its great importance, we beg to ſtate again: it is this:—*When ſeveral pieces of timber, iron, or any other materials are introduced into a machine or ſtructure of any kind, the parts not only of the ſame piece, but of the different pieces in the fabric, ought to be ſo adjuſted with reſpect to magnitude that the ſtrength may be in every part as near as poſſible in a conſtant proportion to the ſtrain to which they will be ſubjected.* Thus, in the conſtruction of any engine, the weight and preſſure upon every part ſhould be inveſtigated, and the ſtrength ſhould be apportioned accordingly. All levers, for inſtance, ſhould be made ſtrongeſt where they are moſt ſtrained: as levers of the firſt kind, at the fulcrum; levers of the ſecond kind, where the weight acts; and thoſe of the third kind, where the power is applied. The axles of wheels and pullies, the teeth of wheels, ropes, &c. muſt be made ſtronger or weaker, as they will be more or leſs acted upon. Let the ſtrength allowed be more than fully competent to the ſtreſs to which the parts can ever be liable; but let not the ſurplus be extravagant: for ſuch an exceſs of ſtrength in any part, inſtead of being ſerviceable, is injurious by increaſing the reſiſtance the machine has to overcome, and thus encumbering, impeding, and often deſtroying the requiſite motion: while, on the other hand, a defect of ſtrength in any one part will cauſe a

failure there, and either render the whole uſeleſs, or call for frequent repairs.

190. The propoſitions we have given on the ſtrength and ſtreſs of materials, however true, according to the principles aſſumed, are of no uſe in practice till the comparative ſtrength of different ſubſtances is aſcertained. And even then they will apply more accurately to ſome ſubſtances than others. Hitherto they have been almoſt excluſively applied to the reſiſting force of beams of timber; though it is probable no materials whatever accord leſs with the theory than timber of all kinds. The reſiſting body is ſuppoſed in the theory to be perfectly homogeneous, or compoſed of parallel fibres, equally diſtributed around the axis, and preſenting uniform reſiſtance to rupture. But this is not the caſe in a beam of timber: for, by tracing the proceſs of vegetation, it has been found that the ligneous coats of a tree, formed by its annual growth, are almoſt concentric; and that they are like ſo many hollow cylinders thruſt into each other, and united by a kind of medullary ſubſtance which offers but little reſiſtance: theſe hollow cylinders, therefore, furniſh the chief reſiſtance to the force which tends to break them. Now, when the trunk of a tree is ſquared in order that it may be converted into a beam it is evident that all the ligneous cylinders greater than the circle inſcribed in the ſquare or rectangle, which is the ſection of the beam, are cut off at the ſides; and therefore, as Montucla remarks, almoſt the whole reſiſtance ariſes from the cylindric trunk inſcribed in the ſolid part of the beam. The portions of the cylindric coats which are towards the angles add a little, it is true, to the ſtrength of that cylinder, as they cannot fail to oppoſe *ſome* reſiſtance to the ſtraining force; but it is far leſs than though the ligneous cylinder were entire. Hence we cannot by legitimate compariſon accurately deduce the ſtrength of a joiſt cut from a ſmall tree, by experiments on another which has been ſawn from a much larger tree or block: the latter is generally weak, and very liable to break. As to the concentric cylinders we have been ſpeaking of, they are evidently not all of equal ſtrength. Thoſe neareſt the centre being the oldeſt, are likewiſe the hardeſt: which again, is contrary to the theory, in which they are ſuppoſed uniform throughout. After all, however, it is ſtill found that in ſome of the moſt important problems the reſults of the theory and well-conducted experiments coincide, even with regard to timber: thus, for example, the experiments of Duhamel on rectangular beams afford reſults deviating but in a ſlight degree from the theorem of Galileo, that the ſtrength is proportional to the product of the breadth into the ſquare of the depth.

190 *. Experiments on the ſtrength of different kinds of wood,

are by no means ſo numerous as might be wiſhed. The moſt uſeful ſeem to be thoſe made by Emerſon, Parent, Banks, and Girard: but it will be at all times highly advantageous to make new experiments on the ſame ſubject; a labour eſpecially reſerved for engineers who poſſeſs ſkill and zeal for the advancement of their profeſſion. It has been found by experiments that the ſame kind of wood, and of the ſame ſhape and dimenſions, will break with very different weights: that one piece is much ſtronger than another not only cut out of the ſame tree, but out of the ſame rod: and, that if a piece of any length planed equally thick throughout, be ſeparated into three or four pieces of an equal length, it will be found that theſe pieces require different weights to break them. Emerſon obſerves that wood from the boughs and branches of trees is far weaker than that of the body: the wood of the great limbs ſtronger than that of the ſmall ones: and the wood in the heart of a ſound tree ſtrongeſt of all. He alſo obſerves that a piece of timber which has borne a great weight for a ſmall time, has broke with a far leſs weight, when left upon it for a much longer time. Wood is likewiſe weaker when it is green, and ſtrongeſt when thoroughly dried; and ſhould be two or three years old, at leaſt. Knots in wood often weaken it very much. And when wood is croſs grained, as often happens in ſawing, this will weaken it in a greater or leſs degree, according as the cut runs more or leſs acroſs the grain. From all which it follows that a conſiderable allowance ought to be made for the ſtrength of wood, when applied to any uſe where ſtrength and durability are required.

Iron is generally much more uniform in its ſtrength than wood: yet experiments ſhew that there is ſome difference occaſioned by different kinds of ore; the difference is not only found in iron from different furnaces; but from the ſame furnace, and the ſame melting; this may ariſe in great meaſure from the different degrees of heat which it has when it is poured into the mould.

Every beam or bar, whether of wood, ſtone, or iron, is more eaſily broken by any tranſverſe ſtrain, when it is ſuſtaining any very great compreſſion endways. Several experiments have been made on this kind of ſtrain: a piece of white marble $\frac{1}{4}$ inch ſquare and three inches between the props, bore 38lbs: when compreſſed endways with 300lbs. it broke with $14\frac{1}{2}$lbs. The effect is much more remarkable in timber, and more elaſtic bodies; but is conſiderable in all. This, therefore, is a point which muſt be attended to in all experiments: as muſt likewiſe the following, namely, that a beam ſupported at both ends, will carry *twice* as much when the ends beyond the props are kept from riſing, as when the beam reſts looſely on the props. The demonſtration of this is given by *Girard*; and many experiments furniſh nearly the ſame reſult.

Some writers, however, ſeem to doubt its truth: and as every one has not opportunities of trying the matter experimentally, *Profeſſor Robiſon*, in order to remove any doubts, gives this familiar repreſentation of the caſe. Let LM (fig. 2. pl. IX.) be a long beam divided into ſix equal parts, in the points D, B, A, C, E; and firmly ſupported at L, B, C, M. Let it be cut through at A, and have compaſs-joints at B and C: FB and GC are two equal uprights, reſting on B and C, but without any connection; AH is a ſimilar and equal piece occaſionally applied at the ſeam A. Now let a thread or wire AGE be extended over the piece GC, and made faſt at A, G, and E: and let the ſame thing be done on the other ſide of A, as at A, F, D. If a weight be now laid on at A, the wires AFD, AGE, may be ſtrained till they are broken. In the inſtant of fracture we may ſuppoſe their ſtrains to be repreſented by Af and Ag. Complete the parallelogram, and Aa repreſents the magnitude of the weight. Nothing, it is plain, is concerned here, but the coheſion of the wires; for the beam is ſawed through at A, and its parts move with perfect freedom round B and C. Inſtead of this proceſs apply the piece AH below A, and keep it there by ſtraining the ſame wire over it in the poſition BHC. If a weight be now laid on, it muſt preſs down the ends of BA and CA, and cauſe the piece AH to ſtrain the wire BHC. In the inſtant of fracture of the ſame wire, ſimilarly poſited, its reſiſtance Hb, Hc, muſt be equal to Af, Ag, and the weight bH which breaks them muſt be equal to Aa. Laſtly, employ all the three pieces FB, AH, GC, with the ſame wire attached as in the two caſes combined: there can be no doubt that the weight which breaks all the wires muſt be $= Aa + bH$, or twice Aa. And it is exceedingly manifeſt that the wires perform the very ſame office with the fibres of an entire beam LM held faſt in the four holes D, B, C, E, of ſome upright poſts.

The ſame thing is ſhewn rather differently at pa. 464, Emerſon's Algebra.

The relative ſtrengths of ſeveral ſorts of wood, and of other bodies, as determined by Mr. Emerſon, are as follow:

Box, yew, plumtree, oak	11
Elm, aſh	8½
Walnut, thorn	7½
Red fir, hollin, elder, plane crabtree, apple-tree	7
Beech, cherry-tree, hazle	6⅔
Alder, aſp, birch, white fir, willow or ſaugh	6
Iron	107
Braſs	50
Bone	22

Lead	$6\frac{1}{2}$
Fine free-ſtone	1

A cylindric rod of good clean fir, of an inch circumference, drawn in length, will bear at extremity 400 lbs., and a ſpear of fir 2 inches diameter, will bear about 7 tons.

A rod of good iron of an inch circumference will bear nearly 3 tons weight.

A good hempen rope of an inch circumference will bear 1000 lbs. at its extremity.

Hence this author concludes that if a rod of fir, or a rope, or a rod of iron, of d inches diameter were to lift $\frac{1}{4}$ the extreme weight, then

The fir would bear $8\frac{4}{5}\, d\, d$ hundred weights.
The rope . . 22 $d\, d$ ditto.
The iron . . $6\frac{3}{4}\, d\, d$ tons.

191. Mr. Banks, an ingenious lecturer on Natural Philoſophy, has at various times made many experiments on the real and comparative ſtrengths of oak, deal, and iron. He found that the worſt or weakeſt piece of dry heart of oak, 1 inch ſquare, and 1 foot long, bore 660 lbs. though it was much bent; and 2 pounds more broke it. The ſtrongeſt piece he tried of the ſame dimenſions, broke with 974 lbs. The worſt piece of deal bore 460 lbs. but broke with 4 more. The beſt piece bore 690 lbs.; but broke with a little more. And with reſpect to *caſt* iron, he concludes that a bar of the weakeſt kind, an inch ſquare and a foot long, would break with about 2190 lbs. The following are ſome of the experiments he mentions. [Banks on Power of Machines, pa. 89.]

" *Experiments on the ſtrength of caſt iron, tried at Ketley,*
" *March* 1795.

☞ " The different bars were all caſt at one time out of the ſame air furnace, and the iron was very ſoft, ſo as to cut or file eaſily.

" *Exp.* I. Two bars of iron, one inch ſquare, and exactly three feet long, were placed upon an horizontal bar ſo as to meet in a cap at the top, from which was ſuſpended a ſcale; theſe bars made each an angle of 45° with the baſe plate, and, of conſequence formed an angle of 90° at the top: from this cap was ſuſpended a weight of ſeven tons, which was left for 16 hours, when the bars were a little bent, and but very little.

" *Exp.* II. Two more bars of the ſame length and thickneſs, were placed in a ſimilar manner making an angle of $22\frac{1}{2}$° with the baſe plate; theſe bore four tons upon the ſcale: a little more weight broke one of them, which was obſerved to be a little crooked when firſt put up. In this caſe the preſſure would be as the ſines of the angles of elevation, viz. as ·3826 to 7071;

and as 3826 : 4 tons : : 7071 : 7 . 6 tons; that is, if the ſecond bars broke with four tons, the firſt ought to have taken 7 . 6 tons to break them; and it is likely that would if tried have been the caſe.

" *Exp.* III. Another bar was placed horizontally upon two ſupporters, exactly three feet diſtant; it bore 6 cwt. 3 qrs. but broke when a little more was added.

" *Exp.* IV. The ſame experiment repeated with the ſame reſult.

" *Exp.* V. The bearings were 2 feet 6 inches apart, the bar bore 9 cwt. and broke." " Three more experiments were tried the next day with the priſms 3 feet diſtant; the average reſult was 6 cwt. 2 qrs. $7\frac{1}{3}$ lbs."

" *Experiments tried at Colebrook-dale, on curved bars or ribs*
" *of caſt iron, April* 1795.

" Rib 29 feet 6 inches ſpan, and 11 inches high in the centre; it ſupported 99 cwt. 1 qr. 14 lbs. : it ſunk in the middle $3\frac{7}{8}$, and roſe again $\frac{3}{4}$ when the weight was removed. The ſame rib was afterwards tried without abutments, and broke with 55 cwt. 0 qrs. 14 lbs.

" Rib 29 feet 3 inches in ſpan, a ſegment of a circle, 3 feet high in the centre; it ſupported 100 cwt. 1 qr. 14 lbs. and ſunk $1\frac{3}{16}$ in the middle. The ſame rib was afterwards tried without abutments, and broke with 64 cwt. 1 qr. 14 lbs."

The thickneſs of theſe ribs, unluckily, is not ſpecified; but the experiments prove that each rib exerted little more than half the ſtrength when the abutments were removed. This correſponds with what we have ſtated in the laſt article relatively to beams being fixed at the ends, as nearly as can be expected, conſidering the neceſſary difference between priſms and theſe arch ribs.

Mr. Banks made ſome experiments on the ſtrength of caſt iron, at Meſſrs. Aydon and Elwell's foundry, Wakefield. " The iron came from their furnace at Shelf, near Bradford, and was caſt from the air furnace; the bars one inch ſquare, and the props exactly a yard diſtant. One yard in length weighs exactly 9lbs. or one was about half an ounce leſs, and another a very little more: they all bent about an inch before they broke.

" 1. The firſt bar broke with . . . . . 963 lbs.
" 2. Bar broke with . . . . 958 lbs.
" 3. Bar broke with . . . . 994 lbs.
" 4. Bar made from the cupola, broke with 864 lbs.
" Bar equally thick in the middle, but the ends formed into a parabola, and weighed 6 lbs. 3 oz. broke with . . . 874 lbs."

This gentleman made many other experiments. He concludes from the whole that caſt iron is from $3\frac{1}{3}$ to $4\frac{1}{2}$ times ſtronger than oak of the ſame dimenſions, and from 5 to $6\frac{1}{2}$ times ſtronger than deal.

With reſpect to the twiſt to which bars or ſhafts placed in an upright poſition are liable, between the wheel which drives them and the reſiſtance they have to overcome, he obſerves that a caſt iron bar an inch ſquare and fixed at one end, and 631 pounds ſuſpended by a wheel of 2 feet diameter fixed on the other end, will break by the twiſt: though ſome have required more than 1000 lbs. in ſimilar ſituations to break by the twiſt. The ſtrength to reſiſt the twiſting ſtrain is as the cube of like lateral dimenſions.

191. * Profeſſor Robiſon found by many trials that a priſm of white marble an inch ſquare and a foot long, bears about 500 lbs. And he has collected from various authors that the coheſive force of a ſquare inch of gold when caſt is about 20000 lbs. of ſilver 40000, caſt iron 40000 to 60000, wrought iron 60000 to 90000, ſoft ſteel 12000, razor ſteel 15000; oak and beech in the direction of their fibres 8000 to 17000, willow 12000, cedar 5000, fir 8000; ivory 16000, bone 5000; rope 20000: and a cylinder of an inch in diameter loaded to one-fourth will carry, if of iron 135 hundred weight, of rope 22, oak 14, fir 9. The ſtrength of ſome metals is doubled or tripled by the operation of forging and wire-drawing; and the coheſive as well as the repulſive force of wood is often increaſed by moderate compreſſion. Oak will *ſuſpend* much more than fir, but fir will *ſupport* twice as much as oak; the curvature of the fibres of oak appears to be the reaſon of the difference; yet oak has been known to ſupport, with ſafety, more than 2 tons for every ſquare inch. Stone will ſupport from 250 to 850 thouſand pounds on a foot ſquare, brick 300; and ſometimes they are practically made to ſupport one-ſixth as much. Stone is ſaid to be capable of bearing a much greater weight in that poſition with reſpect to the horizon in which it is found in the quarry, than in any other poſition.

192. Having now given, in addition to the theoretic propoſitions, the reſults of experiments made at different times by various perſons, it remains to exhibit a few examples of the practical application of the whole.

I. Let it be required to find what weight ſuſpended from the middle of an oak beam (ſupported at each end) will break it, the length of the beam being 8 feet, its end a ſquare, and each ſide ſix inches.

According to Mr. Banks's experiments a bar of oak an inch ſquare and a foot long lying on a prop at each end, will break with 660 lbs. on its middle, taking the loweſt number. And in all caſes where we make ſuch compariſon of ſtrength, *the* *breadth multiplied into the ſquare of the depth and divided by the product of the length and weight muſt be a conſtant quantity:* that is, taking b, d, l, and w, for the breadth, depth, length, and weight, uſed in the experiment, and B, D, L, W, thoſe propoſed in the example, we have $\frac{bd^2}{lw}=\frac{BD^2}{LW}$. Hence $W=\frac{BD^2lw}{bd^2L}$, which in the preſent inſtance gives $W=\frac{6\times6^2\times1\times660}{1\times1^2\times8}=$ 17820 lbs.

II. Let the length of the beam be as above, the breadth 3 inches, and the weight 17820, what muſt be the depth of the beam?

Here b, d, l, and w, alſo L, and W, are as in the preceding example. From the general equation we obtain $D=\sqrt{\frac{bd^2LW}{Blw}}$ $=\sqrt{\frac{1\times1\times8\times17820}{3\times1\times660}}=8\cdot45$ inches, nearly.

Cor. The latter beam though as ſtrong as the former, is but little more than $\frac{2}{3}$ of it in ſize.

III. Required the breadth of a bar of iron 8 feet long, and 4 inches in depth, to ſuſtain the ſame weight at its middle point?

Here b, d, l, L, W, as before, $w=2190$, $D=4$; and from the theorem $B=\frac{bd^2LW}{D^2lw}=\frac{1\times1\times8\times17820}{4^2\times1\times2190}=4\cdot07$ inches, nearly.

IV. Required the length of a piece of oak an inch ſquare ſo that when propped at both ends it may juſt break with its own weight?

Here we adopt the notation in art. 184. cor. 2. and have $l=1$, $w=\frac{2}{5}$ of a lb. $u=660$ lbs. Then $L=\sqrt{l\times\frac{l\times(w+2u)}{w}}$ $=57\cdot45$ feet, nearly.

V. Find the length of an iron bar an inch ſquare, that it may break with its own weight, when it is ſupported at both ends.

Here l as before, $u=2190$, $w=3$. Hence $L=\sqrt{l\times\frac{l\times(w+2u)}{w}}$ $=38\cdot223$ feet, nearly.

Cor. It might have been ſuppoſed this reſult ſhould exceed the preceding one: but it muſt be conſidered that while iron is only $4\frac{1}{2}$ times ſtronger than oak, it is about $7\frac{2}{5}$ times heavier.

VI. When a weight W is ſuſpended from E on the arm of a crane ABCDE (fig. 3. Pl. IX.), it is required to find the preſſure at the end D of the ſpur, and that at B againſt the upright poſt AC.

Here, by the nature of the lever $\frac{CE \times W}{CD}$ = the preſſure at D in the vertical direction DG: but the preſſure at D in the direction DB will be to the vertical preſſure as DB to DG; hence DG : DB : : $\frac{CE \times W}{CD}$: $\frac{DB \times EC \times W}{DG \times CD}$. And the horizontal preſſure againſt the upright poſt CA at B, will be to that in direction DB, as GB to DB; therefore DB : GB, or DB : CD : : $\frac{DB \times EC \times W}{DG \times CD}$: $\frac{EC \times W}{DG} = \frac{EC \times W}{BC}$.

Mr. Banks gives the following among other inſtances of the uſe of this: "Given EC = 12 feet, BC = 6, DC = 6.7, DB = 9, W = 4. Required the preſſure on the ſpur, and the horizontal preſſure againſt the upright. 1. $\frac{DB \times EC \times W}{DG \times DC} = 10.74$, the preſſure againſt the end of the ſpur in direction DB. The preſſure againſt the poſt is $\frac{EC \times W}{BC} = \frac{12 \times 4}{6} = 8$. In this example, let AC and CE be oak beams, each 10 inches ſquare, and the ſpur DB 6 inches ſquare. The ſtrength of EC is $\frac{1000}{10.6}$, or $94\frac{1}{3}$; which multiplied by 660 gives 31132 pounds, which ſuſpended at E would break the beam CE at D. The length of the upright AC is 12 feet, and has its ſtrength expreſſed by $\frac{1000}{12}$; which multiplied by 660, produces 55000 pounds, the weight which would break it at B. But $\frac{31132 \times 12}{6} = 62264$, the preſſure at B, being 7264 pounds more than the beam AC can ſupport. The ſtrength of the ſpur BD is $\frac{6 \times 6 \times 6}{9} = 24$, which multiplied by 2 gives 48 tons for the ſtrength, or 107520 pounds. But $\frac{DG \times EC \times W}{DG \times DC} = \frac{9 \times 12 \times 31132}{6 \times 6.7} = 83638$ pounds, which is 23882 pounds leſs than the force requiſite to break the ſpur. From the above it appears that the upright AC is the weakeſt part; but from the principles already explained, the ingenious mechanic will eaſily proportion the parts ſo as to be equally ſtrong."

VII. "Let it be required to make a crane of caſt iron to bear 4 cwt. but that it may be perfectly ſafe let it be calculated for 10 cwt. and let AC = CE = 3 feet, alſo BC = CD = $1\frac{1}{2}$ foot.

"Let the thickneſs of the iron be half an inch, and put z = depth of CE. Then as 1 : 2190 : : $\frac{z^2 \times \frac{1}{2}}{3}$: 1120, from which we find $z^2 = 3{\cdot}0685$, and $z = 1{\cdot}75$ inches. The preſſure upon the ſpur at D, in the direction DG = 1120 pounds; the length of the ſpur is 2·12 feet, and as DG (15) : DB (2·12) : : 1120 : 1583 for the preſſure in the direction DB. As a bar 1 inch ſquare, and 1 foot long, will bear 15 tons, or 33600 pounds (at the end), we ſay as 1 : 33600 : : $\frac{z^3}{2\cdot 12}$: 1583, from which we find z the ſide of the prop or ſpur = ·46385 of an inch. Next, for the upright, we have $\frac{CE \times W}{BC} = \frac{560 \times 3}{1\cdot 5} = 1120$ pounds, the preſſure againſt B, then as 1 : 2190 : : $\frac{x^2}{3 \times 2}$ (the ſquare of the breadth) : 1120 pounds, the ſame as CE, as they are of the ſame length, and the breadth will be the ſame, that is 1·75 inches."

Many other examples, particularly of the ſtrength of beams to work cylinders, and of caſt iron ſhafts or axles for mills, may be ſeen in Mr. Banks's little Treatiſe on the Power of Machines, &c.

CHAPTER VI.

On Cords, Arches, and Domes.

193. We have already ſpoken of the uſe of cords in their connection with the lever, the pulley, and the axis in peritrochio: but they are ſometimes uſed in a manner more complicated than what we have yet conſidered, and are by ſome authors regarded as a diſtinct machine under the name of *Funicular machine.* We ſhall, therefore, treat of them a little more particularly, and conſider them as perfectly flexible, without gravity and reduced to their axes, unleſs the contrary is ſtated. The *tenſion* of a cord is the force which acts at one end of a cord when the other is fixed, or it is equivalent to it: thus in the caſe of the equilibrium of powers applied to a phyſical point, if we regard that point as fixed, the tenſion of each cord is preciſely the force applied at each cord to move the point: but if the equilibrium does not obtain, as when, for example, a cord has two unequal powers acting at its extremities, the tenſion is the leaſt of the two forces, for the tenſion will obviouſly be the ſame as if the one of the extremities were fixed, and the leaſt of the two forces acted ſolely at the other end.

194. When three, four, or more powers, act at the extremities of different cords, all united at one node, the conditions of equilibrium will be the ſame as we have exhibited in Chap. II. between arts. 46. and 70. whether the cords are all ſituated in one plane, or ſeveral. But if two forces C, C′ (fig. 3. Pl. I.), acting at the extremities of a cord CPC′ paſſing through a ring at P faſtened to a cord OP, and retained by a power O, the conditions of equilibrium are theſe:

1. The line OP when produced muſt biſect the angle CPC′.
2. The forces C, C′ muſt be equal to each other. For, if the angles CPD, CPD, are unequal, the cord will ſlide along the ring; and this condition combined with that of the parallelogram of forces requires the equality of C, and C′. This being premiſed, we proceed to a few uſeful problems.

195. Prop. *Given the length* l *of a cord* EAH, *and the poſition of the points* E, H, *to which its ends are fixed, and ſuppoſe a given weight* P *hangs by a cord* AP, *the latter cord running freely along the former by a ring at the end* A (fig. 4. Pl. IX.), *to determine the*

poſition of the whole when in equilibrio, and the tenſion of the cord EAH *at its extremities.*

Draw through E the horizontal line EG, and through H the vertical FK. Since AL divides the angle EAH into two equal parts, if EA produced meet FK in K, we ſhall, becauſe of the parallels LA, FK, have $K = EAL = LAH = AHN$; therefore the triangle HAK is iſoſceles, and $KA = AH$; alſo $EK = EA + AK = EA + AH = l$. If, therefore, with A as a centre and radius $= l$, we cut the vertical FK in K, and on the middle point N of HK erect the perpendicular NA, it will interſect the line EK in A, the point where the ring will reſt in equilibrio.

The analytical ſolution of the remaining part of the problem is by no means difficult. Thus, put $EF = h$, the angle $EAH = 2a$, and let the tenſions of the cord at the points of ſuſpenſion E, H, be P', P''. Then, ſince P' muſt be equal P'', and $EAL = LAH$, we have (art. 41. cor.) $P = 2P' \cos. a$. In the right-angled triangle EFK, we have $\text{ſin. } EKF = \text{ſin. } a = \frac{EF}{KE} = \frac{h}{l}$. Employing this value in the equation $P = 2P' \cos. a$, there ariſes $P = 2P' \sqrt{\frac{l^2 - h^2}{l^2}}$, whence $P' = \frac{P}{2\sqrt{\frac{l^2-h^2}{l^2}}} = \frac{Pl}{2\sqrt{l^2 - h^2}}$, an equation which gives the tenſion of the cord at E or H.

COR. 1. The diſtances HA and EA may be eaſily found when required: for, ſince the relative poſitions of E and H are given, FH is known as well as EF; and becauſe $FK = KE . \cos. a$, it is known alſo: hence NK is known, being $\frac{FK - FH}{2}$, and we have, by ſim. tri. $KF : KE :: KN : KA\ (= AH) :: NF : AE$.

COR. 2. The locus of all the points A is an ellipſe whoſe foci are E, H, and tranſverſe axis $= l$.

COR. 3. When H falls in the horizontal line EG, it will be $EA = AH$.

196. PROP. *Given the weight* R *attached to the point* A *of a cord which paſſes over two fixed pulleys in given poſitions* B *and* C, *the given weights* P, Q, *hanging at the extremities of the cord, to determine the poſition of* AC *and* AB *when the whole is in equilibrio.* (fig. 5. Pl. IX.)

At a convenient diſtance from B and C, draw the vertical $d\,a$ to repreſent the weight R (fig. 6.), and form the triangle $d\,a\,e$ ſuch that the ſides $a\,e$ and $d\,e$, ſhall be to $d\,a$, as the weights P, and Q, reſpectively to R. By the point B draw BA parallel to $e\,a$, and by the point C draw CA parallel to $d\,e$, then will CAB be the poſition in which the cord will reſt in equilibrio. For, if we take any diſtance AD in the vertical AG to repreſent R, and complete the parallelogram AEDF, we ſee, by the ſimilitude of the triangles ADE, ade, that DE and DF ought to repreſent the tenſions of AC and AB, and that in this poſition they are in equilibrio with the weight R.

By means of the known ſides of the triangle ade, we may calculate the three angles: then, drawing the horizontal line BG, the angle GBC will be given, ſince the points B and C are; the angle GBA is known alſo, being the complement of GAB or dae; and the angle $\text{CBA} = \text{GBA} - \text{GBC}$, will be therefore known: thus, in the triangle CAB we know the ſide CB, and the angles CAB, CBA (and conſequently the third angle), from which we may compute, with eaſe, the ſides CA and AB.

Cor. 1. When $P = Q$, the triangle ade becomes iſoſceles, by which means the computation is much ſimplified.

Cor. 2. When beſides having $P = Q$, we have B and C in the horizontal line, the proceſs is ſtill farther ſimplified. In this caſe AC and AB will be equal to each other; the triangle ade will be iſoſceles, and eia, ſimilar to BGA. We ſhall alſo have $ei = \sqrt{ae^2 - ai^2} \propto \sqrt{P^2 - \frac{1}{4}R^2}$: ſo that if BG be put $= \frac{1}{2}d$ and $\text{GA} = x$, we ſhall expreſs $ai : ie :: \text{AG} : \text{GB}$ thus,

$$\tfrac{1}{2}R : \sqrt{P^2 - \tfrac{1}{4}R^2} :: x : \tfrac{1}{2}d, \text{ whence } x = \frac{\frac{1}{4}dR}{\sqrt{P^2 - \frac{1}{4}R^2}} = \frac{dR}{2\sqrt{4P^2 - R^2}}$$

197. Prop. *To determine the conditions of equilibrium in the funicular polygon when many forces are acting at different points of the cord, but in the ſame plane.*

Let $PNN'N''$ &c. (fig. 7. pl. IX.) be the polygon propoſed, being kept in equilibrio by the powers P', P'', ... P^n, acting in the directions PN, $P'N$, $P''N''$, &c. And call t, t', t'', &c. the reſpective tenſions of the parts of the cord PN, NN', $N'N''$ &c. Now, ſince the equilibrium obtains in the ſyſtem, it muſt neceſſarily have place in each portion of the polygon ſeparately. Hence P, P', and t', muſt be in equilibrio about the node N; and t'' muſt be the reſultant of the component forces P, P': the force which acts on the point N' in direction $N'N$, is therefore equivalent to the two forces P, P', acting ſimultaneouſly at N; and the node N'' is acted upon as though it were ſolicited by the four forces P, P', P'', t'', in directions reſpectively parallel to PN, PN', $P'N''$, $N'N''$. In like manner it may be ſhewn that the node N'' is kept in equilibrio in the ſame way as it would be, if ſubjected to the ſimultaneous action of the powers P, P', P'', P''', t''', in directions parallel to PN, $P'N$, $P''N'$, &c. And ſo on throughout. Hence it follows, that *when a funicular polygon is ſuſtained in equilibrio by any number of forces whatever, if we tranſport theſe powers parallel to their reſpective directions, ſo as all to exert their energies upon one point, it will be kept in equilibrio by their combined action.*

COR. 1. The reſult juſt obtained correſponds with that which was deduced in Chap. II. (art. 85.) with regard to forces acting at different points of ſolid bodies: conſequently the other theorems given there may, when required, be applied to the caſe before us.

COR. 2. If we neglect the two extreme forces P, P^n, and reaſon upon the others as in the propoſition, it is clear we may conceive all of them applied at the ſame point (provided they make reſpectively the ſame directions with any aſſumed axe AX), without at all altering the magnitude of their reſultant: and hence as that reſultant is deſtroyed by the two powers P, P^n, it muſt neceſſarily paſs through the point of concourſe O of the directions PN, $P^n N'''$.

COR. 3. If, therefore, a cord AEB (fig. 10. pl. IX.), fixed at two points A and B, has all its points ſolicited by any forces whatever in the ſame plane, it will aſſume a plane curvature: and the point of interſection O of two tangents will fall upon the direction of the reſultant of all the forces applied to the various points of the cord. And if we transfer theſe forces parallel to their directions, ſo as to apply them all at the point O, their reſultant being reſolved into two others acting according to the directions AO and OB, we ſhall thence obtain the effort exerted upon each of the fixed points.

COR. 4. The caſe of the laſt corol. applies obviouſly to gravity: for, on the one hand, this force exerciſes its action on all the points of the cord, and on the other, theſe efforts may be aſſimilated to the weight diſtributed throughout the length of the heavy cord. Hence, the curve thus formed, and known by the names of *funicular curve*, *chainette*, or *catenary*, is a plane curve.

COR. 5. In the catenary the total effort exerted on the fixed points A and B, is the whole weight of the cord: if therefore, at the point of concourſe O of the two tangents to the curve at A and B, a weight equal to that of the cord were ſuſtained by two threads AO, BO, void of gravity, the points A and B would be acted upon in the ſame manner as they are by the action of gravity upon the cord AEB; viz. the powers P, P′, neceſſary to retain either the heavy cord AEB, or the equal weight at O, would be the ſame in both caſes. As the reſultant is the weight of the cord, if we erect upon O an indefinite perpendicular OE, it will paſs through the centre of gravity (art. 106. IV.), and the forces exerted upon A and B will be proportional to the ſines of the angles BOE and AOE (art. 48.): thus, if W be the entire weight of the cord, we have

$$W : P : P' :: \text{ſin. } AOB : \text{ſin. } EOB : \text{ſin. } AOE.$$

COR. 6. The ſame will obtain, wherever the points A and B are found in the curve, ſince the ſtate of equilibrium allows us to conſider as fixed any two points in the curve. If, therefore, we conſider the point F as fixed inſtead of the point B, the portion AEF of the curve will not change its form: nor will there be any change in the tenſion P exerted at A.

Cor. 7. If ſeveral weights W, W′, W″, &c. (fig. 8.) hang upon a cord PC′CP′, the weight of which is inconſiderable with reſpect to either of thoſe weights, the preſſure upon any angle C of the funicular polygon, will be $\propto \frac{\text{sin.}\, d\,C\,e}{\text{sin.}\, e\,C\,c \times \text{sin.}\, c\,C\,o}$, the line C$c$ being a continuation of the vertical W″C. For, if a ſeries of parallelograms of forces be deſcribed at the angles of the polygon, where the forces C″ o'', e C, . C o, e' C′, &c. being equal and oppoſite, deſtroy each other, we ſhall have, force e'' C″ : C″ o'', or e C : : ſin. $e''\,c''$ C″ or ſin. c'' C″ e'' : ſin. P′ C″ c'' : : $\frac{1}{\text{sin.}\, P' C'' c''} : \frac{1}{\text{sin.}\, c'' C'' e''}$. Again, e C : C o or e' C′ : : ſin. c C o or ſin. e' C′ c' : ſin. c C e : : $\frac{1}{\text{sin.}\, c\,C\,e} : \frac{1}{\text{sin.}\, e'\,C'\,c'}$ or $\frac{1}{\text{sin.}\, c\,C\,o}$; and ſo on. Whence it appears in general that any force e C is as $\frac{1}{\text{sin.}\, c\,C\,e}$. Therefore, becauſe $C\,c = \frac{\text{sin.}\, c\,e\,C \times C\,e}{\text{sin.}\, C\,c\,e}$, we have $C\,c = \frac{\text{sin.}\, c\,e\,C}{\text{sin.}\, C\,c\,e} \times \frac{1}{\text{sin.}\, c\,C\,e} = \frac{\text{sin.}\, d\,C\,e}{\text{sin.}\, c\,C\,e \times \text{sin.}\, c\,C\,o}$.

Cor. 8. If the number of weights hanging from the cord be increaſed, and the diſtances on the cord of the points from which the weights hang be increaſed indefinitely, or if inſtead of the weights we conceive pieces of heavy cord, or of chains, to be hung from different points of the cord PCP′, as in fig. 9. our funicular polygon will then become a curve, being indeed a ſpecies of catenary. The angle e C d will then become the angle of contact formed by the tangent and curve, whoſe ſine is equal to the meaſure of the angle; and the angles c C e, c C o, become equal to the angles c C d, c C d', which are ſupplements to each other. Hence, becauſe the angle of contact is as the curvature of the arch, or reciprocally as the radius of curvature, the weight hanging at any point C, will be reciprocally as the radius of curvature at that point, and the ſquare of the ſine of the angle made by the curve (or its tangent) and the vertical.

Cor. 9. Laſtly, a heavy cord cannot by any force be ſtretched into a right line, except it be in a vertical poſition: for, the weight of the cord may be conſidered as a force applied at its centre of gravity; and then, the cord AEB (fig. 10.) being retained by the two forces P, P′, if W be its weight, we have

(cor. 5.) P : W : : ſin. EOB : ſin. AOB;

where, it is obvious, the more the cord is ſtretched, the greater the angle AOB becomes, and the more nearly the angle EOB approaches to a right angle: ſo that the cord can only be ſtretched ſtraight horizontally, when this analogy obtains, P : W : : 1 : 0, that is, when P is infinite. Thus, however ſmall the weight is, it will cauſe the cord to be curved, unleſs it be placed vertically: which is, indeed, a circumſtance experienced daily.

198. PROP. *To inveſtigate the moſt uſeful equations of the ſimple catenarian curve.*

Let AMC (fig. 11. pl. IX.) be a cord uniformly thick throughout, inextenſible, and perfectly flexible, fixed at the two given points A, C, and ſolicited by gravity in every point. Taking the origin of the rectangular co-ordinates in A, and eſtimating the x, x, &c. horizontally, and y, y, &c. vertically, we ſhall have for any point M in the curve $AP=x$, $PM=y$, $AM=z$. The tenſions exerted at A and M according to the tangents AD, MD, give, weight of AM : tenſion at A : : ſin. ADM, or ſin. ADF : ſin. IDM (cor. 5. art. 197.); where, ſince the tenſion at A is conſtant, and the weight of the arc AM is proportional to its length, the firſt ratio in the analogy, is $=z : a$, a being a conſtant quantity, yet undetermined. Moreover, ſin. IDF $=\frac{\dot{x}}{\dot{z}}$, cos. IDF $=\frac{\dot{y}}{\dot{z}}$, and denoting the angle IAD by s, we have ſin. ADF $=$ ſin. (IDF $-$ IDA) $=\frac{\dot{x}\,\text{sin.}\,s-\dot{y}\,\text{cos.}\,s}{\dot{z}}$: therefore $\dot{z} : a : : \dot{x}\,\text{ſin.}\,s-\dot{y}\,\text{cos.}\,s : \dot{x}$, whence we deduce the fluxional equation.

$$(1.)\quad z\dot{x}=a\dot{x}\,\text{ſin.}\,s-a\dot{y}\,\text{cos.}\,s.$$

In order to eliminate one of theſe variables, we take $\dot{x}$ as conſtant, and thence obtain $-\dot{z}\,\dot{x}=b\,\ddot{y}$, in which $b=a\,\text{cos.}\,s$. Hence, ſubſtituting $\sqrt{\dot{x}^2+\dot{y}^2}$ for $\dot{z}$, we get $-\dot{x}\,\dot{y}=\frac{b\,\dot{y}\,\ddot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}$: here $\dot{x}$ being conſtant, the fluent of the ſecond member is evidently $b\sqrt{\dot{x}^2+\dot{y}^2}$; that of the firſt member is $-y\dot{x}$, or its correct fluent $c\dot{x}-y\dot{x}$; we have therefore $(c-y)\,\dot{x}=b\sqrt{\dot{x}^2+\dot{y}^2}$. But $\frac{\dot{y}}{\dot{x}}$ being the tangent of the angle formed by the axe AX and the tangent to the curve at every point, if we make $y=0$, we thence have $\frac{\dot{y}}{\dot{x}} - \text{tang.}\, s = \frac{\sin.\, s}{\cos.\, s}$: this condition gives $c = a$, and therefore

$$(2.) \quad . \quad . \quad . \quad . \quad \frac{\dot{y}}{\dot{x}} = \frac{\sqrt{(a-y)^2 - b^2}}{\pm b}.$$

Now, ſince $\frac{\dot{y}}{\dot{x}}$ expreſſes the angle formed by AX and the curve or its tangent at any point, in order to find the point where the curve is parallel to AX, or the loweſt point of the curve where the tangent is horizontal, we muſt make $\frac{\dot{y}}{\dot{x}} = 0$, and this will give $y = a - b = \text{HB}$, and from the equa. (1.) gives $z = a$ ſin. s. To determine the relations of the curve to BH as an axis, call BL, X, and LN, Y; then, comparing theſe new co-ordinates with the former, we have $y' = a - b - \text{X}$, and $\text{X} = \text{AH} - \text{Y}$. Subſtituting theſe values in equa. (2.) there ariſes

$$\frac{\dot{\text{X}}}{\dot{\text{Y}}} = \frac{\sqrt{(\text{X}+b)^2 - b^2}}{\pm b}$$: conſequently $\text{Y} = \frac{\pm b\dot{\text{X}}}{\sqrt{(\text{X}+b)^2 - b^2}} = \frac{\pm b\dot{\text{X}}}{\sqrt{\text{X}^2 + 2b\text{X}}}$. Taking the fluents of this equation, we have

$$\text{Y} = \pm b \text{ Hyp. log. } (b + \text{X}\sqrt{\text{X}^2 + 2b\text{X}} + \text{Hyp. log. } d).$$

Now ſuppoſing $\text{Y} = 0$, it will give $\text{X} = 0$, and Hyp. log. $d = \mp b$ Hyp. log. b. whence we obtain the following correct fluent:

$$(3.) \, . \, . \, \text{Y} = \pm b \text{ Hyp. log.} \left(\frac{b + \text{X} + \sqrt{\text{X}^2 + 2b\text{X}}}{b}\right) = \pm b. \text{ Hyp. log.} \frac{\text{Z} + \sqrt{b^2 + \text{Z}^2}}{b}.$$

This equation ſhews that the vertical axe paſſing through the point where the tangent is horizontal is a diameter, ſince to each value of X there are two equal and oppoſite values of Y.

Other algebraic though not finite expreſſions for the ordinate Y might be eaſily obtained, but there will no particular advantage ariſe from purſuing that part of the inveſtigation, we therefore merely exhibit one more equation which flows naturally from the one juſt given; it is this, $\text{Y} = \pm b \text{ Hyp. log.} \frac{\text{Z}+\text{X}}{\text{Z}-\text{X}}$.

The fluxional equation $\dot{\text{Y}} = \frac{b\dot{\text{X}}}{\sqrt{\text{X}^2 + 2b\text{X}}}$, gives $\sqrt{\dot{\text{X}}^2 + \dot{\text{Y}}^2} = \frac{(\text{X}+b)\dot{\text{X}}}{\sqrt{2b\text{X} + \text{X}^2}}$, the fluent of which is $\text{BN} = \sqrt{2b\text{X} + \text{XX}} = \text{Z}$;

from this equation we obtain

$$(4.) \quad . \; . \; . \; . \quad X = -b + \sqrt{b^2 + Z^2}.$$

When we would conſtruct the catenary we muſt conſider as known the points of ſuſpenſion A and C, and the length AMC of the cord; it will be neceſſary to determine the poſition and length of BH, that is, to find AH, and BH or KH and BH, AK and CK being given, and to know the value of the conſtant quantity b. In order to this we apply the equation (3.) to the point A, ſubſtituting for X, BH $-y$, and for Y, AK $-$ AH $-x$, and then introduce this value of X into equation $Z = \sqrt{2bX + X^2}$. Again, in the equation of the curve we put firſt $x = 0$, $y = 0$, then $x = AK$, and $y = CK$; and in the equation $Z = \sqrt{2bX + X^2}$, we put $X = AK$, and $Z = AMC$: by theſe means we obtain three equations compriſing only known quantities and the three unknown ones b, AH, BH, whence each of the unknown quantities may be determined.

If the points A, C′, are in the horizontal line AX, we have AH = HC, and arc AB = arc BC′ = $\frac{1}{2}$ ABC′. In this caſe we retain the origin at the point B, and making $Z = \frac{1}{2}$ ABC′, in the equation $X = -b \pm \sqrt{b^2 + Z^2}$, the value of X thence reſulting being ſubſtituted in the equation of the curve, in which alſo Y will $= \frac{1}{2}$ AC, the equation itſelf will then contain only one unknown quantity b, the determination of which will become a matter of comparative facility: though it muſt ſtill be by a method of approximation, becauſe the curve is tranſcendental.

COR. 1. Inſtead of gravity, if any other force exerts its energy in like manner, acting equally upon every point of the flexible line, the ſame curve will be produced: thus, for example, if the wind be ſuppoſed equable, and blow according to right lines parallel to a given line; the cord thus inflated by the wind would aſſume the ſhape of the catenary: for, ſince all things obtain with reſpect to this other force, as we have ſuppoſed in relation to gravity, the reſults muſt obviouſly be ſimilar.

COR. 2. If the forces acting upon every point of the curve, inſtead of being exerted in parallel lines, were always exerciſed in directions perpendicular to the curve, the forces exerciſed at every point would be inverſely as the radii of curvature at thoſe points: if, therefore, the forces were equal to each other throughout, the radius of curvature would then be conſtant, and the curve would be a circular arc.

N. B. In the Phil. Trans. No. 231, or New Abridgment, Vol. IV. there is a curious paper on the catenary by Dr. David Gregory, in which is given an elegant conſtruction of the curve by means of the parabola and equiliteral hyperbola, beſides the quadrature and cubature of various parts of the catenarian ſpace, and its ſolids of rotation.

II. *On Arches and Piers.*

199. The conſtruction of arches is one of the moſt important and difficult branches of Architecture, particularly when conſidered in relation to the erection of bridges over broad and rapid rivers: it commonly impoſes the double taſk of blending the handſome forms and the decorations of the ordinary architecture, with the firmneſs and durability which ought always to be found in works deſtined not merely for the accommodation of the public but in many caſes for its ſafety. The theory of arches when purſued to the extent its importance and utility demands, would of itſelf fill a volume; but all that we ſhall attempt here will be a conciſe view of the leading particulars.

Defs. By an arch we obviouſly mean an artful diſpoſition of ſeveral ſtones, or bricks, or other ſuitable materials, the under part of which is in a bow-like form, their weight producing a mutual preſſure, ſo that they not only ſupport each other but may be made to carry the moſt enormous weights. Other particulars relating to an arch, the defining of which is neceſſary here, may be ſoon learnt by turning to fig. 1. pl. X. Thus AS′ESB is the ponderating arch; A or B, the *ſpring* of the arch; D its *crown*; AB its *ſpan*; CD its *height* or verſed ſine, or riſe; ADB the *intrados*, or the lower ſurface of the arch (often called the arch); S′ES, the *extrados*, being in bridges the ſuperior ſurface, or the roadway; F′, F, the *flanks* or *hances*; the ſpaces above theſe are called the *ſpandrels*; the portions of wedges which lie in a courſe contiguous to the intrados are called *vouſſoirs*, or arch-ſtones; that which is at D is called the *keyſtone*; the walls or maſſes PQST, P′Q′S′T′, built to ſupport the arches, and from which they ſpring as their baſes, are either called *piers* or *abutments*; *piers* when they ſtand between two neighbouring arches, *abutments* when they ſupport the arches which are contiguous to the ſhore: the part of the pier from which an arch ſprings is called the *impoſt*; the curve formed by the upper ſides of the vouſſoirs the *archivolt*; and the lines FS, F′S′, about the flanks, in which a break is moſt likely to take place, are called *joints of fracture*. The other terms we ſhall uſe will need no explanation.

When we reflect upon the immenſe quantity of heavy materials ſuſpended in the air in a large arch, and compare it with the ſmall coheſion which the firmeſt cement can give to ſuch an edifice, we ſhall be convinced that its parts are not kept together by the force of the cement; the ſtability of the whole is the reſult of the juſt balance and equilibration of all its parts. The principles of this equilibration we ſhall now exhibit: premiſing, that they are founded upon the hypotheſis of the ponderating matter preſſing upon the vouſſoirs in the vertical direction, and that we here pay no regard to any ſmall preſſure in other directions which may be occaſioned by filling up the ſpandrels with rubbles, &c.

200. If we conceive a vault or arch in equilibrio to be compoſed of a ſeries of very ſmall hard ſpheres, of poliſhed ſurfaces, touching one another, and the centres joined by right lines, being in fact an inverted ſimple catenary; and that the ſpheres are ſo connected that they cannot yield to any other impulſion than that of gravity, it is plain the equilibrium will not be diſturbed by ſuch inverſion of the curve; no part of the curve will be thruſt outward or inward by other parts, but the whole will be ſupported if the feet are firmly fixed: for, ſince the ſituation of the points of the catenary is the ſame, and their reſpective inclinations to the horizon, whether in the original or inverted poſition, the curve being in both inſtances in a vertical plane, and the action of gravity the ſame, the relative conditions are the ſame, and therefore the arch will keep its figure unchanged as well in one ſituation as the other.

Thus alſo when arches of other figures are ſupported, it is, as Dr. Gregory juſtly obſerved, "*becauſe in their thickneſs ſome catenaria is included:*" as, for example, if a quantity of heavy but flexible materials fixed only at the points P, P′ (fig. 12. pl. IX.), when left to the ſole action of gravity, ſhould arrange itſelf into the ſhape of the dotted lines PADBP′ in a vertical plane; then, if the whole were completely inverted, the points P, P′, being ſtill fixed, the equilibrium would be retained notwithſtanding the inverſion; the arch with the matter above it would ſtand as in the upper part of the ſame figure; and the correſponding parts of the hanging and of the ſtanding arch would be ſimilarly ſituated with regard to the forces from which the equilibrium is derived. Here, too, there would be no point of contrary flexure in the *intrados;* and this we ſuppoſe throughout.

201. Hence what was deduced at Cor. 8. art. 197. may be transferred to the preſent caſe; that is, *the weight preſſing upon any point* C *will be reciprocally as the radius of curvature at that point, and the ſquare of the ſine of the angle made by the curve, or its tangent, and the vertical.*

If therefore a weight, as a wall or maſs of maſonry, be incumbent on the intrados PVCP′ (fig. 12.) in a vertical plane, and all the parts be kept in equilibrio, then *the height* CI *on any point* C, *is reciprocally as the radius of curvature, and cube of the ſine of the angle in which the vertical cuts the curve in that point, or* *reciprocally as the radius of curvature, and directly as the cube of the ſecant of the curve's inclination to the horizon.*

For the weight on the indefinitely ſmall portion of the curve Ce being inverſely as $R \times \text{ſin.}^2\, d\,CI$, or $R \times \text{ſin.}^2\, d'CI$ (R being the radius of curvature at C), and the weight of the column $CeiI$, as $IC \times wr = IC \times Ce \times \text{ſin.}\, d'CI = IC \times \text{ſin.}\, d'CI$; becauſe Ce is given: therefore $IC \times \text{ſin.}\, d\,CI \propto \frac{1}{R \times \text{sin.}^2 d\,CI}$, and $IC \propto \frac{1}{R \times \text{sin.}^3 d\,CI} \propto \frac{\text{sec.}^3 k\,C\,d'}{R}$.

202. This method of deducing the fundamental theorem of equilibrated arches from inverted catenaries either ſimple or complex, ſuggeſts an eaſy popular mode of aſcertaining the ſhape of a duly balanced arch, when the ſpan, height, and ſhape of the roadway are given; a method which, we believe, was firſt practiſed by M. de la Hire. Let it be propoſed, for example, to determine the form of an arch which ſhall have the ſpan PP′, and the height EV (fig. 12.) and which ſhall have a roadway in the poſition ADB above it. Let the figure PADBP′ be inverted as repreſented by the dotted lines: then let a chain of ſuitable length and uniform thickneſs be hung at the points P, P′, ſo that when it aſſumes the ſhape of the ſimple catenary it ſhall hang a little below the loweſt point V. Divide PP′ into 20, 30, or more equal points, as in the points 1, 2, 3, &c. and let vertical lines through the points 1, 2, 3, &c. interſect the curve PVP′ in the correſponding points 1, 2, 3, &c. Then take pieces of another uniform chain whoſe links are tolerably numerous, and hang on at the points 1, 2, 3, &c. of the chain ſuſpended from P, P′. Cut or trim theſe pieces of chain till their lower ends all coincide with the inverted roadway ADB: the greater lengths which are hung on in the vicinity of P, P′, will draw down that part of the chain, and ſo cauſe the part which hung below V, to riſe to its aſſigned poſition. This proceſs will give us an arch of equilibration, but, as Dr. Robiſon remarks, ſome farther modifications may be neceſſary to make it exactly ſuit the ſpecified purpoſe. It is a balanced arch for a bridge, which is ſo loaded that the weight of the arch ſtones is to the weight of the matter with which the haunches and crown are loaded, as the weight of the chain PVP′, to the ſum of the weights of all the little bits of chain, very nearly. But this proportion is not known beforehand; we muſt, therefore, proceed thus: Adapt to the curve produced in this way a thickneſs of the vouſſoirs as great as may be thought ſufficient to enſure ſtability; then compute the weight of the vouſſoirs and the weight of the gravel, ſtones, &c. which fill up the haunches, &c. to the roadway. If the ratio of theſe two weights be the ſame with that

of the correſponding weights of chain, we may reſt ſatisfied with the curve now found: but if different it may eaſily be calculated how much muſt be added to, or taken from, each piece of chain, in order to make theſe ratios equal; and thus ſhall we at length aſcertain with ſufficient accuracy the ſhape of the curve required, which may be readily transferred from the vertical wall, or framing, near which the chains were ſuſpended, to any other ſituation.

We might now go on to deduce a few uſeful propoſitions in this theory, from the fundamental formula juſt exhibited: but as it may be more ſatisfactory to ſome perſons to have this theorem deduced independently of the catenary, we ſhall demonſtrate the ſame thing, from the conſideration that the vouſſoirs are ſo many fruſtrums of wedges whoſe ſides are perpendicular to the intrados, each of which being urged by its incumbent weight, endeavours by this force to ſplit the arch.

203. PROP. *The force of a vouſſoir depending on the magnitude of the angle formed by its ſides, the impelling force, and the reſiſtance to be overcome, is on the firſt account directly as the radius of curvature of the arch at that point, on the ſecond as the ſquare of the ſine of the angle included between the tangent of the curve at the given point and the vertical paſſing through that point, and on the third, as the ſine of the ſame angle.*

1. Let EABF, *e a b f*, be two ſimilar concentric curves (fig. 2. pl. X.), and AB, *a b*, two vouſſoirs ſimilarly ſituated, whoſe ſides perpendicular to the curve converge to the centre C. The forces of theſe vouſſoirs conſidered as portions of wedges, are inverſely as the ſines of the half vertical angles (164. 165. cor. 1.), or, becauſe each wedge occupies an equal portion of its reſpective arch, directly as the radii of curvature.

2dly, Let H *h* be the invariable breadth of the vouſſoirs on the arch *e a b f*, G *g* H *h* the incumbent weight, which, ſince GH is ſuppoſed given, is as the breadth *h k*, or as the ſine of the angle *h* H *k*: by the reſolution of the force *g* H into two *h* H, HK, the latter is the force impelling the vouſſoir to ſplit the arch, which, ſince *g* H is given, varies as the ſine of H *g* K, or *h* H *k*: wherefore, the force impelling the vouſſoir is as the ſquare of the ſine of *h* H *k*.

3dly, The wedge impelled in a direction perpendicular to the curve endeavours to ſplit the arch, and therefore move one ſegment about the fulcrum *e*, the other about the fulcrum *f*. Hence the force of the vouſſoir acting on the levers H *f*, H *e*, being as either of the perpendiculars *f* P, *e* Q, is as the ſine of the angle *f* CP or *h* H *k*.

We have ſuppoſed the centre of curvature of the arches at the points A, a, hH, to be at C: but this is merely to prevent the figure from being too complex, and makes no alteration in the nature of the demonſtration.

Cor. Hence, if the height of the wall incumbent on any point H of the intrados is inverſely as the cube of the ſine of hHk into radius of curvature at that point, or directly as cube of the ſecant of the angle formed by hH and the horizon, and inverſely as the radius of curvature, all the vouſſoirs will endeavour to ſplit the arch with equal forces, and will be in perfect equilibrium with each other.

204. Prop. *Given the intrados, to find the correſponding extrados.*

In fig. 12. pl. IX. where PVP′ is the intrados and ADB the required extrados, V and D being the reſpective vertices, put A$=$DV, $x=k$V the abſciſſa, $y=k$C the ordinate to any point C, $a=$CI the requiſite altitude at that point, and $z=$the arc VC: then, by the laſt prop. and cor. $a \propto \frac{\text{sec.}^3\, d'\text{C}k}{\text{R}}$, while, by ſimilar triangles, we have $\dot{y} : \dot{z} :: 1$ (radius) $: \frac{\dot{z}}{\dot{y}} =$ ſec. d'Ck; and therefore $a \propto \dot{z}^3 \div \text{R}\dot{y}^3$. But in every curve whoſe ordinate is referred to an axis the radius of curvature is $\text{R} = \frac{\dot{z}^3}{\dot{y}\ddot{x} - \dot{x}\ddot{y}}$ (Simpſon's Fluxions, vol. I. p. 74.); whence $a \propto \frac{\dot{y}\ddot{x} - \dot{x}\ddot{y}}{\dot{y}^3}$, or $a = \text{C} \times \frac{\dot{y}\ddot{x} - \dot{x}\ddot{y}}{\dot{y}^3}$, where C is a conſtant quantity, the value of which may be determined by taking the expreſſion for the given line A at the vertex of the curve.

Example. To find the extrados of a circular arch. Let O be the centre of the circle, and the ſeveral lines in fig. 3. pl. X. being repreſented as above, while PO$=$OV$=r$; then, if we make $\dot{y}$ invariable, we have $x = r - \sqrt{r^2 - y^2}$, $\dot{x} = \frac{y\dot{y}}{\sqrt{r^2 - y^2}}$, and $\ddot{x} = \frac{r^2 y^2}{(r^2 - y^2{}^{\frac{3}{2}}}$. Hence CI, or a, or $\text{C} \times \frac{\dot{y}\ddot{x} - \dot{x}\ddot{y}}{\dot{y}^3}$, becomes $\text{C} \times \frac{\ddot{x}}{\dot{y}^2} = \frac{\text{C}r^2}{(r^2 - y^2)^{\frac{3}{2}}}$: and this when $y = 0$, or at the vertex where A$=a$, gives $\text{A} = \frac{\text{C}}{r}$, or $\text{C} = \text{A}r$: and conſequently the general value of CI, or $a = \frac{\text{C}r^2}{(r^2 - y^2)^{\frac{3}{2}}}$, becomes $a = \text{A} \times \frac{r^3}{(r^2 - y^2)^{\frac{3}{2}}} = \text{DV} \times \frac{\text{OV}^3}{\text{O}k^3} =$ A × cube ſec. of elevation.

Hence this ſimple calculus: to the logarithm of A add thrice the log. ſec. of elevation, the ſum rejecting ^{3} is the log. of *a*.

Hence alſo flows a neat conſtruction: draw the vertical CS interſecting the horizontal diameter in S: join C and the centre O, and on OC let fall from S the perpendicular ST: draw TS parallel to PO, and join ZO: make $Cu = DV$, and parallel to ZO draw ux; then is $Cx = CI$. For, by the ſimilar triangles we have

$$CO : CS :: CS : CT = \frac{CS^2}{CO}$$

$$CS : CT :: CT : CZ = \frac{CS^4}{CO^2 . CS} = \frac{CS^3}{CO^2}$$

and $DV : CI :: Cu : Cx :: CZ : CO :: \frac{CS^3}{CO^2} : CO :: CS^3 : CO^3 ::$ $Ok^3 : OV^3$.

The curve DIA runs up to an infinite height above the ſpring of the arch, if it be ſemicircular; and this muſt evidently be the caſe with every curve that ſprings at right angles to the horizontal line. But for a moderate diſtance on each ſide the vertex the extrados will aſſume a ſhape that may anſwer tolerably well for a roadway.

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205. When the intrados is elliptic the extrados is of the ſame kind as for the circle: but when the longer axe is horizontal the ellipſis may take a ſtraight line at top more ſafely than the circle, becauſe the extrados runs out farther from the vertex, before it takes its rapid riſe. In the cycloidal arch of equilibration, the extrados reſembles that of the circle and flat ellipſe; approaching, however, more nearly to a right line about the vertex, and extending farther from it before it bends upwards. If the intrados is parabolic the extrados is an equal and ſimilar curve, at any given diſtance from the intrados: but in the hyperbola the extrados approaches continually nearer to the intrados. Hence, in ſome caſes it may be quite convenient to have arches ſegments near the vertex of very large parabolas. The reader may find the inveſtigations of theſe particulars in Dr. Hutton's Principles of Bridges: we merely glance at them here.

206. PROP. If AEDIB (fig. 4. pl. X.) *be the extrados and* PVP′ *the intrados of an equilibrated arch, and if any number of vertical lines* EF, DV, IC, *&c. be drawn from the one curve to the other, and theſe lines be divided in a given ratio in the points* a, e, d, i, b, *then the curve drawn through theſe points of diviſion will alſo be a proper extrados, the maſs contained between it and the intrados being duly balanced, as well as that compriſed between* AEDIB *and* PVCP′.

For, ſince the whole is kept in equilibrio by the vertical preſſures of the ſuperincumbent maſs on the intrados, the points F, V, C, &c. are ſuſtained in equilibrio by the preſſures of the parts EF, DV, IC, &c. preſſing upon them: if, therefore, theſe lines be divided in *e*, *d*, *i*, &c. ſo that EF : *e* F : : DV : *d* V : : IC : *i* C, &c. all in the ſame conſtant ratio, then *a e d i b* being conſidered as the extrados the arch will ſtill be in equilibrium; becauſe the load on the intrados being every where leſſened in a conſtant ratio, its tendency to break the arch will be every where in a conſtant ratio to its preceding tendency to cauſe a rupture, and the equilibrium can be no more deſtroyed in the one caſe than in the other. And the ſame kind of reaſoning would apply, if *a e d i b* were above AEDIB, or if the weight and denſity of the materials between the extrados and intrados ſhould be changed *throughout* in any conſtant ratio.

Cor. Hence we may in many caſes give the extrados a pretty regular and practicable form, by diminiſhing the thickneſs over the crown: and hence appears one great advantage of iron as a material for bridges, ſince its requiſite thickneſs at the crown is vaſtly leſs than that of ſtone.

207. Prop. *Having given the extrados of an equilibrated arch, to find the intrados.*

Let ADB (fig. 5. pl. X.) be the extrados propoſed to which the intrados PVP′ is to be adapted; OV being the common axis of both curves: from *c* and C, correſponding points equidiſtant from the axis, draw the ordinates *c b*, CH. Put DV (the thickneſs of the arch at the crown) $=a$, D $b=\dot{X}$, VH $=x$, the equal ordinates $c\,b$, $=$ CH $=y$, and the arch VC $=z$. Then, by the general form (204.) C$c \propto \frac{\dot{y}\ddot{x}-\dot{x}\ddot{y}}{\dot{y}^3} = \frac{\dot{y}\ddot{x}-\dot{x}\ddot{y}}{\dot{y}^3}\times$ C, where C is a conſtant quantity found by taking the actual value of C *c* in V, the vertex of the curve. But it is manifeſt that C $c=$ DV $+$ VH $-$ D $b=a+x-\dot{X}$: conſequently $a+x-\dot{X}=\text{C}\times\frac{\dot{y}\ddot{x}-\dot{x}\ddot{y}}{\dot{y}^3}=\frac{\text{C}}{\dot{y}}\times$ flux. of $\frac{\dot{x}}{\dot{y}}$. If, then, we ſubſtitute the true value of $\dot{X}$ in terms of y (which is given becauſe the form of the extrados is known), the equations thence reſulting will contain only x and y with their firſt and ſecond fluxions, and known quantities; and from this the real relation of x and y muſt be ſtruck out by ſuch means as ſeem moſt naturally to apply to any propoſed inſtance.

This, however, in many caſes will be a matter of conſiderable difficulty: we ſhall here, therefore, ſolely trace the proceſs in the moſt uſeful inſtance, which, happily, admits of a comparatively ſimple inveſtigation. We advert to the caſe in which *the extrados is a ſtraight horizontal line*, which we ſhall now conſider.

Retaining the ſame notation, we have $\mathrm{D}\,h = \dot{\mathrm{X}} = 0$, and conſequently $a + x = \frac{\mathrm{C}}{\dot{y}} \times$ flux. of $\frac{\dot{x}}{\dot{y}}$. Aſſume $\dot{y} = \frac{\dot{x}}{u}$, whence $u = \frac{\dot{x}}{\dot{y}}$, and $\frac{\mathrm{C}}{y} \times$ flux. of $\frac{\dot{x}}{\dot{y}} = \frac{\mathrm{C}\,u\,\dot{u}}{\dot{x}}$, that is, $a + x = \frac{\mathrm{C}\,u\,\dot{u}}{\dot{x}}$; and of courſe $a\dot{x} + x\dot{x} = \mathrm{C}\,u\,\dot{u}$: taking the fluents of this we have $x^2 + 2\,a\,x = \mathrm{C}\,u^2$, and $u = \sqrt{\frac{x^2 + 2\,a\,x}{\mathrm{C}}}$. But becauſe $\dot{y} = \frac{\dot{x}}{u}$ it is alſo $= \dot{x} \div \sqrt{\frac{x^2 + 2\,a\,x}{\mathrm{C}}} = \dot{x}\sqrt{\frac{\mathrm{C}}{x^2 + 2\,a\,x}}$. The fluent of this expreſſion is $y = \sqrt{\mathrm{C}} \times$ hyp. log. $(2\,x^2 + 2\,a\,x + 2\sqrt{x^2 + 2\,a\,x})$. Now at the vertex where $x = 0$, we have $y = \sqrt{\mathrm{C}} \times$ hyp. log. $2\,a$, ſo that the corrected fluent is $y = \sqrt{\mathrm{C}} \times$ hyp. log. $\frac{x + a + \sqrt{x^2 + 2\,a\,x}}{a}$.

We have yet to aſcertain the conſtant quantity C, in order to which we may proceed thus: when H arrives at O we have $x = v\,o = h$, and $y = \mathrm{O\,P} = s$; ſubſtituting theſe in the laſt equation, it becomes $h = \sqrt{\mathrm{C}} \times$ hyp. log. $\frac{s + a + \sqrt{s\,s + 2\,a\,s}}{a}$, and conſequently $\sqrt{\mathrm{C}} = h \div$ hyp. log. $s + a + \frac{1}{a}\sqrt{s^2 + 2\,a\,s}$. Hence, then, we at length obtain this general value of y, that is, $y = h \times$ hyp. log. $\frac{a + x + \sqrt{2\,a\,x + x^2}}{a} \div$ hyp. log. $\frac{a + s + \sqrt{2\,a\,s + s^2}}{a}$.

Comparing theſe equations with equa. (3) art. 198. it will be ſeen that when $\sqrt{\mathrm{C}} = a$, the intrados PVP′ is the catenary, although the extrados is a horizontal right line: but this will require an immenſe thickneſs at the crown; for if $x = 40$, and $y = 50$, we ſhould have $a = 36{\cdot}88$, which is more than $\frac{1}{3}$ the ſpan of the arch, and more than $\frac{9}{10}$ of its height.

As an example of the uſe of the preceding formulæ, we ſubjoin a table calculated by Dr. Hutton, for an arch whoſe ſpan is 100, height 40, and thickneſs at the crown 6: which will anſwer for any other arch whoſe ſpan, height, and thickneſs, are related to each other in like manner, by changing all the values of D H and C H in a conſtant ratio.

HC	DH	HC	DH	HC	DH
0	6·000	21	10·381	36	21·774
2	6·035	22	10·858	37	22·948
4	6·144	23	11·368	38	24·190
6	6·324	24	11·911	39	25·505
8	6·580	25	12·489	40	26·894
10	6·914	26	13·106	41	28·364
12	7·330	27	13·761	42	29·919
13	7·571	28	14·457	43	31·563
14	7·834	29	15·196	44	33·299
15	8·120	30	15·980	45	35·135
16	8·430	31	16·811	46	37·075
17	8·766	32	17·693	47	39·126
18	9·168	33	18·627	48	41·293
19	9·517	34	19·617	49	43·581
20	9·934	35	20·665	50	46·000

208. Prop. *To determine the magnitude of the piers, or abutments, that they may ſuſtain the arch in equilibrium, independently of other arches.*

In order to give a ſolution to this problem we muſt aſſume ſome particulars as having been determined by adequate experiments and admeaſurement: for we do not conſider the piers as priſms ſtanding upon their baſes and reſiſting the preſſure of the arches, though upon ſuch an hypotheſis it would be eaſy to lay down rules for the determination of the centres of gravity both of the arch and the piers; but, ſince the ſtones, &c. of the wall above the vouſſoirs are bonded in with thoſe of the pier, the pier will by theſe means become augmented, and the weight of the arch diminiſhed. We, therefore, regard the piers as extending to the *joints of fracture* (art. 199.), and that portion of the arch which is compriſed between thoſe joints as a ponderating body reſting in a ſtate of equilibrium upon thoſe joints as upon two inclined planes. Let, then, FS, F′S′ (fig. 1. pl. X.), be the joints of fracture, G the centre of gravity of the pier Q T S F B P, *g* that of the half arch D E F S, and let *g o* the perpendicular from *g* upon F S be produced till it meets the horizontal line Q′ Q in I: draw GH perpendicular to Q Q′ and Q *h* perpendicular to *g* I; and let the maſs of the ſemi-arch D E S F be repreſented by A, that of the pier by P, and the force of gravity by *g*: the weight of the former will then be *g* A, and of the latter *g* P. The magnitude of the pier is generally computed on the ſuppoſition that the preſſure of the arch has a tendency to make the pier turn upon Q as a centre or fulcrum; and this hypotheſis is often conſiſtent with fact: but when the height CD is ſmall compared with the ſpan, the weight of the arch has a ſtrong tendency to make the pier ſlide along the horizontal line PI; we ſhall, therefore, ſtate the conditions of equilibrium on this ſuppoſition alſo. Firſt, ſuppoſing the pier ſolely capable of turning upon Q as a centre of rotation: then will the caſe be the ſame as if the body D E S F whoſe weight is g A, by preſſing upon the face E S, tended to move the maſs F S T Q P upon the fulcrum Q. But (art. 156.) the weight g A is to its preſſure upon F S, as ſine of angle included between E D and F S, to ſine of angle E R Q; that is, ſin. I : rad. : : g A : $\frac{g\text{A}}{\text{sin. I.}}$ = preſſure of half arch upon the joint of fracture. Now g being the centre of gravity of the half arch, the preſſure it occaſions is exerted in the direction g I: and G being the centre of gravity of the pier, the force reſulting from its weight acts in the vertical direction G H; therefore in the caſe of equilibrium, we muſt, by the nature of the lever, have, preſſure on S F × Qh = weight of pier × QH, that is, $\frac{g\text{A}}{\text{sin. I}} \times \text{Q}h = g\text{P} \times \text{QH}$, whence we readily obtain

$$\text{(I.)} \quad \ldots\ldots \quad \frac{\text{A}}{\text{P}} = \frac{\text{QH}}{\text{Q}h} \times \text{ſin. I.}$$

This equation compriſes the conditions of the equilibrium of rotation (art. 19.) about the point Q; and we may find by its means any one of the five quantities it contains, when the other four are given.

In the ſecond caſe, in which we ſuppoſe the pier may ſlide along in the horizontal direction, let f be a force which is exerted horizontally in oppoſition to the motion of tranſlation; then fP acting in the direction Ik muſt counterbalance $\frac{g\text{A}}{\text{sin. I.}}$ acting along h I. Here hI being to Ik as radius to coſ. I, we ſhall have, rad. : coſ. I : : $\frac{g\text{A}}{\text{sin. I}}$: fP; whence $g\text{A} \times \frac{\text{sin. I}}{\text{cos. I.}} = f\text{P}$, and for an equation including the conditions of the equilibrium of tranſlation we have

$$\text{(II.)} \quad \ldots\ldots \quad \frac{\text{A}}{\text{P}} = \frac{f}{g} \times \frac{\text{sin. I.}}{\text{cos. I.}} = \frac{f}{g} \times \text{tan. I.}$$

As to the poſition of the joints of rupture, and of the centres of gravity of the ſemi-arch and pier, they may in moſt caſes be determined with tolerable accuracy, thus: having drawn on paſteboard the arch and propoſed pier, upon a pretty large ſcale, and deſcribed the vouſſoirs of the arch, of the intended thickneſs, draw from the middle of the key vouſſoir a tangent to the intrados, and produce it till it again meets the middle of a vouſſoir, as at F, from which point draw F S perpendicular to the intrados; it will be nearly the poſition of a joint of fracture. Next, cut the paſteboard through at the ſeveral outlines, and find by ſome of the methods deſcribed in art. 106, the centres of gravity of the two parts D E S F, S T Q P F. With regard to the ratio of A to P, it may always be found pretty nearly, either by weighing or meaſuring the pieces of paſteboard which repreſent them; and the diſtances Q H, Q h, and angle I, will be aſcertained by the conſtruction. If, when theſe values of A, P, &c. are introduced into the equations, the firſt members are leſs than the ſecond, the piers will be large enough to enſure the equilibrium: if otherwiſe, ſome of theſe particulars muſt be changed until that takes place.

This mode of conſidering the ſubject ſuggeſts, that to diminiſh the thruſt of the arch, or increaſe the ſtability of the pier, the commencement of the flanks ought to be loaded; and that the thickneſs of the vouſſoirs near the key ought to be leſſened conſiderably: in ſhort, to make the arch, inſtead of having a uniform thickneſs throughout its whole extent, to be very thick at its origin, and at the key to be no thicker than is neceſſary to reſiſt the preſſure of the flanks: for by ſuch a procedure a part of the force which tends to move the pier is thrown upon that which reſiſts being overturned, and the latter will gain a great advantage in point of ſtability.

III. *Of Domes.*

209. Def. A *dome* or *cupola* is a roof of a ſpherical, ſpheroidal, or conoidal form, reſembling a bell, or an inverted cup. It is a ſpecies of arch, or vaulting, the erection of which, like the former, is a ſcientific art depending upon the principles of equilibrium.

The moſt ample account of the theory of domes we have yet met with is given by Dr. Robiſon, in the Supplement to the Encyclopædia Britannica; an abſtract of which will be here given, chiefly in the doctor's own words.

Prop. *To determine the thickneſs of a dome vaulting when the curve is given, or the curve when the thickneſs is given.*

"Let B b A (fig. 7. pl. X.) be the curve which produces the dome by revolving round the vertical axis A D. We ſhall ſuppoſe this curve to be drawn through the middle of all the arch-ſtones, and that the courſing or horizontal joints are every where perpendicular to the curve. We ſhall ſuppoſe (as is always the caſe) that the thickneſs K L, H I, &c. of the arch-ſtones is very ſmall, in compariſon with the dimenſions of the arch. If we conſider any portion H A *b* of the dome, it is plain that it preſſes on the courſe, of which H L is an arch-ſtone, in a direction *b* C perpendicular to the joint H I, or in the direction of the next ſuperior element β *b* of the curve. As we proceed downwards, courſe after courſe, we ſee plainly that this direction muſt change, becauſe the weight of each courſe is ſuperadded to that of the portion above it, to complete the preſſure on the courſe below. Through B draw the vertical line B C G, meeting β *b*, produced in C. We may take *b c* to expreſs the preſſure of all that is above it, propagated in this direction to the joint K L. We may alſo ſuppoſe the weight of the courſe H L united in *b*, and acting on the vertical. Let it be repreſented by *b* F. If we form the parallelogram *b* F G C, the diagonal *b* G will repreſent the direction and intenſity of the whole preſſure on the joint K L. Thus it appears that this preſſure is continually changing its direction, and that the line, which will always coincide with it, muſt be a curve concave downward. If this be preciſely the curve of the dome, it will be an equilibrated vaulting; but ſo far from being the ſtrongeſt form, it is the weakeſt, and it is the limit to an infinity of others, which are all ſtronger than it. This will appear evident, if we ſuppoſe that *b* G does not coincide with the curve A *b* B, but paſſes without it. As we ſuppoſe the arch-ſtones to be exceedingly thin from inſide to outſide, it is plain that this dome cannot ſtand, and that the weight of the upper part will preſs it down, and ſpring the vaulting outwards at the joint K L. But let us ſuppoſe, on the other hand, that *b* G falls within the curvilineal element *b* B. This evidently tends to puſh the arch-ſtone inward, toward the axis, and would cauſe it to ſlide in, ſince the joints are ſuppoſed perfectly ſmooth and ſlipping. But ſince this takes place equally in every ſtone of this courſe, they muſt all abut on each other in the vertical joints, ſqueezing them firmly together. Therefore, reſolving the thruſt *b* G into two, one of which is perpendicular to the joint K L, and the other parallel to it, we ſee that this laſt thruſt is withſtood by the vertical joints all around, and there remains only the thruſt in the direction of the curve. Such a dome muſt therefore be firmer than an equilibrated dome, and cannot be ſo eaſily broken by overloading the upper part. When the curve is concave upwards, as in the lower part of the figure, the line *b* C always falls below *b* B, and the point C below B. When the curve is concave downwards, as in the upper part of the figure, *b* C paſſes above, or without *b* B. The curvature may be ſo abrupt, that even $b'G'$ ſhall paſs without $'bB'$, and the point G' is above B'. It is alſo evident that the force which thus binds the ſtones of a horizontal courſe together, by puſhing them towards the axis, will be greater in flat domes than in thoſe that are more convex; that it will be ſtill greater in a cone; and greater ſtill in a curve whoſe convexity is turned inwards: for in this laſt caſe the line bG will deviate moſt remarkably from the curve. Such a dome will ſtand (having poliſhed joints) if the curve ſprings from the baſe with any elevation, however ſmall; nay, ſince the friction of two pieces of ſtone is not leſs than half of their mutual preſſure, ſuch a dome will ſtand, although the tangent to the curve at the bottom ſhould be horizontal, provided that the horizontal thruſt be double the weight of the dome, which may eaſily be the caſe if it do not riſe high.

"Thus we ſee that the ſtability of a dome depends on very different principles from that of a common arch, and is in general much greater. It differs alſo in another very important circumſtance, viz. that it may be open in the middle: for the uppermoſt courſe, by tending equally in every part to ſlide in toward the axis, preſſes all together in the vertical joints, and acts on the next courſe like the key-ſtone of a common arch. Therefore an arch of equilibration, which is the weakeſt of all, may be open in the middle, and carry at top another building, ſuch as a lantern, if its weight do not exceed that of the circular ſegment of the dome that is omitted. A greater load than this would indeed break the dome, by cauſing it to ſpring up in ſome of the lower courſes; but this load may be increaſed if the curve is flatter than the curve of equilibration: and any load whatever, which will not cruſh the ſtones to powder, may be ſet on a truncate cone, or on a dome formed by a curve that is convex toward the axis; provided always that the foundation be effectually prevented from flying out, either by a hoop, or by a ſufficient maſs of ſolid pier on which it is ſet."

"We have ſeen that if bG, the thruſt compounded of the thruſt bC, exerted by all the courſes above $HILK$, and if the force bF, or the weight of that courſe, be every where coincident with bB, the element of the curve, we ſhall have an equilibrated dome; if it falls within it, we have a dome which will bear a greater load; and if it falls without it, the dome will break at the joint. We muſt endeavour to get analytical expreſſions of theſe conditions. Therefore draw the ordinates $b\delta b''$, BDB'', CdC''. Let the tangents at b and b'' meet the axis in M, and make MO, MP, each equal to bc, and complete the parallelogram $MONP$, and draw OQ perpendicular to the axis, and produce bF, cutting the ordinates in E

and *e*. It is plain that M N is to M O as the weight of the arch H A *b* to the thruſt *b c* which it exerts on the joint K L (this thruſt being propagated through the courſe of H I L K); and that M Q, or its equal *b e*, or δd, may repreſent the weight of the half A H.

" Let A D be called x, and D B be called y. Then $be = \dot{x}$, and $eC = \dot{y}$ (becauſe *b c* is in the direction of the element βb). It is alſo plain, that if we make $\dot{y}$ conſtant, B C is the ſecond fluxion of x, or $BC = \ddot{x}$, and *b e* and B E may be conſidered as equal, and taken indiſcriminately for $\dot{x}$. We have alſo $bC = \sqrt{\dot{x}^2 + \dot{y}^2}$. Let d be the depth or thickneſs H I of the arch-ſtones. Then $d\sqrt{\dot{x}^2 + \dot{y}^2}$ will repreſent the trapezium H L; and ſince the circumference of each courſe increaſes in the proportion of the radius y, $dy\sqrt{\dot{x}^2 + \dot{y}^2}$ will expreſs the whole courſe. If $\int$ be taken to repreſent the ſum or aggregate of the quantities annexed to it, the formula will be analogous to the fluent of a fluxion, and $\int dy\sqrt{\dot{x}^2 + \dot{y}^2}$ will repreſent the whole maſs, and alſo the weight of the vaulting, down to the joint H I. Therefore we have this proportion $\int dy\sqrt{\dot{x}^2 + \dot{y}^2} : dy\sqrt{\dot{x}^2 + \dot{y}^2} = be : bF, = be : CG, = \delta d : CG, = \dot{x} : CG$.

Therefore $CG = \frac{dy\dot{x}\sqrt{\dot{x}^2 + \dot{y}^2}}{\int dy\sqrt{\dot{x}^2 + \dot{y}^2}}$.

" If the curvature of the dome be preciſely ſuch as puts it in equilibrium, but without any mutual preſſure in the vertical joints, this value of O G muſt be equal to C B, or to $\ddot{x}$, the point G coinciding with B. This condition will be expreſſed by the equation $\frac{dy\dot{x}\sqrt{\dot{x}^2 + \dot{y}^2}}{\int dy\sqrt{\dot{x}^2 + \dot{y}^2}} = \ddot{x}$, or, more conveniently, by $\frac{dy\sqrt{\dot{x}^2 + \dot{y}^2}}{\int dy\sqrt{\dot{x}^2 + \dot{y}^2}} = \frac{\ddot{x}}{\dot{x}}$. But this form gives only a tottering equilibrium, independent of the friction of the joints and the coheſion of the cement. An equilibrium, accompanied by ſome firm ſtability, produced by the mutual preſſure of the vertical joints, may be expreſſed by the formula $\frac{dy\sqrt{\dot{x}^2 + \dot{y}^2}}{\int dy\sqrt{\dot{x}^2 + \dot{y}^2}} > \frac{\ddot{x}}{\dot{x}}$, or by $\frac{dy\sqrt{\dot{x}^2 + \dot{y}^2}}{\int dy\sqrt{\dot{x}^2 + \dot{y}^2}} = \frac{\ddot{x}}{\dot{x}} + \frac{\dot{t}}{t}$, where

t is ſome variable poſitive quantity, which increaſes when x increaſes. This laſt equation will alſo expreſs the equilibrated dome, if t be a conſtant quantity, becauſe in this caſe $\frac{\dot{t}}{t}$ is $= 0$.

"Since a firm ſtability requires that $\frac{dy\,\dot{x}\sqrt{\dot{x}^2+\dot{y}^2}}{\int dy\sqrt{\dot{x}^2=\dot{y}^2}}$ ſhall be greater than $\ddot{x}$, and C G muſt be greater than C B: Hence we learn, that figures of too great curvature, whoſe ſides deſcend too rapidly, are improper. Alſo, ſince ſtability requires that we have $\frac{dy\,\dot{x}\sqrt{\dot{x}^2+\dot{y}^2}}{\ddot{x}}$ greater than $\int dy\sqrt{\dot{x}^2+\dot{y}^2}$, we learn that the upper part of the dome muſt not be made very heavy. This, by diminiſhing the proportion of b F to b C, diminiſhes the angle $c\,b$ G, and may ſet the point G above B, which will infallibly ſpring the dome in that place. We ſee here alſo, that the algebraic analyſis expreſſes that peculiarity of dome-vaulting, that the weight of the upper part may even be ſuppreſſed.

"The fluent of the equation $\frac{dy\sqrt{\dot{x}^2+\dot{y}^2}}{\int dy\sqrt{\dot{x}^2+\dot{y}^2}} = \frac{\ddot{x}}{\dot{x}}+\frac{\dot{t}}{t}$ is moſt eaſily found: It is $\mathrm{L}\int dy\sqrt{\dot{x}^2+\dot{y}^2} = \mathrm{L}\,\dot{x}+\mathrm{L}\,t$, where L is the hyperbolic logarithm of the quantity annexed to it. If we conſider $\dot{y}$ as conſtant, and correct the fluent ſo as to make it nothing at the vertex, it may be expreſſed thus, $\mathrm{L}\int dy\sqrt{\dot{x}^2+\dot{y}^2}-\mathrm{L}\,a=\mathrm{L}\,\dot{x}-\mathrm{L}\,\dot{y}+\mathrm{L}\,t$. This gives us $\mathrm{L}\int\frac{dy\sqrt{\dot{x}^2+\dot{y}^2}}{a}=\mathrm{L}\,\frac{\dot{x}}{\dot{y}}t$, and therefore $\int\frac{dy\sqrt{\dot{x}^2+\dot{y}^2}}{a}=t\,\frac{\dot{x}}{\dot{y}}$.

"This laſt equation will eaſily give us the depth of vaulting, or thickneſs d of the arch, when the curve is given. For its fluxion is $\frac{dy\sqrt{\dot{x}^2+\dot{y}^2}}{a}=\frac{\dot{t}\,\dot{x}+t\,\ddot{x}}{\dot{y}}$, and $d=\frac{a\,\dot{t}\,\dot{x}+a\,t\,\ddot{x}}{y\,\dot{y}\sqrt{\dot{x}^2+\dot{y}^2}}$, which is all expreſſed in known quantities;

for we may put in place of t any power or function of x or of y, and thus convert the expreſſion into another, which will ſtill be applicable to all ſorts of curves.

" Inſtead of the ſecond member $\frac{\ddot{x}}{\dot{x}} + \frac{\dot{t}}{t}$ we might employ $\frac{p\,\ddot{x}}{\dot{x}}$, where p is ſome number greater than unity. This will evidently give a dome having ſtability; becauſe the original formula $\frac{d\,y\,\ddot{x}\sqrt{\dot{x}^2+\dot{y}^2}}{\int d\,y\sqrt{\dot{x}^2+\dot{y}^2}}$ will then be greater than $\ddot{x}$. This will give $d = \frac{p\,a\,\dot{x}^{p-1}\,\ddot{x}}{y\,\dot{y}^{p}\sqrt{\dot{x}^2+\dot{y}^2}}$. Each of theſe forms has its advantages when applied to particular caſes. Each of them alſo gives $d = \frac{a\,\ddot{x}}{y\,\dot{y}\sqrt{\dot{x}^2+\dot{y}^2}}$ when the curvature is ſuch as is in preciſe equilibrium. And, laſtly, if d be conſtant, that is, if the vaulting be of uniform thickneſs, we obtain the form of the curve, becauſe then the relation of $\ddot{x}$ to $\dot{x}$ and to $\dot{y}$ is given.

" The chief uſe of this analyſis is to diſcover what curves are improper for domes, or what portions of given curves may be employed with ſafety. Domes are generally built for ornament; and we ſee that there is great room for indulging our fancy in the choice. All curves which are concave outwards will give domes of great firmneſs: they are alſo beautiful. The Gothic dome, whoſe outline is an undulated curve, may be made abundantly firm, eſpecially if the upper part be convex and the lower concave outwards.

" The chief difficulty in the caſe of this analyſis ariſes from the neceſſity of expreſſing the weight of the incumbent part, or $\int d\,y\sqrt{x^2+y^2}$. This requires the meaſurement of the conoidal ſurface, which, in moſt caſes, can be had only by approximation by means of infinite ſerieſes. We cannot expect that the generality of practical builders are familiar with this branch of mathematics, and therefore will not engage in it here; but content ourſelves with giving ſuch inſtances as can be underſtood by ſuch as have that moderate mathematical knowledge which every man ſhould poſſeſs who takes the name of engineer.

" The ſurface of any circular portion of a ſphere is very eaſily had, being equal to the circle deſcribed with a radius equal to the chord of half the arch. This radius is evidently = $\sqrt{x^2 + y^2}$.

“ In order to diſcover what portion of a hemiſphere may be employed (for it is evident we cannot employ the whole) when the thickneſs of the vaulting is uniform, we may recur to the equation or formula $\frac{d y \dot{x} \sqrt{\dot{x}^2 - \dot{y}^2}}{\ddot{x}} = \int d y \sqrt{x^2 + y^2}$. Let a be the radius of the hemiſphere. We have $\dot{x} = \frac{a y \dot{y}}{\sqrt{a^2 - y^2}}$, and $\ddot{x} = \frac{a^2 \dot{y}^2}{\overline{a^2 - y^2}\rceil^{\frac{3}{2}}}$. Subſtituting theſe values in the formula, we obtain the equation $y^2 \sqrt{a^2 - y^2} = \int \frac{a^2 y \dot{y}}{\sqrt{a^2 - y^2}}$. We eaſily obtain the fluent of the ſecond member $= a^3 - a^2 \sqrt{a^2 - y^2}$, and $y = a \sqrt{-\frac{1}{2} + \sqrt{\frac{5}{4}}}$. Therefore if the radius of the ſphere be 1, the half breadth of the dome muſt not exceed $\sqrt{-\frac{1}{2} \times \sqrt{\frac{5}{4}}}$, or 0·786, and the height will be ·618. The arch from the vertex is about 51° 49′. Much more of the hemiſphere cannot ſtand, even though aided by the cement, and by the friction of the courſing joints. This laſt circumſtance, by giving connection to the upper parts, cauſes the whole to preſs more vertically on the courſe below, and thus diminiſhes the outward thruſt; but it at the ſame time diminiſhes the mutual abutment of the vertical joints, which is a great cauſe of firmneſs in the vaulting. A Gothic dome, of which the upper part is a portion of a ſphere not exceeding 45° from the vertex, and the lower part is concave outwards, will be very ſtrong, and not ungraceful.

“ Perſuaded that what has been ſaid on the ſubject convinces the reader that a vaulting perfectly equilibrated throughout is by no means the beſt form, provided that the baſe is ſecured from ſeparating, we think it unneceſſary to give the inveſtigation of that form, which has a conſiderable intricacy, and ſhall merely give its dimenſions. The thickneſs is ſuppoſed uniform. The numbers in the firſt column of the table expreſs the portion of the axis counted from the vertex, and thoſe of the ſecond column are the length of the ordinates.

AD	DB	AD	DB	AD	DB
0·4	100	610·4	1080	2990	1560
3·4	200	744	1140	3442	1600
11·4	300	904	1200	3972	1640
26·6	400	1100	1260	4432	1670
52·4	500	1336	1320	4952	1700
91·4	600	1522	1360	5336	1720
146·8	700	1738	1400	5756	1740
223·4	800	1984	1440	6214	1760
326·6	900	2270	1480	6714	1780
475·4	1000	2602	1520	7260	1800

"The curve formed according to theſe dimenſions will not appear very graceful, becauſe there is an abrupt change in its curvature at a ſmall diſtance from its vertex; if, however, the middle be occupied by a lantern of equal or of ſmaller weight than the part whoſe place it ſupplies, the whole will be elegant, and free from this defect."

Profeſſor Robiſon concludes with obſerving, that "The connexion of the parts ariſing from cement and from friction has a great effect on dome vaulting. In the ſame way as in common arches and cylindrical vaulting, it enables an overload on one place to break the dome in a diſtant place. But the reſiſtance to this effect is much greater in dome vaulting, becauſe it operates all round the overloaded part. Hence it happens that domes are much leſs ſhattered by partial violence, ſuch as the falling of a bomb, or the like. Large holes may be broken in them without much affecting the reſt; but, on the other hand, it greatly diminiſhes the ſtrength which ſhould be derived from the mutual preſſure in the vertical joints. Friction prevents the ſliding in of the arch-ſtones which produces this mutual preſſure in the vertical joints, except in the very higheſt courſes, and even there it greatly diminiſhes it. Theſe cauſes make a great change in the form, which gives the greateſt ſtrength; and as their laws of action are but very imperfectly underſtood as yet, it is perhaps impoſſible, in the preſent ſtate of our knowledge, to determine this form with tolerable preciſion. We ſee plainly, however, that it allows a greater deviation from the beſt form than the other kind of vaulting; and domes may be made to riſe perpendicular to the horizon at the baſe, although of no great thickneſs; a thing which muſt not be attempted in a plane arch. The immenſe addition of ſtrength which may be derived from hooping largely compenſates for all defects; and there is hardly any bounds to the extent to which a very thin dome-vaulting may be carried, when it is hooped or framed in the direction of the horizontal courſes. The roof of the Halle du Bled at Paris is but a foot thick, and its diameter is more than 200, yet it appears to have abundant ſtrength."

END OF BOOK THE FIRST.

DYNAMICS.

BOOK II.

Introductory Definitions and Remarks.

210. DYNAMICS is that branch of Mechanics which has for its object the action of forces on ſolid bodies, when the reſult of that action is motion; and in which, ſince all motion occupies ſome portion of time, we introduce time into our inveſtigations. This department of ſcience preſents a wide and varied field of diſcuſſion, and when viewed in its full extent it exhibits many queſtions of conſiderable difficulty: we ſhall not attempt to give the whole of theſe, as ſuch a procedure would draw us far beyond the limits which muſt be aſſigned to this part of the work; but ſhall ſelect thoſe chiefly which appear neceſſary as a preparation to the knowledge of the powers and effects of machinery.

211. The ſum of the material particles of which a body is compoſed, is what we denote by the word *Maſs*. This maſs depends on the volume of the body and that which we call *Denſity*. We have already obſerved (art. 10.) that denſity is directly as the quantity of matter, and inverſely as the magnitude of the body: but it will not be improper to deduce conciſely the general theorem which compriſes this relation. To this end it muſt be conſidered that as all bodies are penetrated with a great number of void ſpaces or pores, their quantity of matter is not proportional to their volume; but under the ſame volume there will be more or leſs matter as the particles are nearer or further aſunder; and we ſay that a body has a greater or leſs Denſity, according as there ſubſiſts a greater or leſs proximity between its moleculæ. Thus we ſay a body is more *denſe* than another when in an equal volume the former contains more matter than the latter: we ſay, on the contrary, that it is leſs *denſe* or more *rare* (for *denſity* and *rarity* are reciprocal qualities) when in an equal volume it compriſes leſs matter. The denſity ſerves, therefore, to judge of the number of material particles when the volume is known: thus, we may regard the denſity as repreſenting the number of equal moleculæ in a determinate

volume; as when, for example, we ſay that gold is 19 times denſer than water, we wiſh to be underſtood that gold contains 19 times the number of particles that water does in the ſame ſpace.

Since we repreſent the denſity as expreſſing the number of moleculæ in a determinate volume which we aſſume as the *unit of magnitude*; it is obvious that to obtain the maſs, or the total number of moleculæ, of any body of which the magnitude is known, we muſt take the rectangle of the denſity and magnitude. Thus, if we repreſent generally the body or maſs by B, its volume or magnitude by M, and its denſity by D, we ſhall have $B=MD$: whence, it will be eaſy to compare the maſſes, the magnitudes, and the denſities of bodies.

In ſimilar bodies the maſſes are as the denſities and cubes of the diameters, or depths, or lengths, or of any like linear dimenſions. Hence, if L denote the lineal dimenſion, we readily deduce theſe general proportions:

$$B \propto MD \propto DL^3.$$

$$M \propto \frac{B}{D} \propto L^3.$$

$$D \propto \frac{B}{M} \propto \frac{B}{L^3}.$$

$$L \propto \sqrt[3]{\frac{B}{D}} \propto \sqrt[3]{M}.$$

212. *Force* according to our definition (art. 17.) is that which cauſes a change in the ſtate of a body; or, it is that which either moves or tends to move a body: forces, as we there obſerved, are no further known to us than by their effects; it is only therefore by the effect any force produces that we can meaſure it. Now the effect of a force is to give to every material particle of a body a certain velocity: if, therefore, all the parts of a body receive the ſame velocity as we ſuppoſe here, the effect of the moving cauſe has for its meaſure the product of the velocity into the number of moleculæ moved, or the product of the velocity and maſs: *A force therefore is proportional to the velocity which it can impreſs on a known maſs, and that maſs conjointly.*

Def. *Momentum*, or *Quantity of Motion*, is the rectangle of the maſs of a body and its velocity.

Conſequently, *forces are meaſured by the quantities of motion they are capable of producing.*

Thus, if F denote the motive or moving force, B the body moved, and V the velocity imparted to it, we have $F \propto BV$. From this we deduce $V \propto \frac{F}{B}$, and $B \propto \frac{F}{V}$: therefore, 1. *The* *velocity of a body in motion, is as the moving force directly, and the maſs inverſely.* 2dly, *The body or maſs is as the moving force directly, and the velocity inverſely.*

If now, two bodies be repreſented by B, and b, the forces by which they are moved by F, and f, and the velocities imparted to them by V and v, we have $F \propto BV$, and $f \propto bv$. Let B be ſuppoſed equal to b, then will $F \propto V$, and $f \propto v$: hence, *when the maſſes are equal the moving forces are as the velocities.* Again, ſuppoſing $V = v$, we have $F \propto B$ and $f \propto b$: therefore, *when the velocities are equal the moving forces are as the maſſes.* Laſtly, making $F = f$, we have $\frac{1}{V} \propto B$, and $\frac{1}{v} \propto b$: conſequently, *when the moving forces are equal, the velocities are in the inverſe ratio of the maſſes.*

213. The ſeveral deductions in the preceding article depend, manifeſtly, upon our having aſſumed a juſt meaſure of moving force? we ſay, that forces are meaſured by the quantities of motion they are capable of producing, and that theſe quantities of motion are proportional to the products of the maſſes and their velocities. It follows, therefore, that when theſe products are equal the quantities of motion, or the equivalent forces, are equal: but equal forces acting in oppoſite directions produce an equilibrium (art. 35.); ſo that if bodies which move with equal quantities of motion (according to our definition) in oppoſite directions, are in equilibrio after they meet, ſuch a fact being proved will at the ſame time evince the truth of the foregoing deductions. In order then to ſhew the truth of this principle, we extract a paragraph from M. Laplace's *Expoſition du Syſtême du Monde.* L. III. Ch. 3.

"The moſt ſimple caſe of the equilibrium of many bodies, is that of two phyſical points which rencounter with velocities equal and directly contrary. Their mutual impenetrability, that property of matter in virtue of which two bodies cannot occupy the ſame place at the ſame inſtant, evidently annihilates their velocities, and reduces them to the ſtate of reſt. But, *if two bodies of different maſſes ſtrike each other with oppoſite velocities, what is the relation of the velocities to the maſſes in the caſe of equilibrium?* To reſolve this problem, imagine a ſyſtem of contiguous phyſical points, ranged on the ſame right line, and animated with a common velocity in the direction of that line; conceive, in like manner, a ſecond ſyſtem of contiguous material points, diſpoſed on the ſame right line, and animated with a common velocity and contrary to the preceding, in ſuch a manner that the two ſyſtems ſtriking mutually ſhall be in equilibrio. It is obvious that if the firſt ſyſtem were compoſed of only one material point, each point of the ſecond ſyſtem would extinguiſh in the point ſtruck a part of its velocity equal to the velocity of the ſyſtem; the velocity of the point ſtruck ought therefore in the caſe of equilibrium to be equal to the product of the velocity of the ſecond ſyſtem by the number of its points, and we may ſubſtitute for the firſt ſyſtem one point ſingly incited to a velocity equal to this product. We may likewiſe ſubſtitute for the ſecond ſyſtem a material point animated with a velocity equal to the product of the firſt ſyſtem by the number of its points. Thus, inſtead of theſe two ſyſtems we have two points which bring themſelves to an equilibrium by ſtriking with contrary velocities, of which the one will be the product of the velocity of the firſt ſyſtem by the number of its points, and the other will be the product of the velocity of the points of the ſecond ſyſtem, by their number: theſe products, therefore, muſt be equal in the caſe of equilibrium."—"This product of the maſs by the velocity we name *quantity of motion*; it is this alſo which we mean by the *force of bodies*: for the equilibrium of two bodies or of two ſyſtems of material points which ſtrike each other in contrary directions, the quantities of motion, or the forces oppoſed, muſt be equal, and of conſequence the velocities muſt be reciprocals to the maſſes."

214. We muſt not omit obſerving, that about a century ago there was a warm diſpute among the mathematicians, in order to determine whether we ought to conſider the force of bodies in motion proportional to the velocity or to the ſquare of the velocity: it is eaſy, from what has preceded, to reduce this queſtion to a ſimple enunciation which will remove all difficulty. The word *force* denoting any cauſe of which the nature is unknown, and of which the effects are the only things we can meaſure, it is evident that by the term *meaſure of force*, we can only mean that of its effects: now the effects may be conſidered under different aſpects, each comporting with a ſpecies of meaſure particular and conformable to its nature. If we conſider the effect of the force as conſiſting in the deſtruction of a certain ſum of obſtacles or of quantities of motion, this ſum muſt, as above ſhewn, be expreſſed by $BV + bv +$ &c. that is, it is proportional to the velocity ſimply. But if we conſider the effect of the force, not with relation to the ſum of the obſtacles, but to their number, this number, as will appear further on, will be repreſented by $\frac{1}{2} B v^2$, and will be proportional to the ſquare of the velocity when all the obſtacles are equal. Hence this famous queſtion is only a diſpute about words: and as ſuch we need dwell no longer upon it.

CHAPTER I.

On Motion, uniform and variable.

215. The motion of a phyſical point, or of a body, is *uniform* (art. 15.) when it moves always in the ſame manner, or when it paſſes over equal ſpaces in any equal intervals of time whatever. This motion, the moſt ſimple of all, and the moſt eaſy to conceive, probably exiſts in no part of nature, but is only a pure abſtraction of the mind: it is, notwithſtanding, important to conſider it, becauſe were it not for obſtructions motions in general would be uniform (art. 21.), and becauſe it conduces greatly to the analyſis of all other motions.

That affection of motion which we call *velocity* being meaſured by the ſpace uniformly deſcribed in a given time (art. 15.) is in fact a meaſure of motion itſelf, and is that which characteriſes each ſpecies of uniform motion: it is cuſtomary in mathematical diſcuſſions to fix upon a ſmall period of time, a ſecond, for example, as a unit, and to call *the velocity of a moving body at any inſtant the ſpace which the body would deſcribe uniformly during a unit of time.*

Hence it follows, that in the uniform motion of a body *the ſpaces run over are proportional to the times employed.* For if it deſcribe V feet in one ſecond, it will deſcribe 2V feet in two ſeconds, 3V feet in three ſeconds, and TV feet in T ſeconds, T being any number whole or fractional. This being granted, we may now ſtate a propoſition from which the whole doctrine of uniform motions will readily flow.

216. Prop. *When bodies have different uniform motions, the ſpaces deſcribed are proportional to the times and velocities jointly.*

Let V and v be the velocities of the two bodies B and b, T and t the times of their motions, S and s the ſpaces deſcribed, likewiſe let s' be the ſpace deſcribed by b in the time T:

$$\text{Then } S : s' :: V : v$$
$$s' : s :: T : t$$
$$\text{And, comp. } S : s :: TV : tv. \quad \text{That is } S \propto TV.$$

Cor. 1. The velocity is as the ſpace divided by the time: for the preceding expreſſion gives $V \propto \frac{S}{T}$. Or, ſince the ſame will hold in any correſponding indefinitely minute portions of the ſpace and time, we ſhall have $V = \frac{\dot{S}}{\dot{T}}$.

Cor. 2. *The velocities of two bodies moving uniformly are directly as the ſpaces and inverſely as the times:* for we have $V : v :: \frac{S}{T} : \frac{s}{t}$.

Cor. 3. *In equal times the velocities are proportional to the ſpaces run over:* for $T = t$, gives $V : v :: S : s$.

Cor. 4. *If the velocities are equal, the ſpaces paſſed over are proportional to the times:* for $V = v$, gives $St = sT$, or $S : s :: T : t$.

Cor. 5. *If the ſpaces paſſed over are equal, the velocities are reciprocally as the times:* for when $S = s$, we have $V : v :: \frac{1}{T} : \frac{1}{t} :: t : T$.

Cor. 6. Since the areas of rectangles are in the ratio compounded of the ratios of their ſides, if the baſes repreſent the velocities of two motions, and altitudes the times, the areas will repreſent the ſpaces deſcribed.

Cor. 7. Since it has been ſhewn that the forces which give motion to bodies are proportional to their quantities of motion, and theſe to the rectangles of the maſſes and velocities (art. 212.), that is, $F \propto Q \propto BV$; we may, by combining this with the preſent propoſition, have the following formulæ of relation of the ſix quantities, force F, momentum or quantity of motion Q, maſſes or quantity of matter B, time T, ſpace S, and velocity V; the forces being ſuppoſed inſtantaneous or impulſive, and the motions uniform:

$$F \propto Q \propto BV \propto \frac{BS}{T}.$$

$$Q \propto F \propto BV \propto \frac{BS}{T}.$$

$$B \propto \frac{F}{V} \propto \frac{Q}{V} \propto \frac{FT}{S} \propto \frac{QT}{S}.$$

$$T \propto \frac{S}{V} \propto \frac{BS}{F} \propto \frac{BS}{Q}.$$

$$S \propto TV \propto \frac{TF}{B} \propto \frac{TQ}{B}.$$

$$V \propto \frac{S}{T} \propto \frac{F}{B} \propto \frac{Q}{B}.$$

SCHOLIUM.

217. We have before ſaid that we know nothing more of forces than by their effects in moving bodies: we call thoſe equal forces, however different they may be in their nature, which give to bodies equal momenta, or which when the bodies are equal give to them equal velocities; and we ſay that forces

are greater or leſs when they give to bodies greater or leſs momenta, or when they impreſs upon equal bodies greater or leſs velocities. Therefore when different forces act upon equal bodies, *the forces are*, cæteris paribus, *proportional to the velocities imparted:* the velocity then being proportional to the force, theſe two quantities may be repreſented the one by the other, and all which we have eſtabliſhed in Chap. I. of the firſt Book on the compoſition and reſolution of forces may be applied to the compoſition and reſolution of velocities. Hence it might ſeem unneceſſary to give here the demonſtration of the moſt extenſive propoſition, as it relates to velocities and directions; but, ſince it may be proved ſatisfactorily in ſmall compaſs, and admits of two or three uſeful deductions, we are unwilling to omit it.

PROP. *The ſimultaneous action of two impulſive forces* P, P′, *on a body* A, *which would impreſs upon it ſeparately the velocities* V, V′, *in the directions* AC, AC′, *will cauſe that body to move uniformly over the diagonal of the parallelogram whoſe ſides are in the directions of thoſe forces.*

Imagine that the body A (fig. 6. pl. X.) is placed on a plane ACC′ which moves uniformly in the direction AC′ with ſuch a velocity as in each unit of time will carry it over a ſpace equal to the line AC′: it is certain that this body conſidered with relation to the plane on which it is placed has no motion; yet if a ſpectator fixed immoveably out of that plane obſerves the body A, he will attribute to it a motion equal and parallel to that of the plane. Now, if we conceive that any impulſive force whatever, P, acts upon the body A in the direction PAC, and impreſſes upon it ſuch a velocity that in a unit of time it would paſs over a ſpace equal to AC, there can be no doubt that if the body were acted upon by this force only it would be found at the point C at the termination of the unit of time. But ſince, in conſequence of the motion of the plane, the line AC advances towards C′B by a motion uniform and parallel, ſo that it would really coincide with C′B at the end of a unit of time, it is obvious that the point C will then coincide with the point B, and that, of conſequence, the body A which partakes of the motion of the plane ought to be found in B at the end of the firſt unit of time. We may prove, in like manner, that at the end of any part or multiple whatever, T, of this unit, the body A, animated with the ſame velocity AC, ought to run over a proportional ſpace $Ac = T \times AC$, while the common motion conſtrains the line Ac to paſs parallel to itſelf over a diſtance $Ac' = T \times AC'$. This line coincides, therefore, with $c'b$, and conſequently b is the place of the body A at the end of the time T. And it is manifeſt that all the points b, b, that may be determined by the ſame reaſoning, are found on the ſame diagonal AB, ſince Ac : cb : : AC : CB. The body A, therefore, actually deſcribes the diagonal AB. But beſides this, its motion along this line muſt be uniform: for Ab : AB : : Ac : AC : : T × AC : AC : : T : 1; that is to ſay, Ab is to AB as the time employed in paſſing through Ab to that occupied in paſſing over AB. Conſequently the motion of the body A along the diagonal AB is uniform. Since a body at reſt on a moveable plane has the ſame motion as the plane, it is clear that if the plane were at reſt, but that the body moved uniformly according to the right line P′AC′, with the velocity AC′ equal to that which would be impreſſed upon it by the force P′, and received at the point A from the force P a velocity AC in the direction PAC, it would deſcribe uniformly the diagonal AB of a parallelogram formed upon the ſides AC, AC′, which repreſent the velocities of the body in thoſe reſpective directions, while the diagonal AB repreſents its new velocity. Q. E. D.

218. We may likewiſe ſhew that, *if a body be acted on by two ſimilar variable forces (for the ſame time) whoſe directions and magnitudes are expreſſed by the adjacent ſides of a parallelogram concurring in the body, it will deſcribe the diagonal of the parallelogram.*

Let the forces act by impulſes, at the beginning of equal particles of time, and let Ac', c'C′, C′c', and Ac, cC, Cc, be the relative magnitudes of correſponding impulſes. Then, by the action of the two firſt impulſes the body will, by the preceding article, deſcribe the diagonal Ab; and by the next two the diagonal bB, of the parallelogram $d\,d'$ whoſe ſides $b\,d'$, $b\,d$, are equal and parallel to the repreſentatives of thoſe new impulſes: but the forces are ſimilar, therefore the parallelograms $c\,c'$ $d\,d'$, are ſimilar; and, having parallel ſides and a common point b, they exiſt about the ſame diagonal AB. The ſame may be ſhewn for a third pair of impulſes: and ſo on, *ad libitum*. Let now the particles of time be evaneſcent and the forces inceſſant, and the ſame demonſtration will obtain.

Cor. If the forces by which the body is urged in the directions AC, AC′, be not ſimilar, it will move in ſome curve line, whoſe nature will depend on the relation of the forces. Of this many inſtances will occur as we proceed.

219. There remains another general theorem, which it will not be amiſs to exhibit in this place; viz. *If a number of bodies is moving in any manner whatever, and an equal force acts on each particle of matter, in the ſame or parallel directions, their relative motions will not be affected.*

The motion of any body A (fig. 8. pl. X.) with reſpect to another moving body B, is compounded of the real motion of A and the oppoſite to the real motion of B: for, let A move uniformly from A to C, while B moves uniformly from B to D; draw AE equal and parallel to BD, join AB, EC, DC, and ED. The motion of A with regard to B conſiſts in its change of poſition and diſtance. If while A had deſcribed AE, B deſcribed BD, there would be no change of relative place or diſtance: but A is now at C, and DC is its new direction and diſtance: therefore, the relative (art. 12.) or apparent motion of A is EC. Let the parallelogram ACFE be completed: then it is evident that the motion EC is compounded of EF, which is equal and parallel to AC, the real motion of A; and of EA, the equal and oppoſite to BD, the real motion of B.

Now let the motions of A and B ſuſtain the ſame change; let the equal and parallel motions AG, BH, be compounded with the motions AC and BD; or, ſuppoſe forces to act at once on A and B, in the parallel directions AG, BH, and with equal intenſities: on either ſuppoſition the reſulting motions will be AC′, BD′, the diagonals of the parallelograms AGC′C, and BHD′D. Then conſtructing the figure as before, we ſee that the relative motion is E′C′, and that it is equivalent to EC both with reſpect to magnitude and direction.

Here we may again remark the conſtant analogy between the compoſition of motions and that of forces. In the former the relative motions of things are not changed, whatever common motion be compounded with them all: and in the latter the relative motions and actions are not changed by any external force, however conſiderable, when equally exerted on all the moleculæ in parallel directions.

By means of this it is that we account for the circumſtance of the evolutions of a fleet in a uniform current being the ſame, with relation to the ſeveral ſhips, and produced by the ſame means, as in ſtill water: alſo, that the motions and operations in a ſhip, ſailing ſmoothly and regularly along, are performed in the ſame manner as though the veſſel were at reſt: and again, univerſally, of all bodies included in a given ſpace (for inſtance, thoſe on the ſurface of the earth,) their motions amongſt themſelves will be the ſame, their congreſs the ſame, the force of their percuſſion the ſame, and all their mutual operations, whether the ſpace they occupy is at reſt, or whether it moves in a trajectory compounded of the diurnal and annual motions about the centre of the ſolar ſyſtem; or laſtly, whether theſe are combined with a motion about ſome far more diſtant centre of force *.

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* Thoſe who are deſirous of purſuing further the theory of apparent and relative motions, particularly as it regards the phenomena of the motions of the planets, are referred to Chap. IX. of my *Treatiſe on Aſtronomy*.

220. Prop. *Given the velocities* V, V′, *with which two bodies move in the ſame line and towards the ſame point, and the diſtance* s' *of one from the other at the commencement of their motion, to find the time and place of their rencounter.*

If any point A be aſſumed as a fixed point in a line, from which we would eſtimate any diſtances upon that line through which a body paſſes; then, if s' be the diſtance from this fixed point to the point where a body commences its motion, or, as we may call it, *the initial ſpace*, the general equation for uniform motions will become $S'=s'+T'V'$; let this be the equation for the body which commences its motion at the diſtance s' before the other body; let alſo $S=s+TV$, be the equation for that other body, in which we have $s=0$, and conſequently $S=TV$. Here the ſpaces being both eſtimated from the ſame point A, and the times of moving being equal, we have $S=S'$, and $T=T'$: we have therefore

$$\text{(I.)} \;\ldots\; T=T'=\frac{s'}{V-V'}, \text{ and } S=S'=\frac{s'V}{V-V'}.$$

Theſe equations manifeſtly ſolve the problem.

221. Prop. *Let it be propoſed to find at what time the two moving bodies will be at any given diſtance* D *from each other.*

In this caſe it will be neceſſary to have $S-S'=\pm D$, where we put the double ſign ±, becauſe the bodies may be at the diſtance D from each other either before or after their rencounter: that is, we may either have $S>S'$, or $S<S'$. Now from this equation of condition conjointly with the preceding ones we obtain the following:

$$\text{(II.)} \;\ldots\; T=\frac{s'\pm D}{V-V'}, \;\ldots\; S=V\,.\,\frac{s'\pm D}{V-V'}, \;\ldots\; S'=\frac{Vs'\pm V'D}{V-V'}.$$

222. Prop. *Having two bodies moving uniformly in the perimeter of any complete curve, to find their points of rencounter.*

Suppoſing the curve to be rectified, it is obvious that the ſolution to this problem will be compriſed in the equations I. art. 220. But beſides the point of meeting which may be found thus, the bodies may have ſeveral others, ſince they may continue to run along the curve and paſs again and again over the ſame points, and in the courſe of this motion have many different points of meeting: now, in this ſtate, the firſt point of rencounter is taken as a new point of departure, and we may conſider the two bodies as diſtant from each other the whole perimeter p of the curve. Here p will obviouſly correſpond with s' in equa. I. above, and the inſtant of the ſecond rencounter will be diſtant from that of the firſt an interval $T'=\frac{p}{V-V'}$:

we have, therefore, eſtimating the time from the origin of the motion to that of the ſecond rencounter, $\mathrm{T}=\frac{s'}{\mathrm{V}-\mathrm{V}'}+\frac{p}{\mathrm{V}-\mathrm{V}'}=\frac{s'+p}{\mathrm{V}-\mathrm{V}'}$. Reaſoning in the ſame manner, for the third rencounter we ſhall have $\mathrm{T}=\frac{s'+2p}{\mathrm{V}-\mathrm{V}'}$; and in general the *n*th rencounter will happen at the end of a time $\mathrm{T}=\frac{s'+(n-1)p}{\mathrm{V}-\mathrm{V}'}$.

In like manner we obtain an equation for the time employed by the ſecond body in reaching the *n*th point of rencounter: and, comparing the two equations, we find

$$\text{(III.)} \ldots \mathrm{S}=\mathrm{V}\,.\,\frac{s'+(n-1)p}{\mathrm{V}-\mathrm{V}'}, \ldots \mathrm{S}'=\frac{\mathrm{V}s+\mathrm{V}'p(n-1)}{\mathrm{V}-\mathrm{V}'}$$

If ſeveral bodies move uniformly over the ſame curve, we have ſimilar equations for each of them, which muſt be compared two by two; and the *n*th rencounter of the firſt and third body will be given by the equation

$$\mathrm{T}=\frac{s''+(n'-1)p}{\mathrm{V}-\mathrm{V}''}.$$

Thus to find the point where three bodies will meet, it is neceſſary to make this value of T equal to the former, which furniſhes this equation

$$\frac{\frac{1}{p}s''+n'-1}{\mathrm{V}-\mathrm{V}''}=\frac{\frac{1}{p}s'+n-1}{\mathrm{V}-\mathrm{V}'}, \text{ or } \frac{\frac{1}{p}s''+n'-1}{\frac{1}{p}s'+n-1}=\frac{\mathrm{V}-\mathrm{V}''}{\mathrm{V}-\mathrm{V}'}.$$

The problem in this caſe obviouſly becomes indeterminate: but as the values of *n* and *n'* muſt be whole numbers, the number of ſolutions is leſs conſiderable than might at firſt be ſuppoſed. The matter, however, need not be purſued in this place.

223. To give an example of the uſe of ſome of theſe equations: *Suppoſe that we have a clock ſhewing the hours, minutes, and ſeconds:* we may conceive the extremities of the three hands as three moveable points moving over the ſame circumference; the determination of the rencounter two and two, or all three of the hands, will not be attended with any difficulty. For, putting V, V', V'', the reſpective velocities of the ſecond, minute, and hour hands; taking alſo the minute for the unit of time, and the perimeter for the unit of ſpace, we thence have $p=1$, $\mathrm{V}=1$, $\mathrm{V}'=\frac{1}{60}$, $\mathrm{V}''=\frac{1}{720}$; and the preceding formulæ give

For the rencounter 2 and 2 of the . . $\begin{cases} \text{ſecond and minute hands} & \mathcal{T}=\dfrac{60\,(s'+n-1)}{59}. \\ \text{hour and ſecond hands} & \mathcal{T}=\dfrac{720\,(s''+n'-1)}{719}. \end{cases}$

Rencounter of all three hands . . . $\left\{ \dfrac{60\,(s'+n-1)}{59}=\dfrac{720\,(s''+n'-1)}{719}. \right.$

If therefore we put for s' and s'' their values, which are the diſtances of the hands at the commencement of the time, we ſhall reſolve the problem completely, if we ſatiſfy theſe equations by means of any whole numbers n, n', whatever.

II. *On Motions uniformly varied.*

224. A body which has received only a ſingle impulſion will, according to the firſt law of motion, perſevere in its motion with the ſame velocity and in the ſame direction it had at the firſt inſtant: but if it receives a new impulſion, either in the ſame direction or in a direction contrary to the firſt, it will then move with a velocity equal to either the ſum or the difference of the two velocities which it received ſucceſſively. If, therefore, we conceive that at ſucceſſive intervals of time the body receives new impreſſions, either in the ſame or contrary directions, it will be transferred to different parts of ſpace with a *varied* or unequable motion; its velocity will be different at the commencement of each interval of time. In variable motions the velocity undergoing repeated changes, it is uſual to eſtimate it at any time whatever by the ſpace it is capable of paſſing over during a unit of time, if its motion for that interval continued the ſame as at the inſtant where we would conſider the velocity. Or in variable motions, the velocity of a body at any determinate inſtant is the ſpace which it would run over in every unit of time, if at that inſtant the action of the power ceaſed, and the motion became uniform.

DEFS. We call in general any force which acts on a body ſo as to makes it vary its motion an *accelerating force:* when, in equal intervals of time, it acts equably, or the velocity undergoes equal mutations, we call it *a conſtant or uniform accelerating force, or a conſtant retarding force,* according as it tends to augment or diminiſh the actual velocity of the moving body.

When a ſingle body is acted upon by a conſtant force there are four quantities which become the objects of mechanical conſideration, viz. the ſpace deſcribed, the time of deſcription, the velocity acquired, and the force which produces it; any three of which being given, the other may be aſcertained. But when different forces act upon bodies of different maſſes, theſe are two additional quantities for conſideration, making in the whole ſix kinds of magnitudes which affect the diſcuſſion.

225. PROP. *The velocities generated in equal bodies by the action of conſtant forces are in the compound ratio of the forces and times of acting.*

For, when the times are the ſame the velocities generated each inſtant are as the forces of acceleration, and conſequently the velocities generated at the end of equal times are as thoſe forces; and if the forces are the ſame the velocities generated are as the times wherein the forces act; becauſe, when the force is given equal velocities are generated in equal times, and conſequently the whole velocities acquired are as the times wherein the given force acts: wherefore, both times and accelerating forces being different, the velocities generated will be as the forces and times of action, jointly.

COR. 1. *The momenta generated in unequal bodies are alſo conjointly as the forces and their times of action.* This is evident, becauſe momenta in unequal bodies may be ſubſtituted for proportional velocities in equal bodies, throughout the whole reaſoning.

COR. 2. *The momenta loſt or deſtroyed in any times are likewiſe conjointly as the retarding forces and their times of action.* For, whatever momenta any force generates in a given time would an equal force deſtroy in an equal time, by acting in a contrary direction.

And the ſame is true of the increaſe or decreaſe of motion, by forces that either conſpire with, or oppoſe, the motions of bodies.

COR. 3. *The velocities generated or deſtroyed in any times are directly as the forces and times, and reciprocally as the bodies or maſſes.* For, ſince the compound ratios of the bodies and their velocities are as thoſe of the forces and times, the velocities are as the forces and times divided by the bodies.

226. PROP. *In motions uniformly accelerated, when the force and body are given, the ſpace deſcribed during a certain time is the half of that which the body, moving uniformly with the laſt acquired velocity, would deſcribe in an equal time.*

Since the velocities are as the times of deſcription, when the body and force are given, the velocities which a given body is found to have ſucceſſively for the duration of each conſecutive interval form an arithmetical progreſſion, g, $2g$, $3g$, &c. of which the laſt term is gt or v, the number of terms being t, that is to ſay, being marked by the number of ſolicitations of the accelerating force. And ſince each of the velocities is nothing elſe than the ſpace which the body would deſcribe uniformly during the correſponding interval, the total ſpace deſcribed during the time t will therefore be the ſum of the terms of this arithmetical progreſſion; which, becauſe g and v are the extremes and t the number of terms, will be expreſſed by $\frac{1}{2}t\,(g+v)$. Or if s be the total ſpace deſcribed by the body, then will $s=\frac{1}{2}t\,(g+v)$. Conceive now that the accelerating force acts (as by hyp.) without intermiſſion, or, which comes to the ſame, imagine that the time t is divided into an indefinite number of infinitely ſmall parts, or inſtants, and that at the beginning of each inſtant the accelerating force gives an impulſion to the body. Then g being infinitely minute in relation to v, which is the velocity acquired during the indefinite number of inſtants denoted by t, muſt be omitted in the equation $s=\frac{1}{2}\,t\,(g+v)$, which will become ſimply $s=\frac{1}{2}\,t\,v$, the ſpace actually deſcribed.

This granted, imagine that at the end of the time t the accelerating force ceaſes to act; then, by the firſt axiom, the body will perſevere in its motion with the velocity v it has acquired: but in uniform motions the ſpaces deſcribed are as the times and velocities jointly (art. 216.), therefore the body moving with the velocity v, during the time t, will deſcribe a ſpace $s'=t\,v$; which is evidently double the ſpace $\frac{1}{2}\,t\,u$ deſcribed by the body in an equal time, by the conſtant action of the accelerating force. Q. E. D.

227. Prop. *The ſpaces deſcribed by a body uniformly accelerated are as the ſquares of the times.*

Since the velocities acquired increaſe as the time expired, if ϕ be the velocity at the end of one ſecond, then the velocity acquired after a number t of ſeconds will be $\phi\,t$; thus we have $v=\phi\,t$. The equation $s=\frac{1}{2}\,v\,t$, found in the preceding article, becomes therefore $s=\frac{1}{2}\,\phi\,t^2$. If, in like manner, we repreſent another ſpace by S, which is deſcribed by uniform acceleration during the time T, we ſhall have $S=\frac{1}{2}\,\phi\,T^2$. Hence we ſee that $s:S::\frac{1}{2}\,\phi\,t^2:\frac{1}{2}\,\phi\,T^2::t^2:T^2$. Q. E. D.

Cor. 1. Becauſe the velocities acquired are as the times, we have alſo *the ſpaces deſcribed as the ſquares of the velocities.*

Cor. 2. *Therefore either the velocities or the times are as the ſquare roots of the ſpaces deſcribed from the commencement of the motion.*

Cor. 3. All that has been ſhewn here applies equally to motions *uniformly retarded;* provided that by the times we mean thoſe which are to elapſe before the extinction of the velocity, and by the ſpaces thoſe which remain to be deſcribed until the body is brought to reſt. Similar propoſitions have, therefore, been applied to the motions of balls reſiſted by banks of earths, blocks of wood, &c. See *Dr. Hutton's Select Exerciſes*, and *Atwood on Motion.*

COR. 4. *The velocity ϕ which will be acquired at the end of a ſecond is that which the accelerating force can generate in a ſecond choſen as a unit of time; it is therefore a meaſure of the accelerating force, and may of courſe be ſafely ſubſtituted for that force in any of our ſubſequent inveſtigations.*

SCHOLIUM.

228. We may now exhibit general theorems for reſolving all problems relating to motions uniformly accelerated or retarded. In order to this, put B = any body or maſs, F = the force acting conſtantly upon it, T = the time of its action, V = the velocity generated (or loſt) in the time T, S = the whole ſpace deſcribed, Q = the momentum or quantity of motion, at the end (or beginning) of the time: then the fundamental relations are $S \propto TV$, $Q \propto BV$, $Q \propto FT$, from which and the propoſitions and corollaries juſt laid down, we have—

$$B \propto \frac{Q}{V} \propto \frac{FT}{V} \propto \frac{QT}{S} \propto \frac{FT^2}{S} \propto \frac{FT^3}{QS} \propto \frac{Q^2}{FS} \propto \frac{Q^2}{FTV} \propto \frac{FS}{V^2}.$$

$$Q \propto BV \propto FT \propto \frac{BS}{T} \propto \frac{FS}{V} \propto \frac{FT^2V}{S} \propto \sqrt{BFS} \propto \sqrt{BFTV}.$$

$$F \propto \frac{Q}{T} \propto \frac{BV}{T} \propto \frac{QV}{S} \propto \frac{QS}{T^2V} \propto \frac{Q^2}{BTV} \propto \frac{BV^2}{S} \propto \frac{BS}{T_2}$$

$$V \propto \frac{S}{T} \propto \frac{FT}{B} \propto \frac{Q}{B} \propto \frac{QS}{FT^2} \propto \frac{FS}{Q} \propto \frac{Q^2}{BFT} \propto \sqrt{\frac{FS}{B}} \propto \sqrt{\frac{F^2ST}{Q^2}}$$

$$S \propto TV \propto \frac{FT^2}{B} \propto \frac{QT}{B} \propto \frac{FT^2V}{Q} \propto \frac{QV}{F} \propto \frac{Q^2}{BF} \propto \frac{Q^2V}{F^2T} \propto \frac{BV^2}{F}$$

$$T \propto \frac{S}{V} \propto \frac{Q}{F} \propto \frac{BV}{F} \propto \frac{BS}{Q} \propto \sqrt{\frac{BS}{F}} \propto \sqrt{\frac{QS}{FV}} \propto \frac{Q^2}{BVF}.$$

When any quantities are given, or their relations to ſome fixed quantities of the ſame kind known, they are to be left out in the general theorems: thus, if the body be proportional to the force, we ſhall have $S \propto TV \propto F'T^2 \propto \frac{V^2}{F^1}$, where $F^1 = \frac{F}{B}$.

229. As it is ſometimes neceſſary to conſider the effect of accelerating forces upon bodies already in motion, it will be worth while to deduce a general formula for that purpoſe. To this end let g repreſent, as in art. 226., the velocity due to the acceleration during each unit of time, gt will then be the

total velocity acquired at the end of the time t: then, if we denote by v the velocity which the body has at the commencement of the time, and by V the velocity at the end of the time t, we have $V = v + gt$. But $V = \frac{\dot{s}}{\dot{t}}$, both in uniform and variable motions, the force or correſponding velocity being ſuppoſed conſtant for the indefinitely ſmall time $\dot{t}$. Therefore $\frac{\dot{s}}{\dot{t}} = v + gt$, and taking the correct fluents, we have

$$\text{(IV.)} \quad S = s + vt + \tfrac{1}{2}gt^2.$$

Here the conſtant quantity s evidently repreſents the initial ſpace, viz. the diſtance between the point of departure and that in relation to which we conſider the ſeveral poſitions of the moving body: for $t = 0$, gives $S = s$.

230. The general equation given in the laſt article may be readily conſtructed: for it is plain, from the theory of conic ſections, that its locus is a common parabola. For, changing S into $x + a$, and t into $y + b$, we have

$$x + a = s + vy + vb + \tfrac{1}{2}g\,(y + b)^2$$

Then determining the conſtant quantities a and b, by the equations

$$v + bg = 0, \text{ and } a = s + vb + \tfrac{1}{2}gb^2$$

We thence find $b = -\frac{v}{g}$, $a = s - \frac{v^2}{2g}$ and the equation (IV.) in the laſt article will become $y^2 = \frac{2}{g}x$, which is an equation to a parabola whoſe parameter is $\frac{2}{g}$.

Now the nature of the motion being ſuppoſed given, and the conſtant quantities s, v, g, being known, if AE (fig. 1. pl. XI.) is the line paſſed over by the moveable body, and we take $AB = s$, the initial ſpace B will be the point of departure. Here it reſults from the preceding values of a and b, that if we make $CA = -\frac{v}{g}$, $CD = s - \frac{v^2}{2g}$, and then conſtruct on DF as an axis a parabola DMmm', whoſe vertex is D, and parameter $\frac{2}{g}$, it will be the curve required; or that in which the ordinates AP, Ap, &c. will repreſent the times, and the correſponding abſciſſas PM, pm, &c. the ſpaces.

If we draw a tangent BR to the point B of the parabola, it will form with AT an angle RBN, of which the tangent will be $= v$. Moreover, if we ſet off from any point P in AT, the axis of the times, two conſecutive ſeconds, or units of time, Pp, pp', and draw to the correſponding points M, m, of the locus, the tangents M r, $m\ r'$, we ſhall have $n\,r = v + g\,t$, $n'r' = v + g\,(t + 1)$, and conſequently $n'r' - n\,r = g$. Or, we may find the length of g by another method: for, ſince $\text{P M} = s + v\,t + \frac{1}{2}g\,t^2$, and $p\,m = s + v\,(t + 1) + \frac{1}{2}g\,(t + 1)^2$, therefore $m\,n = v + g\,t + \frac{1}{2}g$; but $n\,r = v + g\,t$, whence $m\,r = \frac{1}{2}g$, and $2\,m\,r = g$.

COR. When $s = o$, A coincides with B: and when $v = o$; B coincides with D, and R N vaniſhes.

III. *Variable Motions in general.*

231. When a moving body is ſubjected to the energy of a force which acts on it without interruption, but in a different manner at each inſtant, the motion is called in general, *variable motion*. We have inſtances of variable motions in the unbending of ſprings: although the velocity continues to be augmented, yet the degrees by which the augmentation proceeds are diminiſhing. It is the ſame with regard to the degrees by which the motion of a ſhip arrives at uniformity: the action of the wind on the ſails diminiſhes in proportion as the veſſel acquires greater velocity, becauſe the action of the wind varies as the difference between its velocity and that of the ſail on which it acts.

The different natures of conſtant and variable accelerating forces, and their correſponding motions, has been illuſtrated by Dr. Hutton in the following manner. " Let two weights, W, w, be connected by a thread paſſing over a pulley at A, B, or C (figs. 2, 3, 4, pl. XI.); and let the weight W deſcend perpendicularly down, while it draws the ſmaller weight w up the line AD, or BE, or CF, the firſt being a ſtraight inclined plane, and the other two curves, the one convex, and the other concave to the perpendicular. Then the ſmall weight w will always make ſome certain reſiſtance to the free deſcent of the large weight W, and that reſiſtance will be conſtantly the ſame in every part of the plane AD, the difficulty to draw it up being the ſame in every point of it, becauſe every part of it has the ſame inclination to the horizon, or to the perpendicular; and conſequently the acceſſions to the velocity of the deſcending weight W will be always equal in equal times; that is, in this caſe W deſcends by a uniformly accelerating force. But in the two curves BE, CF, the reſiſtance or oppoſition of the ſmall weight w will be conſtantly altering as it is drawn up the curves, becauſe every part of them has a different inclination to the horizon, or to the perpendicular: in the former curve the direction becomes more and more upright, or nearer perpendicular, as the ſmall weight w aſcends, and the oppoſition it makes to the deſcent of W becomes more and more, and conſequently the acceſſions to the velocity of W will be always leſs and leſs in equal times; that is, W deſcends by a decreaſing accelerating force; but in the latter curve CF, as w aſcends, the direction of the curve becomes leſs and leſs upright, and the oppoſition it makes to the deſcent of W becomes always leſs and leſs; and conſequently the acceſſions to the velocity of W will be always more and more in equal times; that is, W deſcends by an increaſing accelerating force. So that although the velocity continually increaſes in all theſe caſes, yet whilſt it increaſes in a conſtant ratio to the times of motion, in the plane AD; the velocity increaſes in a leſs ratio than the time it aſcended up BE, and in a greater ratio than the time increaſes in the other curve CF." *Hutton's Math. Dict.* art. *Acceleration.*

The principles neceſſary for the determination of the circumſtances of variable motions are eaſily deducible from what has been done with reſpect to uniform motions, and thoſe which are uniformly accelerated or retarded, as will be ſeen in the next propoſition.

232. Prop. *To find the fundamental equations which apply to variable motions.*

In whatever manner any motion is varied, if we conſider it with relation to evaneſcent inſtants, we may conceive its velocity to be invariable during any ſuch indefinitely minute interval. But when the motion is uniform the velocity is expreſſed by the quotient of the ſpace s, deſcribed during the interval of time t, divided by that time (216.): Therefore, when the velocity is only uniform for the evaneſcent inſtant $\dot{t}$, the velocity muſt be expreſſed by the indefinitely ſmall ſpace $\dot{s}$, deſcribed during this inſtant, divided by the inſtant itſelf. We have, therefore,

$$\text{(I.)} \ldots v = \frac{\dot{s}}{\dot{t}} \text{ or } \dot{s} = v\,\dot{t}.$$

The equation $v = \varphi t$ (art. 227.) which expreſſes the relation of the velocities to the times, in motions uniformly accelerated, gives $\varphi = \frac{v}{t}$; that is to ſay, when the accelerating force, or, rather the quantity φ by which it is meaſured (227. cor. 4.), is conſtant, it has for its expreſſion the quotient of the velocity v, which it generates during a certain time t, divided by that time; therefore, if the accelerating force φ acts differently at each inſtant, we imagine it to be conſtant only for the evaneſcent inſtant $\dot{t}$, in which it would generate the velocity $\dot{v}$, and conſequently,

$$\text{(II.)} \ldots \varphi = \frac{\dot{v}}{\dot{t}}, \text{ or } \dot{v} = \varphi \dot{t}.$$

In the equation $v = \varphi t$ we underſtand φ to denote the velocity that the accelerating force generates in the moving body during a determinate unit of time, as a ſecond, by an action continued and always equal. In the equation $\dot{v} = \varphi \dot{t}$ we ought to underſtand the ſame thing. But it is neceſſary to obſerve, that the accelerating force being ſuppoſed variable, the quantity φ which repreſents the velocity which it would be capable of generating if it acted as a conſtant accelerating force during a ſecond is different for every inſtant of its motion. Thus we eaſily conceive that when the accelerating force becomes ſmaller, the velocity which it will be capable of generating in a ſecond, by its action repeated uniformly during each inſtant of this ſecond, muſt be ſmaller, and *vicè verſâ.*

The two preceding equations readily furniſh a third, which may often be advantageouſly adopted: for, from the equa. $\dot{s} = v \dot{t}$, we deduce $\dot{t} = \frac{\dot{s}}{v}$: ſubſtituting this value of $\dot{t}$ in the equation $\dot{v} = \varphi \dot{t}$, we readily find

$$\text{(III.)} \ldots \varphi \dot{s} = v \dot{v}, \text{ or } \varphi = \frac{v \dot{v}}{\dot{s}}.$$

Again, employing the ſame equations, ſince $\dot{v} = \varphi \dot{t}$, and $v \dot{t} = \dot{s}$, we have by multiplication $v \dot{t} \dot{v} = \varphi \dot{t} \dot{s}$; whence, ſtriking out $\dot{t}$, there remains $v \dot{v} = \varphi \dot{s}$. But $v \dot{v} = \frac{1}{2}(v v)^{\cdot}$, conſequently,

$$\text{(IV.)} \quad \varphi \dot{s} = \tfrac{1}{2}(v v)^{\cdot}.$$

In the reaſoning by which we found the equations $\dot{v} = \varphi \dot{t}$, we have conſidered the velocity as increaſing. If, therefore, caſes ariſe in which the velocity diminiſhes, its fluxion will become negative, and the equations $\dot{v} = \varphi \dot{t}$, and $\varphi \dot{s} = v \dot{v}$, in order to accommodate them to all caſes which may ariſe, muſt be written with the double ſign: viz. $\pm \dot{v} = \varphi \dot{t}$, and $\varphi \dot{s} = \pm v \dot{v}$, the ſuperior ſign obtaining when the motion is accelerated, and the lower one when it is retarded.

The equation $\dot{s} = v\dot{t}$, or $v = \frac{\dot{s}}{\dot{t}}$, being fluxed gives $\dot{v} = \left(\frac{\dot{s}}{\dot{t}}\right)^{\cdot}$: if this value be ſubſtituted for $\dot{v}$ in the equation $\varphi\dot{t} = \pm\dot{v}$, it will become

$$\text{(V.)} \ldots \varphi\dot{t} = \pm\left(\frac{\dot{s}}{\dot{t}}\right)^{\cdot}$$

And this equation muſt be employed when $\dot{t}$ is ſuppoſed variable: but if we imagine, as it is often right to do, that $\dot{t}$ is conſtant, we have $\varphi\dot{t}^2 = \pm\frac{\ddot{s}}{\dot{t}}$; wherefore

$$\text{(VI.)} \ldots \varphi\dot{t}^2 = \pm\ddot{s}, \text{ or } \varphi = \frac{\ddot{s}}{\dot{t}^2}.$$

SCHOLIUM.

233. According to whatever law the motions of bodies may be varied, we may conſtruct curves as loci of the equations which compriſe the relations of the times and ſpaces: but ſince there will be as many kinds of curves as there may ariſe equations compriſing the law of the variations, it will be impoſſible to attend to them minutely here. All, therefore, that will be remarked in this place is, that when the motion is accelerated, the correſponding curve will preſent its convexity to the axis of the time; while, if the motion is retarded, the concavity of the curve will be preſented to that axis: and if in any inſtant whatever the motion becomes uniform, the curve will then degenerate into a right line, which will be a tangent to that point of the curve which correſponds with the inſtant of time in which the uniformity of the motion commences.

234. As the formula $\varphi = \frac{\ddot{s}}{\dot{t}^2}$ is of the utmoſt importance in the theory of varied motions, and as the manner in which we have deduced it above has been ſometimes objected to; we ſhall here preſent a more rigorous demonſtration of the ſame, which was firſt given by the celebrated D'Alembert, and is deduced from the known theory of curves.

Let A P, A P′, &c. repreſent the times, and P M, P′ M′, &c. the ſpaces deſcribed (fig. 6. pl. XI.): call A P, t, P M, s; and imagine the three ordinates P M, P′ M′, P″ M″, to be infinitely near to each other; make $\dot{t}$ conſtant, or P P′ = P′ P″, and P′ M′ = s. This done, it is evident that K M′ = $\dot{s}$ will be paſſed over in the time $PP' = \dot{t}$, and that $QM'' = \dot{S}$, will be run over in the time $P'P'' = \dot{t}$: now there are two hypotheſes reſpecting the acceleration which has place in the ſpaces $\dot{s}$, $\dot{S}$, which we ought to diſtinguiſh with great care. 1. The augmentation of the velocity which obtains in theſe ſpaces may be gradually acquired during the inſtant $\dot{t}$, viz. the motion may be continually accelerated from P to P′, and from P′ to P″. 2dly, The augmentation of the velocity which takes place in the ſpace KM′ may be given it all at once in M′, in ſuch a manner that KM will be deſcribed by a uniform motion; in like manner, the augmentation which obtains in QM″ may be reckoned to be acquired all at once in M″, ſo that QM″ be deſcribed by a uniform motion, &c. In this ſecond hypotheſis, the motion is made as it were by ſmall leaps, which take place at the end of each inſtant $\dot{t}$, the motion retaining its uniformity during that inſtant.

The effect produced at the termination of $\dot{t}$ is the ſame on either hypotheſis: the effect conſiſting in running over the ſpaces $\dot{s}$, $\dot{S}$, &c. But it is not the ſame with regard to what takes place during the exiſtence of each inſtant: for, according to the one or the other hypotheſis the elements MM′, M′M″, &c. are different. In the firſt, where the acceleration of the motion is conceived to be always taking place throughout the inſtant $\dot{t}$, the correſponding element of the curve is a real curve M*g*M′, M′*n*M″, &c. different from the cords MM′, M′M″, &c. In the ſecond, the motion being conſidered as uniform during the inſtant $\dot{t}$, the elements M*g*M′, M′*n*M″, &c. become rectilinear, and differ not from the cords MM′, M′M″, &c. Let us firſt ſhew what reſults from the former caſe.

Draw to the point M′ of the curve the rigorous tangent M′R, then will the ſpace QR be that which the body would run over during the inſtant $\dot{t}$, if the velocity acquired at the point M′ were continued uniformly; the ſpace RM″ will be that run over in virtue of the acceleration which obtains between P′ and P″, and the ſpace *nr* will be that paſſed over, in conſequence, during the time P′*d*; we will examine the relation which exiſts between *nr* and M″R with regard to the times P′*d*, P′P″. The arc M′M″, being infinitely ſmall, may be conſidered as appertaining to any curve whatever, and of conſequence to a parabola; where, by the property of this curve, M′R being a tangent, we have the proportion $nr : M''R :: P'd^2 : P'P''^2$; for, by Hutton's Conics, Prop. ix.

Parab. $nr : M''R :: M'r^2 : M'R^2$, and it is obvious that $M'r^2 : M'R^2 :: P'd^2 : P'P''^2$. Therefore the ſpaces nr, $M''R$, are as the ſquares of the times $P'd$, $P'P''$, employed in deſcribing them; therefore during the inſtant $P'P'' = \dot{t}$, the motion is uniformly accelerated. Whence we have $M''R = \Psi\dot{t}^2$, Ψ being a conſtant quantity during the inſtant $P'P''$; but it is different in the ſucceeding inſtants.

If we prolong the cord MM' to X, we have $QX = KM' = \dot{s}$, and $XM'' = QM'' - QX = \dot{S} - \dot{s} = \ddot{s}$: but, by a well-known general property of curves, $XM'' = 2M''R$ (ſee Emerſon on Curve Lines, book ii. pr. 2.), therefore $M''R = \frac{\ddot{s}}{2}$; and, ſubſtituting this in the equation $M''R = \Psi\dot{t}^2$, we have $\Psi = \frac{\ddot{s}}{2\dot{t}^2}$. This expreſſion denotes that the ſpace which will be deſcribed in a unit of time, in virtue of the uniform acceleration which has place during the inſtant $\dot{t}$ is equal to $\frac{\ddot{s}}{2\dot{t}^2}$. Thus we ſee, on the firſt hypotheſis, the means of meaſuring the acceleration at each inſtant.

In the ſecond hypotheſis the motion being uniform during the inſtant PP', and the element of the curve MgM' coinciding with the ſide MM' of the inſcribed polygon, if the body continues to be moved with the velocity acquired in M', the ſpace which it will deſcribe during the following inſtant will be QX, ſince, in this caſe, the tangent ought to be the prolongation of the infinitely little ſide MM'. Hence it follows, that the ſpace paſſed over during the inſtant $P'P''$ in virtue of the acceleration which has place at the point M', is XM'', which, as we have ſeen, is $= 2M''R = \ddot{s} = 2\Psi\dot{t}^2$. If we make $2\Psi = \phi$, we have, for the meaſure of the acceleration, the equation $\phi = \frac{\ddot{s}}{\dot{t}^2}$, ϕ expreſſing here the double of the ſpace which would be run over in a unit of time, in virtue of the acceleration of the firſt hypotheſis. Now the double of that ſpace is preciſely the velocity acquired by a like acceleration (art. 226); therefore ϕ expreſſes the velocity that the moving body would acquire in a unit of time, if the motion continued to be uniformly accelerated by the quantity with which it was augmented during the inſtant $P'P''$ in the hypotheſis of the rigorous curve; therefore, the formulæ obtained by the two methods are identical.

We ſee, therefore, that whether, ſuppoſing the curve rigorous, the acceleration is meaſured by $\frac{\ddot{s}}{2\dot{t}^2}$, or, ſuppoſing it polygonal, it is meaſured by $\frac{\ddot{s}_{,}}{\dot{t}^2}$; either of theſe meaſures is indifferent, provided that we always eſtimate by the ſame the different effects which we would compare. The ſpace deſcribed by the moving body is the ſame on either hypotheſis, that is to ſay, QM″ during the inſtant P′P″; only in the firſt QR is run over in virtue of the motion previouſly acquired, and RM″ in virtue of the acceleration; while in the ſecond theſe are QX and XM″. It is, otherwiſe, eaſy to aſſign a direct reaſon why the elementary ſpace run over, in the hypotheſis of the polygonal curve, is the double of that deſcribed on the ſuppoſition that the curve is rigorous: namely this;—in the firſt caſe the body acquires all at once the increment of its velocity, while it is obtained in the ſecond by a uniform acceleration; it ought, therefore, during the ſame time $\dot{t}$, to deſcribe a double ſpace.

235. Having now deduced the chief formulæ in variable motions, it remains for us to preſent an example or two of their uſe and application.

I. Suppoſe that a material point, or very ſmall globe, placed at A (fig. 7. pl. XI.) is ſolicited by two forces; the one tending to make it move from A towards B, with a motion uniformly varied; the other tending, on the contrary, to puſh it back from A towards D: the circumſtances of the motion of the globule are required, on the ſuppoſition that the repulſive power impreſſes upon it an accelerating force varying inverſely as the diſtance from the point B. Let $AB = a$, $AN = s =$ the ſpace paſſed over at the end of the time t; the accelerating force which ariſes from the repulſion from A towards D will be $= \frac{m}{a+s}$, m being a conſtant quantity depending upon the law according to which the repulſive force acts. Laſtly, let $g =$ the conſtant accelerating force which ariſes from the impulſion of the moving point from A towards D. The force accelerating the motion which we conſider as being the difference of theſe two forces, we have by the equation (232. VI.) which gives

$$\frac{\ddot{s}}{\dot{t}^2} = \frac{m}{a+s} - g.$$

To find the fluent of this equation, we muſt multiply by $\dot{s}$, whence will ariſe $\frac{\dot{s}}{\dot{t}} \times \left(\frac{\dot{s}}{\dot{t}}\right)^{\cdot} = v\dot{v} = \frac{m\dot{s}}{a+s} - g\dot{s}$; and conſequently, by a well-known form, $\frac{1}{2}v^2 = m\,\text{H·L·}(a+s) - gs + \text{C}$. Where, ſince at the point A we have $v = 0$, and $s = 0$, we conclude that $\text{C} = -m\,\text{H·L·}\,a$. Therefore

$$v = \pm\sqrt{(2m\,\text{H·L·}\frac{a+s}{a} - 2gs)}.$$

This equation determines the velocity that the moving body has when it has run over the ſpace s: here if we put for v its value $\frac{\dot{s}}{\dot{t}}$ (art. 232. i.) and find the correſponding fluent, we ſhall obtain the time t in terms of the ſpace: but this is ſometimes attended with conſiderable difficulty.

The problem juſt reſolved finds its application in a caſe which we ſhall now ſtate: if a heavy body, as a piſton, is forced into a cylinder or vertical tube BD, open only at the extremity D, which the piſton cloſely fits, and if the part AB is full of a compreſſed elaſtic fluid, or of an expanſive vapour; then, not conſidering the friction of the piſton againſt the ſides of the tube, it is obvious that this piſton will be ſubjected to the action of gravity which tends to make it deſcend and impreſſes a conſtant accelerating force g, and at the ſame time to the repulſive force of the elaſtic fluid: but this fluid having leſs ſpring as it is leſs compreſſed, viz. as the piſton is further diſtant from the extremity B, the accelerating force thence ariſing varies inverſely as the diſtance of the moveable piſton from the bottom of the tube.

We have an example of this ſpecies of motion in the balls of guns, and pieces of cannon, driven by the inflammation of the powder: this produces inſtantaneouſly a great quantity of an aeriform fluid, of which the repulſive force is inverſely as the ſpace in which it is contained. We here neglect the conſideration of the weight of the ball, ſince it has but little effect upon the velocity up to the mouth of the piece, the weight being nothing in theory when the axis of the piece is horizontal. We therefore make $g = 0$, or, which amounts to the ſame, we conſider at the commencement of the calculation the accelerating force as $= \frac{m}{a+s}$: conſequently,

$$v = \sqrt{2m\,\text{H·L·}\frac{a+s}{a}}.$$

Making s to equal the diſtance of the point A from the orifice, this equation gives us the velocity with which the ball iſſues from the piece.

If the weight of the powder, and of the ball, be taken into the computation, it will of courſe become more intricate: the general principle, however, is ſtill the ſame. Theſe particulars, with other minutiæ affecting the inveſtigation, are conſidered by Dr. Hutton, in a ſolution which may be ſeen in his *Select Exerciſes.*

236. II. Let there be at D (fig. 7.) a material point, or globule, ſolicited by an accelerating force varying inverſely as the ſquare of the diſtance of the moveable from the point B; it is required to find the equation of its motion.

Put B D $= a$, D N $= s =$ the ſpace paſſed over at the end of the time t: when the globule has arrived at N, its diſtance from B will be $a - s$, and the accelerating force is $\frac{m}{(a-s)^2}$, m being again a conſtant quantity which depends upon the nature of this force; viz. its magnitude at a unit of diſtance from the centre of attraction. Here, then, we have from the equation at 232. vi.

$$\frac{\ddot{s}}{\dot{t}^2} = \frac{m}{(a-s)^2}$$

Multiplying, as before, by $\dot{s}$, we obtain $\frac{\dot{s}}{\dot{t}} \times \left(\frac{\dot{s}}{\dot{t}}\right)^{\cdot} = v\dot{v} = \frac{m\dot{s}}{(a-s)^2}$; whence $v^2 = \frac{2m}{a-s} + C$. Suppoſing that at the origin D the globule were not animated with any velocity, we ſhould have at the ſame time $s = 0$, and $v = 0$; therefore $C = -\frac{2m}{a}$. Subſtituting this, and reducing, we have

$$v = \sqrt{\frac{2m}{a}} \times \sqrt{\frac{s}{a-s}}.$$

Now, to obtain from this equation of the relations between v and s that which obtains between s and t, we muſt ſubſtitute $\frac{\dot{s}}{\dot{t}}$ for v: then taking the reciprocal of the expreſſion, multiplying by $\dot{s}$, and the quantity affected with the radical by $a-s$, we have $t = \sqrt{\frac{a}{2m}} \times \frac{a-s}{\sqrt{as-s^2}}\,\dot{s}$, where the laſt factor is equivalent to $\frac{\frac{1}{2}a - s}{\sqrt{as-s^2}}\,\dot{s} + \frac{1}{2}a \times \frac{\dot{s}}{\sqrt{as-s^2}}$. The firſt term has for its fluent $\sqrt{as-s^2}$: that of the ſecond is found by transforming s into $\frac{1}{2}a - z$; it is then $\frac{1}{2}a \times$ arc $\left(\cos. = \frac{2z}{a}\right)$: conſequently

$$t = \sqrt{\frac{a}{2m}} \times \left\{ \sqrt{as-s^2} + \tfrac{1}{2}a.\ \text{arc}\left(\cos. = \frac{a-2s}{a}\right)\right\}$$

This requires no correction, becauſe s ought to be nothing at the ſame time that t is.

The preceding values of v and t reſolve the problem propoſed in the moſt general manner, compriſing all the particular circumſtances of the motion: we remark, for example, that $s = a$, gives $v = \infty$, and $t = \sqrt{\frac{a}{2m}} \times \frac{1}{2}a\pi$, where π is the circumference of the circle whoſe diameter is unity: the firſt of theſe expreſſions indicates that the velocity of the moving point is infinite at the centre of attraction; which is natural to conceive, becauſe the force is ſo much the more intenſe as the moveable is nearer the centre. The ſecond expreſſion is proportional to $a\sqrt{a}$, or $a^{\frac{3}{2}}$; whence it follows, that *the times employed by two bodies in deſcending from repoſe to the centre of attraction are reſpectively as the ſquare roots of the cubes of the heights fallen from.*

In the caſe where bodies fall by their own gravity towards the earth, the attracting body being conſidered as a point with regard to the diſtance s, we ſhall have $m = 32\frac{1}{6}$ (art. 242.) and $t = \cdot 785398\, a \sqrt{\frac{2a}{m}}$.

237. III. It is required to determine the circumſtances of velocity, time, and ſpace, with relation to a body, which moves from quieſcence in conſequence of an attracting force which varies directly as the diſtance from the centre of force.

Let the point from which the body commences its motion be P (fig. 5, pl. XI.), and let P C $= a$, its diſtance from C the centre of force; let $v =$ the velocity at any variable diſtance A C $= s$, and at any diſtance d from c let f be the force compared with that unit of force whoſe repreſentative is m. Then, by the nature of the problem, it will be $d : s :: f : \frac{fs}{d}$, the force at the diſtance s, compared with unity, or $\frac{mfs}{d}$ will be that force with reſpect to the meaſure m, correſponding with ϕ in our equation 232. iii. Hence, ſince v increaſes as s decreaſes, we ſhall have

$$v\dot{v} = -\frac{mfs}{d}\dot{s}.$$

This equation gives us $v^2 = -\frac{mf}{d}s^2 + C$. Here when $v = 0$, $s = a$, and $0 = -\frac{mf}{d}a^2 + C$; therefore $C = \frac{mf}{d}a^2$; and, conſequently $v^2 = \frac{mf}{d} \times (a^2 - s^2)$, and $v = \sqrt{\frac{mf}{d}} \times \sqrt{a^2 - s^2}$. Hence then, if with centre C and radius CP, the quadrant PDB be deſcribed, and at the point A, whoſe diſtance from C is $= s$, the ordinate AD be drawn, becauſe $AD = \sqrt{CD^2 - CA^2} = \sqrt{a^2 - s^2}$, we ſhall have $v = AD\sqrt{\frac{mf}{d}}$.

In order to find t, we muſt adopt the equation (1) or $t\dot{v} = -\dot{s}$, whence ariſes $\dot{t} = -\frac{\dot{s}}{v} = \sqrt{\frac{d}{mf}} \times \frac{-\dot{s}}{\sqrt{a^2 - s^2}}$. Now, if $z = \text{arc PD}$, we have $\dot{z} : -\dot{s} :: a : \sqrt{a^2 - s^2}$; therefore $\frac{-\dot{s}}{\sqrt{a^2 - s^2}} = \frac{\dot{z}}{a}$, and conſequently $\dot{t} = \frac{\dot{z}}{a}\sqrt{\frac{d}{mf}}$: the fluent of this expreſſion is $t = \frac{z}{a}\sqrt{\frac{d}{mf}} = \frac{PD}{CP}\sqrt{\frac{d}{mf}}$; which wants no correction, becauſe when $t = 0$, $z = 0$. So that, while *the velocity at any point* A *is as the correſponding ſine* AD, *the time of deſcent to that point is as the arc* PD. When A arrives at C we have $t = \frac{PDB}{PC}\sqrt{\frac{d}{mf}} = \frac{1}{2}\pi\sqrt{\frac{d}{mf}}$, for the time of falling to the centre. Hence, from whatever altitude CP the body begins to fall towards the centre, its whole time of deſcent will be the ſame, $\frac{PDB}{PC}$ being in all caſes $= \frac{1}{2}\pi = 1{\cdot}570796$, a conſtant quantity.

COR. If a body be acted upon by a force which is every where as the diſtance from C, the time of its deſcent to that centre from any point P is to the time in which it would deſcend through that ſame ſpace, if impelled by half the firſt force uniformly continued, as the circumference of a circle to four diameters. For, on the firſt ſuppoſition the time is as the quadrantal arc PB; and on the latter the time is as 2PC; and PB : 2PC :: circumf. : 8PC or 4 diameters.

238. On the ſuppoſition that the earth were a homogeneous ſphere, the force of attraction to which any body below its

ſurface would be ſubjected varies as the diſtance from the centre: if, therefore, a perforation were made in a right line from the ſurface to the centre, the circumſtances of a body falling from the ſurface will be determined from the preceding inveſtigation. In this caſe d, the diſtance at which the effects of the force are known would be $= 20935200$ feet, and f the effect of gravity compared with unity, or mf, compared with the aſſumed meaſure m, would be repreſented by $32\frac{1}{6}$ feet, the velocity acquired by a falling body after one ſecond. Hence, when the body has fallen to C, we ſhall have $v = \text{CB}\sqrt{\frac{mf}{d}} = d\sqrt{\frac{mf}{d}} = \sqrt{mfd} = 25950$ feet, or 4·9148 miles per ſecond, for its velocity there: and $t = 1{\cdot}570796\sqrt{\frac{d}{mf}} = 1267\frac{1}{4}^{s} = 21^{m}\,7\frac{1}{4}^{s}$, for the time of falling to the centre.

239. We cloſe the ſubject of variable motions with obſerving that if the velocity v be as any power n of s the ſpace deſcribed, the time may be found by the method of fluxions: for, ſince v is as s^{n}, and $\dot{t}$ as $\frac{\dot{s}}{v}$, it will alſo be as $\frac{\dot{s}}{s^{n}}$; conſequently $t = \frac{s^{1-n}}{1-n} + C$, the correction. The application of this theorem to any caſes which the ſtudent wiſhes to purſue muſt be left to his own induſtry. It has been our object to exhibit here only two or three of the moſt uſeful examples, to ſhew the utility of the doctrine of variable forces: it is now time to advert to other topics.

CHAPTER II.

On the Descent and Ascent of Heavy Bodies, in vertical Lines; the Motion of Projectiles; Descents along inclined Planes, and Curves; the Vibrations of Pendulums, &c.

240. THE motion of heavy bodies at or near the ſurface of the earth occupies an extenſive portion of the theory of mechanics; and its numerous applications to the various purpoſes of life renders it highly worthy the attention of the ſtudent. At the beginning of Chap. III. of our firſt Book we made a few ſuch obſervations on the nature of gravity as were requiſite in diſcuſſing the ſubject of the centre of inertia: in addition to what was there ſtated, we preſent a remark or two, more immediately connected with the buſineſs before us. Gravity being that force which ſolicits all bodies to deſcend in vertical lines, or thoſe which are perpendicular to the ſurface of the earth, it would follow that, if that ſurface, as compoſed of land and ſea, were perfectly ſpherical, the directions of gravity would all concur at its centre. The earth, however, is not perfectly ſpherical; yet is its variation from that ſhape ſo trifling, that, with reſpect to the objects we now mean to conſider, it need not be regarded. We obſerved in art. 106, that, in moſt mechanical enquiries, the directions of gravity may be conſidered as parallel: that it may be ſeen to what extent this remark may be applied, let it be conſidered that a circle whoſe radius is 20935200 feet (art. 238.) will have more than 6000 feet for the meaſure of a minute of a degree, and upwards of 100 feet for that of a ſecond; ſo that *the directions of gravity at two places on the earth's ſurface, a mile aſunder, will not vary one minute from paralleliſm.*

As to the magnitude of the gravitating force, ſtrictly ſpeaking, it is different at different diſtances from the equator, and at different diſtances from the centre of the earth: but the quantities of theſe differences, ſo far as they depend upon the variety of ſituation on the earth's ſurface, are very ſmall, and need not yet be attended to; and the differences reſulting from different diſtances from the centre of the earth will not be ſenſible in any caſes we ſhall have to conſider here with regard to falling bodies. Thus, if two bodies were ſituated, the one at the ſurface of the earth, and the other at a mile above it, the difference in the attractions to which they would be ſubjected would be 3966^2-3965^2, or 7931 compared with 3965^2, or nearly one part in 2000 of the whole attraction: ſo that as a mile is greater than any altitude, or any depth from the ſurface, with regard to which we ſhall have to trace the effects of falling or riſing bodies, we may conſider the force of gravity as conſtant. We conſider therefore this force as acting inceſſantly, and acting equally at each inſtant upon every particle of matter. Now it is clear that, if every particle of a body receive the ſame velocity, the aggregate of the body will move with the ſame velocity as would have been impreſſed upon a ſingle molecule: conſequently the velocity which gravity impreſſes upon any maſs whatever does not depend upon the magnitude of that maſs; but is the ſame with reſpect to the ſmalleſt maſs as the greateſt. It is true, we do not, when bodies of different maſſes and denſities deſcend through the air, obſerve them all to fall from the ſame heights in equal times; but this is occaſioned by the reſiſtance of the medium, and when that is taken away, as in the receiver of an air-pump, the moſt denſe and the moſt rare bodies fall through equal ſpaces in equal times. Theſe premiſes being admitted, we may readily eſtabliſh the enſuing propoſition.

241. Prop. *The chief properties of motion delivered in arts.* 226...230, *with reſpect to conſtant forces, have place in the motions of bodies deſcending freely in conſequence of the action of gravity.*

This is evident from the preceding remarks, granting the aſſumption of gravity being a conſtant force: for it is no more than ſaying that the laws which are ſhewn to obtain with regard to conſtant forces in general, apply directly to any individual conſtant force propoſed.

Or, if we adopt the univerſally received hypotheſis, that the gravitating force towards the earth varies inverſely as the ſquare of the diſtance from its centre, we may deduce the truth of this propoſition from what was done in art. 236. in relation to that ſpecies of variable motion. Retaining the notation of that article, we have $v=\sqrt{\frac{2m}{a}}\times\sqrt{\frac{s}{a-s}}$, and $\dot{t}=\sqrt{\frac{a}{2m}}\times\frac{a-s}{\sqrt{as-s^2}}\dot{s}$: in which equations, in order that they may apply to the caſe of heavy bodies falling at or near the ſurface of the earth, we ſuppoſe a to repreſent the radius of the earth, and that s, the ſpace deſcribed, is indefinitely ſmall compared with

a; the firſt equation then becomes $v = \frac{\sqrt{2ms}}{a}$, and conſequently $\dot{t} = \frac{as}{\sqrt{2ms}}$. Now if g be the value, or repreſentative, of the accelerating force at the diſtance a from the centre of the earth, we have $m : g :: a^2 : 1$, whence $m = ga^2$. This value of m put for it in the laſt given expreſſion for v makes it become $v = \sqrt{2gs}$: hence $t = \frac{1}{\sqrt{2g}} \times \int \frac{\dot{s}}{\sqrt{s}} = \frac{2\sqrt{s}}{\sqrt{2g}} = \sqrt{\frac{2s}{g}}$, and conſequently $s = \frac{1}{2}gt^2$. This value of s agrees exactly with that at art. 227. the general repreſentative ϕ of the accelerating force in that expreſſion being here ſupplied by g, the particular repreſentative of the force of gravity at the earth's ſurface. Gravity may, therefore, be regarded as impreſſing on falling bodies a conſtant accelerating force; and the propoſition is true.

242. We muſt now aſcertain the real value of g, the meaſure of the force of gravity on the earth's ſurface: this, as we have already remarked, varies at different diſtances from the earth's equator; but it will be ſufficient at preſent if we determine it for ſome one place, London, for example. Now, it is found by means of accurate experiments with the pendulum (art. 271.), and by other means which need not here be deſcribed, that a heavy body in the latitude of London falls nearly $16\frac{1}{12}$ feet in the firſt ſecond of time from its quieſcent ſtate, and has then (art. 226.) acquired a velocity which, if uniformly continued, would carry it over twice $16\frac{1}{12}$, or $32\frac{1}{6}$ feet in the next ſecond; but the action of gravity upon the body continuing, the motion will be ſuch that at the end of the next ſecond of time the body will altogether have paſſed over 2^2, or four times $16\frac{1}{12}$ feet (art. 227.), that is, $64\frac{1}{3}$ feet; and will have acquired a velocity (art. 225.) of twice $32\frac{1}{6}$, or $64\frac{1}{3}$ feet per ſecond: and in a ſimilar manner will the force of gravity operate in the ſucceeding ſeconds; ſo that

If the times in ſeconds be	1,	2,	3,	4,	&c.
The vel. acq. in feet will be	$32\frac{1}{6}$,	$64\frac{1}{3}$,	$96\frac{1}{2}$,	$128\frac{2}{3}$,	&c.
The ſpaces in the whole times	$16\frac{1}{12}$,	$64\frac{1}{3}$,	$144\frac{3}{4}$,	$257\frac{1}{3}$,	&c.
The ſpaces for each ſecond	$16\frac{1}{12}$,	$48\frac{1}{4}$,	$80\frac{5}{12}$,	$112\frac{7}{12}$,	&c.

Thus, then, the action of gravity continues to change the ſtate of the body: at the end of the firſt ſecond from quieſcence it had impreſſed upon it a velocity which, if continued uniformly, would carry it through $32\frac{1}{6}$ feet in the next ſecond: but the ſolicitations of gravity being ſtill exerted on the body, it actually deſcribes $48\frac{1}{4}$ feet, and acquires a velocity of $64\frac{1}{3}$ feet; at the end of the third ſecond it has acquired a velocity of $96\frac{1}{2}$; and at the end of the fourth a velocity of $128\frac{2}{3}$ feet. Now $128\frac{2}{3}-96\frac{1}{2}=96\frac{1}{2}-64\frac{1}{3}=64\frac{1}{3}-32\frac{1}{6}=32\frac{1}{6}$, is the invariable difference of the determinations to motion at the end of each ſucceſſive ſecond of time: conſequently $32\frac{1}{6}$ feet, being a conſtant quantity naturally ariſing from the free motions occaſioned by gravity, is a juſt indication and proper meaſure of its conſtant intenſity upon bodies ſubjected to its operation; then g, in our theorems, is $= 32\frac{1}{6}$ feet. The ſpace paſſed over in the firſt ſecond may, it is true, be uſed, becauſe it is the half of $32\frac{1}{6}$, and becauſe halves have the proportion of the wholes; but it ſhould not be forgotten that this is only *half* the true meaſure of gravitating force.

243. Hence, if bodies ſimply fall from a quieſcent ſtate, not being projected downwards by any additional force, the velocities acquired will be as the times, and the whole ſpaces deſcribed as the ſquares of either; in ſuch manner, that

If the times be as the Nos.	1,	2,	3,	4,	5,	6,	&c.
The velocities acq. will be, as	2,	4,	6,	8,	10,	12,	&c.
The whole ſpaces, as	1,	4,	9,	16,	25,	36,	&c.
The ſpace for each time, as	1,	3,	5,	7,	9,	11,	&c.
Their conſtant differences,		2,	2,	2,	2,	2,	&c.

The latter anſwering to $32\frac{1}{6}$, the value of g. And if, while this value of g is retained, we put v for the velocity acquired at the end of any time t, and s for the ſpace deſcribed from quieſcence during that time, we ſhall, by comparing the latter equations in art. 241. with the general formula in art. 230., have the following general equations for the free deſcent of heavy bodies; viz.

$$s = \tfrac{1}{2}gt^2 = \frac{v^2}{2g} = \tfrac{1}{2}tv.$$

$$t = \frac{v}{g} = \sqrt{\frac{2s}{g}} = \frac{2s}{v}.$$

$$v = \sqrt{2gs} = \frac{2s}{t} = gt.$$

$$g = \frac{v}{t} = \frac{2s}{t^2} = \frac{v^2}{2s}.$$

To theſe theorems reference muſt be made in all caſes where great accuracy is required: but in many practical inſtances the fraction $\frac{1}{6}$ may be dropped, and the computation much facilitated by taking $g = 32$ feet: the theorems may then take this form:

$$s = 16\,t^2 = \frac{v^2}{64} = \tfrac{1}{2}\,t\,v.$$

$$t = \frac{v}{32} = \tfrac{1}{4}\sqrt{s} = \frac{2\,s}{v}.$$

$$v = 8\sqrt{s} = \frac{2\,s}{t} = 32\,t.$$

244. DEFS. *The height due to a given velocity* is the height from which a body muſt fall freely to acquire that velocity; and *the velocity due to a given height* is the velocity which a body will have acquired after deſcending freely through that height.

It will now be eaſy to ſolve the common queſtions relating to the deſcent of heavy bodies. Thus, if it were required to find the velocity due to a given height 400 feet, and the time of falling through it; we ſhould have $v = 8\sqrt{s} = 8 \times 20 = 160$ feet: and $t = \frac{v}{32} = \frac{160}{32} = 5$ ſeconds, the time of deſcent.

245. But if the body inſtead of falling from quieſcence be impelled vertically with any initial velocity V, we ſhall then have the equation IV. art. 229. for that comprehending the circumſtances of its motion, in which, if we reckon the initial ſpace $=0$, we ſhall have

$$s = V\,t + \tfrac{1}{2}\,g\,t^2, \text{ or } s = V\,t - \tfrac{1}{2}\,g\,t^2$$

according as the velocity is impreſſed downwards or upwards, s being eſtimated in the direction of that velocity. In the ſecond caſe we have $v = V - g\,t$, as is evident, becauſe the velocity impreſſed by gravity downwards is $g\,t$ at the end of the time t. From this expreſſion it appears that the body continues to riſe ſo long as V exceeds $g\,t$; and at the point where $V = g\,t$ the body will have attained its greateſt elevation: laſtly, when V becomes leſs than $g\,t$, v will be negative; that is, the body is deſcending again, becauſe the initial velocity has been extinguiſhed by gravity. The maximum of elevation is $\frac{V^2}{2\,g}$, and the time employed to attain it is $\frac{V}{g}$. The body will deſcend from the ſtate of reſt at its greateſt height according to the preceding laws, and the equation of its motion will be $s = \tfrac{1}{2}\,g\,t^2$, eſtimating the ſpaces downwards. At the end of the time $t = \frac{V}{g}$ the body will have run over the ſpace $\frac{V^2}{2\,g}$; thus it will employ to re-deſcend to the point of departure the ſame extent of time as it took to mount to its greateſt height: and finally, at the termination of its fall it will have acquired the velocity V of projection.

Hence we ſee,

1ſt. That a heavy body thrown upwards vertically with any velocity will loſe equal velocities in equal times.

2d. That the heights to which bodies projected upwards will riſe, before their motions are extinguiſhed, are as the ſquares of the initial velocities, or as the ſquares of the times.

3dly. That thoſe heights are the halves of the ſpaces which they would uniformly deſcribe, in the ſame time, with the initial velocities.

4thly. If a body be projected upwards with the velocity it acquired in any time by deſcending freely, it will loſe all its velocity in an equal time; it will aſcend to the height from whence it fell, and will deſcribe equal ſpaces in equal times both in riſing and falling, but in an inverſe order; it will alſo have equal velocities at any one and the ſame point of the line deſcribed, both in aſcending and deſcending.

We may, therefore, find to what elevation a body thrown vertically has ariſen, when we know the time elapſed between the commencement of its motion and its return to the point from which it was projected. For example, a body thrown vertically upwards returns at the end of 18 ſeconds; it therefore occupied 9^s in aſcending; conſequently $s=16t^2=16\times 81=1296$ feet nearly the height to which it aſcended; and $v=32t=32\times 9=288$ feet per ſecond, the initial velocity.

SCHOLIUM.

246. The equation expreſſing the conditions of falling bodies has for its locus the common parabola; being in fact the ſame as the locus for conſtant forces, the conſtruction of which we have already explained in art. 230.

The true theory of falling and riſing bodies was firſt given by Galileo, who may, indeed, be looked upon as the father of the ſcience of Dynamics: his diſcoveries on theſe topics were publiſhed in a work entitled *Dialoghi delle Scienze nuove*, &c. printed for the firſt time, at Leyden, in 1637. His method of inveſtigation (excepting that he repreſents the ſpaces deſcribed by the motions of bodies to be areas, which is rather unnatural) is very elegant: and as he deduces the laws of their motions from two theorems only, we ſhall, for the ſatisfaction of the ſtudent, inſert them here.

THEOR. I. *The time in which any ſpace is paſſed over by a moving body, with a motion uniformly accelerated from reſt, is equal to the time in which the ſame ſpace would be paſſed over by the ſame moveable, carried with a uniform velocity which is half the greateſt and ultimate velocity of the former uniformly accelerated motion.*

Let the line AB (fig. 8. pl. XI.) repreſent the time in which the ſpace CD is deſcribed by a moveable, with a uniformly accelerated motion from quieſcence at C, and let the velocity acquired at the end B of the time AB be repreſented by the line EB, drawn at pleaſure to AB, and join AE. Divide AB into any number of equal parts, and through the points of diviſion draw as many lines parallel to EB; theſe will repreſent the increaſing degrees of velocity after the firſt inſtant A. Biſect BE in F, and draw FG, AG, parallel to BA and BF reſpectively: then will the parallelogram AGFB be equal to the triangle AEB, and the ſides GF, AE, biſect each other in I. For if the parallels in the triangle AEB be extended to GF, the aggregate of all thoſe in the quadrilateral will be equal to the aggregate of thoſe in the triangle AEB; thoſe in the triangle IEF being equal to thoſe in GAI, and thoſe in the trapezoid AIFB being common. Now, ſince all and each of the inſtants of the time AB anſwer to all and each of the points of the line AB, and ſince parallels drawn from theſe points comprehended in the triangle AEB repreſent the increaſing degrees of augmented velocity, while the parallels compriſed within the parallelogram repreſent in like manner as many degrees of equable velocity, it appears that as many momenta or effects of velocity were produced in the accelerated motion according to the increaſing parallels of the triangle AEB, as in the uniform motion according to the parallels of the quadrilateral GB: the deficiency in the firſt half of the motion, repreſented by the triangle AGI, being made up by the ſupernumerary triangle IEF in the latter half of the time. It is manifeſt, therefore, that thoſe ſpaces will be equal which will be paſſed over in the ſame time by two moveables, one of which moves with a velocity uniformly accelerated from reſt, but the other with an equable motion according to a velocity which is half the greateſt velocity of the accelerated motion.

THEOR. II. *If a moveable body deſcend from reſt with a uniformly accelerated motion, the ſpaces it paſſes over in any times whatſoever are to each other as the ſquares of thoſe times.*

Let the line AB (fig. 9. pl. XI.) repreſent an interval of time from any firſt inſtant A, in which take any two times AD and AE; and let HI be a line in which the moveable deſcends from reſt at the point H with a uniformly accelerated motion; the ſpace HL being paſſed over in the firſt time AD, and the ſpace HM in the time AE: then is the ſpace HM to the ſpace HL in the duplicate proportion of the time AE to the time AD. Draw the line AC, making any angle with AB, and from the points D and E draw the parallels DO, EP; the former repreſenting the velocity acquired at the end of the time AD, and the latter the velocity acquired at the end of the time AE: then, it is manifeſt, from the laſt theorem, that the ſpaces MH and LH are the ſame which with equable motions whoſe velocities would be as the halves of PE, and OD, would be paſſed over in the times EA and DA. Now it has been demonſtrated (art. 216.) that of bodies carried with an equable motion the ſpaces paſſed over are in proportion to each other in the ratio compounded of that of the velocities and that of the times: but here the ratio of the velocities is the ſame as that of the times (for that proportion which the half of PE has to the half of OD, the ſame has PE to OD, and the ſame has AE to AD), therefore the ſpaces are in a duplicate ratio of the times, or $HM : HL :: AE^2 : AD^2$.

Hence it is manifeſt that the ratio of the ſpaces is likewiſe the duplicate of the ratio of the ultimate velocities; that is, $HM : HL :: PE^2 : OD^2$.

Cor. 1. Hence, if there were how many ſoever equal times taken ſucceſſively from the firſt inſtant of motion, as, for example, AD, DE, EF, FG, in which are paſſed over the ſpaces HL, LM, MN, NI; thoſe ſpaces will be to one another as the uneven numbers, viz. 1, 3, 5, 7: for this is the ratio of the exceſſes of the ſquares of the lines exceeding one another equally, the exceſs of which lines is equal to the leaſt of them: whilſt, therefore, the velocities are increaſed according to the ſimple ſeries of numbers in equal times, the ſpaces run through in the ſame times increaſe according to the ſeries of the uneven numbers.

Cor. 2. Hence again, it is inferred, that, if from the beginning of the motion any two ſpaces are taken, paſſed through in any times, thoſe times ſhall be to each other as either of the ſaid ſpaces is to a mean proportional between them.

247. Galileo likewiſe ſhews that the ſame laws of acceleration obtain in the motion of bodies along inclined planes; and he illuſtrates the conformity of his theory with nature by a relation of ſome of his experiments. "We took," ſays he, "a priſm of wood about twelve yards long, half a yard wide, and about three inches thick; in which thickneſs we made a very ſtraight groove a little more than an inch wide, and to render it very ſmooth and ſleek we glued within it a piece of vellum poliſhed as much as poſſible. In this groove we let fall a ball made of the hardeſt braſs, round, and well poliſhed. Then, elevating one end of this priſm at pleaſure a yard or two above the plane of the horizon, we let the ball deſcend along the groove, obſerving in the manner I ſhall tell you preſently the time ſpent in its fall from top to bottom.

"We repeated this often, in order to be certain of the quantity of time ſpent in the deſcent; and in theſe times we never found any difference worth mentioning, not even the tenth part of a ſecond: this being thoroughly eſtabliſhed, we let the ſame ball deſcend but a fourth part of the length of the groove, and meaſuring the time of the deſcent we found it to be exactly the half of the other: and then making trial of the other parts, by comparing the time of its deſcent through the whole length with the times in which it ran through $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, or, in a word, with the time of its running through any part of its length, by experiments nearly a hundred times repeated, we always found the ſpaces run through to be to one another as the ſquares of the times; and this in all inclinations of the plane, *i. e.* of the groove in which the ball was made to deſcend. We alſo obſerved the times of the deſcents along different inclinations to retain the ſame proportion to one another which we ſhall find demonſtrated and aſſigned to them.

"Now as to the meaſuring of time we hung on high a large veſſel of water, which, by a very ſmall hole through the bottom, emitted a ſmall thread of water, which was received by a ſmall cup, all the time the ball was falling through the ſeveral parts of the groove: then the ſmall portions of water thus collected were weighed every time in a very exact pair of ſcales, and the differences and proportions of the weight exhibited to us the differences and proportions of the times, and that, as I ſaid before, ſo accurately, that thoſe trials often repeated never differed any thing worth ſpeaking of." *Dial.* III.

II. *On the Motion of Projectiles, in Vacuo.*

248. It is not our intention to enter at all largely into the ſubject of gunnery: but merely to preſent a brief view of the theory of the motion of bodies any way projected from the ſurface of the earth in an unreſiſting medium, and then add a few remarks on the diſparity between this theory and the circumſtances attending bodies actually projected into the atmoſphere. In our diſcuſſions on this ſubject we ſhall pay no regard to the motions of the earth, nor to the variation either in the direction or magnitude of the force of gravity: for we are to determine the path of the projectile with reſpect to the ſurface of the earth; and the force of gravity upon any body that can be projected by human contrivances is at all times ſo nearly equal and in parallel directions (art. 240.) that it would be a uſeleſs refinement to attend to the deviations from equality and paralleliſm. When bodies are projected either directly upwards or directly downwards the circumſtances of their motion may be aſcertained by means of the theorems in art. 245. we ſhall now conſider the circumſtances reſulting from their being projected obliquely; obſerving that the theorems we ſhall deduce are only ſo far of conſequence as that they ſhew what would be the nature of the

motion of balls and ſhells fired from pieces of ordnance, in free ſpace.

249. Prop. *The line deſcribed by a heavy body thrown in any direction not perpendicular to the horizon is a parabola.*

Let the body be projected from the point B in the direction BH (fig 11. pl. XI.) with the velocity it would have acquired by falling from A to B, then will the continual action of gravity cauſe the body to be continually deflected from the line ſo as to deſcribe a curve line concave towards the horizon. Produce AB both ways as to K and T; and through any two points V and G of the curve draw VE, GK, parallel to BH, and VC, GH, parallel to AB the direction of gravity. Now it follows from the compoſition of motions, that the body will arrive at the point V of the curve in the ſame time that it would have deſcribed the right line BC by the uniform projectile motion alone, or the vertical BE by the force of gravity alone; and in like manner the body will have arrived at G in ſuch time as it would have uniformly deſcribed the line BH with the original velocity at B, or the vertical BK by falling freely. But the motion along BH being uniform, we have $BC : BH :: t . BC$ $t . BH$ (the letter t denoting the time of deſcribing BC, BH, &c.) $:: t . BE : t . BK$: and, becauſe the motion along BK is uniformly accelerated, we have $BE : BK :: t^2 . BE : t^2 . BK :: BC^2 : BH^2 :: EV^2 : KG^2$. Therefore, the curve BVG is ſuch that the abſciſſæ BE, BK, are as the ſquares of the correſponding ordinates EV, KG; or that the parts VC, GH, of the parallel lines are as the ſquares BC, BH of the intercepted parts of the line BH: and conſequently the curve is a parabola to which BC is a tangent.

Cor. 1. *The horizontal line* AD *d drawn through the point* A *is the directrix of the parabola:* for, let BE be taken = AB. Then $t . BE = t . AB$; but BC is deſcribed in the ſame time with the velocity acquired by falling through AB; therefore (arts. 226. 242.) $BC = 2\ AB$, and $EV = 2\ BE$: and hence $EV^2 = 4\ BE^2 = 4 \times BE \times BA = BE \times 4\ BA$: ſo that 4 AB is the parameter of the parabola BVG, and conſequently AD d is the directrix, becauſe AB is one-fourth of the parameter of the diameter BK.

Cor. 2. *The times of deſcribing the different portions* BV, VG, *of the curve, are as the correſponding parts* BC, CH, *of the tangent at* B, *or the intercepted parts* AD, D d, *of the directrix.* For $t . BV = t . BC$, and $t . VG = t . CH$, by the demon. of the prop. and becauſe D d cuts the three parallels BT, VC, GH, BC is to CH, as AD to D d.

Cor. 3. *The velocity eſtimated horizontally is uniform.*

Cor. 4. *The velocity of the projectile at any point* G *of the curve,* *eſtimated in the direction of the tangent to the parabola at that point, is as the ſecant of the angle contained between that tangent and the horizon.* For the motion in the horizontal direction A *d* is uniform; and A *d* is to TG as radius to the coſecant of HGT, or ſecant of the angle included between the horizon and GT.

COR. 5. *The velocity at the point* G *in the direction* TG *is equal to that which a heavy body would acquire by falling freely from the directrix through* d G. For, let the points *a*, *f*, be equidiſtant from A and *d*, but indefinitely near them, and draw the verticals *a b*, *f g*: conceive theſe to approach towards AB, *d* G, and ultimately to coincide with them; then will B *b* be ultimately to *g* G as the velocity in the curve at B to the velocity at G. But B *b* : G *g* : : BH : TG : : vel. at B : vel. at G; and, by the nature of the parabola $BH^2 : TG^2 :: AB : dG$: alſo AB is to *d* G as the ſquare of the velocity acquired by falling through AB to the ſquare of the velocity acquired by falling through *d* G (art. 227. cor. 1.): but the velocity in BH or in the point B of the parabola is = the velocity acquired by falling through AB (by hyp.); therefore the velocity in TG, or in the point G of the curve, is = the velocity acquired by falling through *d* G.

COR. 6. *The velocity in the vertical direction at any point* G *of the curve is to the original projectile velocity at* B *as* 2 HG *to* BH. For the times in BH and HG being equal, and the velocity acquired by falling freely through HG being ſuch as would carry the body uniformly over 2 HG in an equal time; we have (art. 216. cor. 3.) BH to 2 HG, as the projectile velocity at B to the vertical velocity at G.

250. DEFS. The height due to the original velocity of the projectile, as AB (fig. 11.) or CA (fig. 12.), is called the *Impetus*. In our general theorems it will be denoted by I.

The diſtance AB (fig. 12.) between the point of projection and the point where the body falls, meaſured on the plane AB, is called the *Amplitude*, *Random*, or *Range*: it will be repreſented by R.

Some authors reſtrict the word *Amplitude* to the range on a horizontal plane.

The angle DAB made by the axis of the gun, or the tangent to the curve at A, and the direction of the object at B, is called the angle of *Elevation* above the plane AB: it will be denoted by E.

The angle DBA included between the vertical DB (paſſing through the object) and the plane AB, is called its *angle of poſition*: we repreſent it by P.

The angle ZAD made by the vertical ZA (paſſing through the point of projection) and the direction of the piece, is called the *zenith diſtance*: it will be denoted by Z.

251. PROP. *To give a general view of the relative ſituation and*

properties of ſuch lines as are moſt uſeful in conſidering the motions of projectiles—

Let a body be projected from B (fig. 13. pl. XI.) in any direction BC with the velocity acquired by falling through AB: this body by art. 249. will deſcribe a parabola BVPM. Produce AB both ways till OA = AB = BK; and with centres A, B, and radius AB, deſcribe the ſemicircles ODB, AHK: then with axis AB and ſemiaxis GE = AB = AD, deſcribe the ſemi-ellipſe AEB; and, with focus B, vertex A, diameter AB, and tangent AD, parallel to the horizon, draw the ſemiparabola APS. This done, we obſerve

I. *The horizontal line* ADL *is the directrix of every parabola which can be deſcribed by a body projected from* B *with the velocity acquired by falling through* AB. This is evident from the foregoing propoſition, and the uſual definitions of the directrix.

II. *The ſemicircle* AHK *is the locus of all the foci of theſe parabolas.* For BA, the diſtance of a point B of any parabola from the directrix AD, is equal to BF, its diſtance from the focus F of that parabola: conſequently, the foci of all the parabolas which paſs through B, and have AD for their directrix, muſt be in the circumference of the circle whoſe centre is B and radius AB.

III. *If* BC, *the line of direction, cut the upper ſemicircle in* C, *and the vertical* CF *be drawn interſecting the lower ſemicircle in* F, *then is* F *the focus of the parabola, which is deſcribed by the body whoſe impetus is* AB, *and initial direction* BC. For, if AC, BF, be joined, then is ACFB evidently a rhombus, and the angle ABF is biſected by BC: conſequently the focus is ſomewhere upon the line BF. But it is alſo upon the circumference of AHK; and muſt, therefore, be at F the point where they interſect.

When C is in the ſuperior quadrant of ODB, F is in the ſuperior quadrant of AFK: and when *c* is in the inferior quadrant of ODB, as when B *c* is the initial direction, then the focus *f* of the correſponding parabola B *v* M is in the inferior quadrant of AHK.

IV. *The ſemiellipſe* AEB *is the locus of the vertices of all the parabolas, and the vertex* V *of any one of them* BVPM *is in the point where this ellipſe interſects the vertical* CF. For, let this vertical interſect the horizontal lines AD, GE, BN, in θ, λ, N: then, it is manifeſt that $\lambda\theta = \frac{1}{2}N\theta = N\lambda$, and that $\lambda V = \frac{1}{2}\theta C = \frac{1}{2}NF$; therefore, $\lambda\theta - \lambda V = \frac{1}{2}(N\theta - NF)$, or $\theta V = \frac{1}{2}\theta F$; conſequently $\theta V = \frac{1}{2}VF$, and V is the vertex of the parabola.

The vertex is at V or *v*, in the upper or lower quadrant of the ellipſe, according as the focus is in the upper or lower quadrant of the ſemicircle AHK.

V. *If from the point of projection* B *the line* BFP *be drawn through the focus of any one of the parabolas, as of* BVM, *cutting* *the parabola* APS *in* P, *then will the curve* BVM *touch the curve* APS *in* P. For, if we draw $P\delta z$ parallel to AB, cutting the directrix Oz of the parabola APS in z, and the directrix AD of the parabola BVM in δ, then will $PB = Pz$, by the nature of the parabola: but $BF = BA = AO = \delta z$; therefore, $PB - BF = Pz - \delta z$, or $PF = P\delta$, and the point P is in the parabola BVM. And ſince the tangents to both parabolas in P biſect the angle zPB, they coincide; conſequently the two parabolas having a common tangent in P touch each other at that point.

Cor. All the parabolas which can be deſcribed by a body projected from B, with the velocity acquired by falling from A to B, will touch the concavity of the parabola APS, and lie wholly within it.

VI. P *is the moſt diſtant point of the line* PB *which can be ſtruck by a body projected from* B *with the velocity due to the height* AB. For, if the direction be more elevated than BC, the focus of the parabola which the body would deſcribe will lie between F and A, and the parabola will touch APS in ſome point between A and P; ſo that, being wholly within the parabola APS, it muſt interſect the line BP in ſome point nearer to B than P is. If, on the contrary, the direction is leſs elevated than BC, the parabola deſcribed by the projectile will touch APS in ſome point below P, and will, therefore, cut BP in ſome point not ſo far diſtant from B as P.

VII. *The parabola* APS *is the locus of the greateſt ranges on any planes* BP, BS, *&c. and no point lying without this parabola can be ſtruck while the initial velocity remains unchanged.*

VIII. *To produce the greateſt range on any plane* BP, *the line of direction* BC *muſt biſect the angle* OBP *formed by that plane and the vertical:* for in this caſe the parabola deſcribed by the body touches APS in P, and its focus is in the line BP; conſequently the tangent BC biſects the angle OBP.

Cor. *On a horizontal plane the greateſt range is made when the elevation is 45° or half a right angle.*

IX. *A point* M *in any plane* PS, *lying between* B *and* S, *may be ſtruck with two directions* BC *and* Bc; *and theſe directions are equidiſtant from the direction* Bt *which gives the greateſt random on that plane.* For, with centre M and radius ML (the neareſt diſtance to the directrix) let a circle LFf be deſcribed; it will manifeſtly either touch or cut the circle AHK in two points F and f, which are the foci of two parabolas BVM, BvM, having the directrix AL, and diameter ABK. Here the interſections of the circle ODB with the verticals FC, fc, determine the directions BC, Bc, of the tangents. Now, draw At parallel to BS, and join tB, Cc, Ff; then is $OBt = \frac{1}{2}OBS$, and Bt the direction which produces the greateſt range on the plane BS;

but F f, being a chord of the circles deſcribed about the centres B and M, is perpendicular to BM; as is likewiſe C c to A t; and arch C t = arch ct; and therefore the angle CB t = angle c B t.

Cor. On a horizontal plane, the ranges are equal at angles of direction, equidiſtant from 45°, as at 40° and 60°, 35° and 65°, &c. the impetus being given.

X. *Every velocity of projection has a correſponding ſet of parabolas, with their directions and ranges, and,* cæteris paribus, *the ranges on any plane are as the ſquares of the initial velocities.* For, every magnitude of velocity has an impetus AB correſponding to it: and, as the height due to any velocity increaſes or diminiſhes in the duplicate ratio of that velocity, it is obvious that all the ranges with a given direction will vary in the ſame ratio.

252. Prop. *To exhibit the relations between all the circumſtances of velocity, elevation, poſition, range, greateſt height, and time.*

Suppoſe a ball or ſhell projected from A (fig. 12. pl. XI.) with the velocity due to the height CA, in order to ſtrike the mark B ſituated in the given line AB. Make AZ = 4AC, and draw BD parallel to it, or perpendicular to the horizon. On ZA deſcribe a circular ſegment ZAD containing an angle ZDA = ∠ DBA, and draw AD to the interſection of this circle with BD: then will a body projected from A, in the direction AD, with the velocity due to the height CA, ſtrike the object B. For, produce CA downwards till BF drawn parallel to DA meets it in F; and join ZD: then, it is manifeſt from the conſtruction that the angles ADZ and DBA are equal, as likewiſe AZD and DAB; that AB touches the circle in A, and that the triangles ZAD, ADB, are ſimilar.

$$\text{Hence BD} : \text{DA} :: \text{DA} : \text{AZ}$$

$$\text{And DA}^2 = \text{BD} \times \text{AZ}$$

$$\text{Conſequently BF}^2 = \text{AF} \times \text{AZ} = \text{AF} \times 4\ \text{AC}:$$

Therefore a parabola which has AF for a diameter, and AZ its parameter, will paſs through B, and this parabola will be the path of the projectile.

But when BD cuts the circle ZDA, it cuts in two points D, d; ſo that there are two directions which will ſolve the problem. If B′ D′ only touches the circle in D′ there is but one direction, and AB′ is the maximum range with this velocity. If the vertical through B does not meet the circle, the problem is impoſſible; the initial velocity being too ſmall. When B′ D′ touches the circle, the two directions AD′, A d', coaleſce into one, producing the greateſt range, and biſecting the angle ZAB; and the other two directions AD, A d, producing the ſame range AB, are equidiſtant from AD′; agreeably to Nos. viii, ix, of the preceding article.

Now, to deduce the relations acquired, let us confine our attention to the direction AD in the figure; where we ſee

That AZ : AD : : sin. ADZ : sin. AZD : : sin. DBA : sin. DAB : : sin. P : sin. E
AD : DB : : sin. DBA : sin. DAB : : .... sin. P : sin. E
DB : AB : : sin. DAB : sin. ADB : : .... sin. E : sin. Z
Theref. comp. AZ : AB : : $\text{sin.}^2\text{P} \times \text{sin. E} : \text{sin.}^2\text{E} \times \text{sin. Z} :: \text{sin.}^2\text{P} : \text{sin. E} \times \text{sin. Z}$.

$$\text{Or } 4\,\text{I} : \text{R} :: \text{ſin.}^2\text{P} : \text{ſin. E} \times \text{ſin. Z},$$

$$\text{whence } 4\,\text{I} \times \text{ſin. E} \times \text{ſin. Z} = \text{R} \times \text{ſin.}^2\text{P}.$$

This equation obviouſly compriſes the relations of E, I, P, R, and Z.

Let us next determine the time of flight; which is equal to the time of deſcribing AD uniformly, or of falling through DB. Now, ſince AB : DB : : ſin. ADB : ſin. DAB : : ſin. Z : ſin. E, we have $\text{DB} = \frac{\text{R} \times \text{sin. E}}{\text{sin. Z}}$. Here DB is equal to s in the general equation $s = \frac{1}{2} g t^2$ (art. 243.): therefore $\frac{\text{R} \times \text{sin. E}}{\text{sin. Z}} = \frac{1}{2} g t^2$; and conſequently $t = \sqrt{\frac{\text{R} \times \text{sin. E}}{\frac{1}{2} g \times \text{sin. Z}}}$.

Laſtly, to find the greateſt height (H) of the projectile above the plane; draw from Q the middle point of AB, the line QT t parallel to BD d: then, by a well-known property of the parabola, QP=PT, or QT=2 QP; and by ſimilar triangles AQ : QT : : AB (=2 AQ) : BD=2 QT=4 QP=4 H. But AB : BD : : ſin. Z : ſin. E : : R : 4 H; conſequently 4 H × ſin. Z=R × ſin. E, and $\text{H} = \frac{\text{R} \times \text{sin. E}}{4\,\text{sin. Z}}$.

Let us now take from the general equations, art. 243. the value of $v = \sqrt{2 g s}$, from which, becauſe s=I in the preſent caſe, and v=V, we ſhall have $\text{V} = \sqrt{2 g \text{I}}$; and comparing this with the preceding equations in this article, we may throw together the general theorems relating to projectiles thus:

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$$\text{R} = \frac{\text{sin E} \times \text{sin. Z}}{\text{sin.}^2\text{P}}\, 4\,\text{I} = \frac{\text{sin. E} \times \text{sin. Z}}{\frac{1}{2} g \times \text{sin.}^2\text{P}}\, \text{V}^2 = \frac{g \times \text{sin. Z}}{2\,\text{sin. E}}\, \text{T}^2 = \frac{4\,\text{sin. Z}}{\text{sin. E}}\, \text{H}.$$

$$\text{V} = \sqrt{2 g \text{I}} = \text{ſin. P} \sqrt{\frac{g \text{R}}{2\,\text{sin. E} \times \text{sin. Z}}} = \frac{g \times \text{sin. P}}{2\,\text{sin. E}}\, \text{T} = \frac{2\,\text{sin. P}}{\text{sin. E}} \sqrt{\tfrac{1}{2} g \text{H}}.$$

$$\text{T} = \frac{2\,\text{sin. E}}{\text{sin. P}} \sqrt{\frac{2\,\text{I}}{g}} = \frac{2\,\text{sin. E}}{g \times \text{sin. P}}\, \text{V} = \sqrt{\frac{2\,\text{sin. E} \times \text{R}}{g \times \text{sin. Z}}} = \sqrt{\frac{2\,\text{H}}{g}}.$$

$$\text{H} = \frac{\text{sin. E}}{4\,\text{sin. Z}}\, \text{R} = \frac{\text{sin.}^2\text{E}}{\text{sin.}^2\text{P}}\, \text{I} = \frac{\text{sin.}^2\text{E}}{2\,g \times \text{sin.}^2\text{P}}\, \text{V}^2 = \tfrac{1}{8} g\, \text{T}^2.$$

$$\text{I} = \frac{\text{sin.}^2\text{P}}{4\,\text{sin. E} \times \text{sin. Z}}\, \text{R} = \frac{\text{V}^2}{2 g} = \frac{g \times \text{sin.}^2\text{P}}{8\,\text{sin.}^2\text{E}}\, \text{T}^2 = \frac{\text{sin.}^2\text{P}}{\text{sin.}^2\text{E}}\, \text{H}.$$

And from many of theſe the angle of elevation may readily be found.

253. When the plane AB is horizontal, the angle DBA=P =90°, and its ſine radius; alſo angle ZAD=Z, becomes= comp. E; therefore, ſince ſin. E × cos. E = 2 ſin. 2 E, ſin. ^{2}E = 2 vers. ſin. 2 E, and ſin. ÷ cos. = tan. the preceding theorems will aſſume the following ſhape when applied to

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$$R=\text{ſin. }E\times\text{cos. }E\times 4\,I=\text{ſin. }2\,E\times 2\,I=\frac{\text{sin.}2\,E}{g}V^2=\frac{g\,T^2}{2\,\text{tan.}E}=\frac{4\,H}{\text{tan. }E}.$$

$$V=\sqrt{2\,g\,I}=\sqrt{\frac{g\,R}{\text{sin. }2\,E}=\frac{g\,T}{2\text{ sin. }E}=\frac{2}{\text{sin. }E}}\sqrt{\tfrac{1}{2}\,g\,H}.$$

$$T=2\,\text{ſin. }E\sqrt{\frac{2\,I}{g}}=\sqrt{\frac{2\,R\times\text{tan. }E}{g}}=\frac{2\text{ sin. }E}{g}V=2\sqrt{\frac{2\,H}{g}}.$$

$$H=\tfrac{1}{4}\,\text{tan. }E\times R=\text{ſin. }^2E\times I=\frac{\text{vers. sin. }2\,E}{4\,g}V^2=\tfrac{1}{8}\,g\,T^2.$$

$$I=\frac{R}{8\,\text{sin. }2\,E}=\frac{V^2}{2\,g}=\frac{g}{16\,\text{vers. sin. }2\,E}T^2=\frac{H}{\text{sin. }^2E}=\frac{H}{2\text{ vers. sin. }2\,E}.$$

And from many of theſe again the angle of elevation may ſoon be found. Here too, as in the inſtance of falling bodies, *g* may be taken = 32, without leading to any important error.

Cor. 1. In horizontal ranges, if E=45°, we have R=4 H= 2 I: conſequently, *the maximum range on a horizontal plane is equal to double the impetus, or to four times the greateſt height of the projectile above the plane;* the impetus being, in that caſe, double the greateſt height.

Cor. 2. When E = 45°, its ſine is $=\sqrt{\tfrac{1}{2}}$; therefore $T=2\sqrt{\frac{I}{g}}=\tfrac{1}{2}\sqrt{\tfrac{1}{2}}\,I$ nearly $={}^{\cdot}35356\sqrt{I}$ nearly.

Cor. 3. When the initial velocity is given in feet, and E= 45°, we have T^{s} nearly $=\frac{\sqrt{\frac{1}{2}}}{16}V={}^{\cdot}044194\,V$.

The expreſſions in this and the former corollary may be uſeful to the bombardier in cutting the fuzes of his ſhells to proper lengths, ſo that they may juſt burſt at the very inſtant they come to the object fired at.

Cor. 4. When the elevation is 15°, we have ſin. 2 E= ſin. 30° = ½: conſequently R = ſin. 2 E × 2 I = ½ × 2 I = I. That is, *at an elevation of* 15°, *the range on a horizontal plane is equal to the impetus.*

Cor. 5. *When the elevation is 75° the horizontal range is equal to the impetus.* For 15° and 75° are equidiſtant from 45°, and therefore the ranges at thoſe elevations are equal.

Cor. 6. In many caſes, where in appearance a body has not received any impulſion, but is abandoned to the ſole action of gravity, the body, notwithſtanding, deſcribes a parabola like a projectile. This, for example, is the caſe with a body let fall from the top of a ſhip's maſt when the veſſel is in motion. If we notice the point on the deck where the body falls, we ſhall find it no further diſtant from the maſt than it would have been had it fallen from the top when the ſhip was at anchor; ſo that with regard to the maſt the body has deſcribed a right line parallel to it: but with reſpect to a ſpectator ſituated at reſt out of the veſſel, it has really deſcribed a parabola, abſtracting, as in the caſe of projectiles, from the reſiſtance of the air. For, when the body is abandoned to the action of gravity, it has the ſame velocity as the veſſel has, ſince they were previouſly moving on together: its actual motion is, therefore, the ſame as though the veſſel were immoveable, and the body were projected horizontally with a velocity equal to that with which the veſſel moves, and in the ſame direction.

Other corollaries which flow naturally from the preceding propoſitions and theorems may be deduced by the ſtudent at his leiſure.

On the Ricochet.

254. Def. The *Ricochet* is a motion by which a projectile, after having ſtruck any obſtacle whatever, is ſo reflected as to recommence a motion ſimilar to that which it had at firſt. The word ſignifies duck-and-drake, or rebounding; becauſe when a ball has this motion it goes bounding along, ſtriking the ground ſeveral times, like the bounding of a flat ſtone along the ſurface of water, when thrown almoſt horizontally. In *Ricochet* firing the ball is projected with ſmall charges, and at elevations generally between 3 and 7 degrees. What little we ſhall advance on this ſpecies of projectile motion will be chiefly illuſtrative.

I. The leſs the angle of elevation at which the ball is projected, the more *(cæteris paribus)* is it in the ſtate for the ricochet: for then the force of projection being exerciſed more entirely in the horizontal direction, will take much more time before it is extinguiſhed by the reſiſtance of the air and other obſtacles. If the projectile were entirely void of elaſticity, and the ſurface on which it falls were horizontal, and inflexible, there would then be no ricochet: becauſe, the velocity of the projectile when it arrives at C (fig. 1. pl. XII.) according to any direction MC, may be decompoſed into two others: of which the one QC perpendicular to the ſurface will be ſimply deſtroyed without any reſtitution, the body having no ſpringineſs; while the other velocity PC ſubſiſts alone, (abſtracting from friction and the air's reſiſtance), and the body, inſtead of bounding, will ſlide along CZ.

II. But if there ſhould be an eminence CE (fig. 2. pl. XII.) at the point C where the body ſtrikes the ſurface, the motion according to MC will then be decompoſed into a motion QC perpendicular to the ſurface CE of this obſtacle, and another PC in the direction of that ſurface, by which the moveable will advance in the direction PE, and may deſcribe after quitting the point E a new curve of the ſame nature with that which it would deſcribe if it were projected from the gun at E with the ſame velocity; and this in ſuch manner, that after it is elevated to a certain point it may again deſcend and ſtrike the earth in a new point I, where it may recommence a ſimilar motion if the circumſtances are ſimilar: and ſo on.

III. If the obſtacle is flexible or moveable, as the earth, water, &c. there may be a ricochet even when the ſurface is perfectly horizontal. For the moveable tends by its vertical velocity QC (fig. 3. pl. XII.) to plunge itſelf more or leſs, according to the nature of the obſtacle; while with the velocity repreſented by PC it ploughs its way, and forms a furrow of which the depth continues to increaſe till the vertical velocity QC is extinguiſhed. Then, by virtue of the velocity remaining in the horizontal direction, it thruſts before it the matter which oppoſes it, and becomes gradually turned towards the ſide where it has the leaſt reſiſtance: thus, in its return from its greateſt depth, the cavity of the furrow operates, with reſpect to the body, in the ſame manner as the obſtacle CE (fig. 2.) in the preceding caſe. Now as the facility of the eſcape from the earth or other obſtacle is the greater, (*cæt. par.*) as the total depth of the furrow is leſs, and as this depth depends on the vertical velocity QC, which will be always ſmaller as the angle MCP is ſmaller, or as the angle of projection is ſmaller, we ſee how the facility of the ricochet depends upon the angle of projection being ſmall.

IV. The ricochet likewiſe depends much upon the figure of the projectile. If we want, for example, a ricochet on the water, and the projectile were globular; in order to this the velocity MC muſt be ſuch that the vertical velocity QC may be entirely conſumed before the vertical diameter of the ſphere is completely immerſed: for, if once it is covered with water, the reſiſtance of the fluid will act equally on all ſides the direction of the projectile, in ſuch a manner that it can only be turned in conſequence of the action of gravity, which will, therefore, be a direct impediment to the ricochet.

V. As the plunging can only be made ſucceſſively or gradually, it is manifeſt that during the time in which it is going on the centre of the ball will deſcribe a curve line; for the direction in which the reſiſtance is made changes continually. If, for example, when the centre C (fig. 10. pl. XI.), after having deſcribed any track whatever PC, tends to move according to the prolongation CI of its actual direction, we conceive two tangents BR, DS, parallel to that direction; it is evident that only the part BVL is ſubjected to the reſiſtance; and that, the body being ſpherical, the reſultant CK of all the reſiſtances offered to the different points of the ſurface BVL has a direction which tends to elevate the body above CI; in ſuch manner that CIEK may be the parallelogram of forces, and CE will be the path the body will take, inſtead of CI, abſtracting from the effect of gravity.

VI. Laſtly, if both the projectile and the obſtacle are flexible, or both ſpringy, theſe circumſtances will ſtill further contribute to the facility of the ricochet. To take a very ſimple inſtance—Suppoſe that the projectile only is flexible, and that it is perfectly elaſtic, or reſtores itſelf with a force equal to that of compreſſion; ſtill abſtracting from the conſideration of gravity. At the inſtant in which the body moving in the direction AC (fig. 4. pl. XII.) touches the ſurface, its velocity is reſolved into a horizontal velocity QC, which ſubſiſts always the ſame (diſregarding friction), and the reſiſtance of the medium in which the body moves: but the vertical velocity PC tends gradually to compreſs the body itſelf, becoming gradually extinguiſhed, while that repreſented by QC ſubſiſts unchanged; ſo that it is clear that the centre C approaches the plane HZ by degrees which are continually diminiſhing, while the degrees by which it advances parallel to that plane remain conſtant: if, therefore, we conceive at every inſtant a parallelogram of forces conſtructed, of which the horizontal ſide is to the vertical ſide as the horizontal velocity to that which remains in the vertical direction, the diagonal of this parallelogram, which will indicate for each inſtant the route of the centre C, will be differently ſituated, ſo that the centre will approach to ZH by deſcribing a curve line CR during the time of the compreſſion: when the compreſſion ceaſes the centre C will move for an inſtant on a tangent parallel to HZ; after which the elaſtic force of the body counteracting that of compreſſion reſtores to the body gradually the velocity by which it moves from the plane, in the inverſe order of the degrees by which it approached the plane; and ſo cauſes its centre to deſcribe the ſecond branch RO of the curve, equal and ſimilar to CR. Finally, when it has arrived at the point O, whoſe diſtance from the plane HZ is equal to the radius CI, it will move on according to the tangent OT ſituated in the ſame manner as AC. Hence, then, the oblique ſtroke of an elaſtic body upon an inflexible and immoveable plane is made (abſtracting from gravity) in ſuch manner that the angle of reflexion is equal to the angle of incidence, theſe two angles being meaſured by thoſe contained between the horizontal plane and the tangents at the extremities C and O of the curve deſcribed by the centre during the compreſſion and the reſtitution of the ſpring; and this curve is always the ſmaller as the compreſſion and reſtitution approach the more nearly to being inſtantaneous.

VII. If we pay regard to gravity, and BD is the line according to which the body is projected, it will deſcribe the portion DC of the parabola whoſe tangent is BD juſt when the body touches the plane; and when the compreſſion, &c. has ceaſed and the centre arrives at O, it will deſcribe another portion SO of a parabola, equal to the former, and ſimilarly ſituated.

VIII. Friction likewiſe contributes to the facility of the Ricochet; becauſe it gives to the moveable a rotation which enables it to ſurmount more eaſily the different obſtacles it meets with. Such are the cauſes and the principal circumſtances of the ricochet: we have attended to them rather minutely, becauſe the principles here applied may often be directed by the ſtudent to other motions which frequently occur.

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255. The theory of the motion of projectiles juſt exhibited depends upon three ſuppoſitions, all of which are inaccurate. 1ſt. That the force of gravity is the ſame in every point of the curve. 2dly. That it acts in parallel lines. 3dly. That the body moves in a non-reſiſting medium. Of theſe ſuppoſitions, however, the two firſt, as we obſerved in art. 248. produce no error which deſerves notice: but the third is a ſource of conſiderable difference between this theory and experiments, particularly when the initial velocity of the projectile is great. The reſiſtance of the air is variable, according to the different velocities and magnitudes of the projectiles: on this account the trajectory of the projectile is not a parabola, nor any known and regular curve; its vertex is not in the middle, but more remote from the point of projection than from the other extremity: the time of deſcent alſo, though through a leſs portion of the curve, is longer than the time of aſcent; and the part of the curve through which the body deſcends is leſs curved than that through which it aſcends. Theſe circumſtances are very perceptible to the ſight in the motion of ſtones, arrows, balls, and ſhells; and even in a jet of water or mercury we may trace the ſame particulars, unleſs the velocity be ſmall, when the path

nearly coincides with a parabola. Beſides this, a body projected with any conſiderable velocity is not only deflected from a parabolic path in a vertical direction, but is made to deviate laterally, and change the plane of motion: in ſome experiments indeed this deviation has been equal to $\frac{1}{6}$ or $\frac{1}{5}$ of the actual range. Such material diſcrepancies between the theory and practice have induced ſeveral philoſophers, at different times, to inſtitute courſes of experiments, in order to improve the theory by a compariſon with their reſults: the moſt extenſive and important of theſe are the experiments by Meſſrs. Robins, Thompſon, and Dr. Hutton; for accounts of them the reader may advantageouſly conſult Robins's Gunnery, or Brown's Tranſlation of Euler's Gunnery, Phil. Tranſ. for 1778 and 1781, and Hutton's Tracts, vol. I. Dr. Hutton has, likewiſe, touched upon this ſubject in the ſecond volume of his Courſe of Mathematics; and as what he has there given contains much practical information in a ſmall compaſs, we ſhall extract an article or two, as peculiarly appropriate to the ſubject before us.

256. "Before the propoſitions can be applied in reſolving the ſeveral caſes in the practice of gunnery, it is neceſſary that ſome more data be laid down, as derived from good experiments made with balls or ſhells diſcharged from cannon or mortars, by gunpowder, under different circumſtances. For without theſe thoſe theorems can be of very little uſe in real practice, on account of the imperfections and irregularities in the firing of gunpowder, and the expulſion of balls from guns, but more eſpecially on account of the enormous reſiſtance of the air to all projectiles that are made with any conſiderable velocities. As to the caſes in which projectiles move with ſmall velocities, or ſuch as do not exceed 200, or 300, or 400 feet per ſecond, they may be reſolved tolerably near the truth by the parabolic theory laid down above. But in caſes of great projectile velocities it is quite inadequate, without the help of ſeveral data drawn from many and good experiments. For ſo great is the effect of the reſiſtance of the air to projectiles of conſiderable velocity, that ſome of thoſe which in the air range only between 2 and 3 miles at the moſt would in vacuo range about ten times as far, or between 20 and 30 miles.

"The effects of this reſiſtance are alſo various, according to the velocity, the diameter, and the weight of the projectile. So that the experiments made with one ſize of ball or ſhell will not ſerve for another ſize, though the velocity ſhould be the ſame; nor will the experiments with one velocity ſerve for other velocities, though the ball be the ſame. And therefore it is plain that, to form rules for practical gunnery, we ought to have good experiments made with each ſize of mortar, and with every variety of charge, from the leaſt to the greateſt. And not only ſo, but theſe ought alſo to be repeated at many different angles of elevation, namely, for every ſingle degree between 30° and 60° elevation, and at intervals of 5° above 60° and below 30°, from the vertical direction to point-blank. By ſuch a courſe of experiments it will be found that the greateſt range, inſtead of being conſtantly that for an elevation of 45°, as in the parabolic theory, will be at all intermediate degrees between 45° and 30°, being more or leſs both according to the velocity and the weight of the projectile; the ſmaller velocities and larger ſhells ranging fartheſt when projected almoſt at an elevation of 45°; while the greateſt velocities, eſpecially with the ſmaller ſhells, range fartheſt with an elevation of about 30°.

257. "There have, at different times, been made certain ſmall parts of ſuch a courſe of experiments as is hinted at above. Such as the experiments or practice carried on in the year 1773, on Woolwich Common; in which all the ſizes of mortars were uſed, and a variety of ſmall charges of powder. But they were all at the elevation of 45°; and conſequently theſe are defective in the higher charges, and in all the other angles of elevation.

"Other experiments were alſo carried on in the ſame place in the years 1784 and 1786, with various angles of elevation indeed, but with only one ſize of mortar, and one only charge of powder, and that but a ſmall one too: ſo that all thoſe nearly agree with the parabolic theory. Other experiments have alſo been carried on with the balliſtic pendulum, at different times; from which have been obtained ſome of the laws for the quantity of powder, the weight and velocity of the ball, the length of the gun, &c. Namely, that the velocity of the ball varies as the ſquare root of the charge directly, and as the ſquare root of the weight of ball reciprocally; and that ſome rounds being fired with a medium length of one-pounder gun, at 15° and 45° elevation, and with 2, 4, 8, and 12 ounces of powder, gave nearly the velocities, ranges, and times of flight, as they are here ſet down in the following Table.

Powder.	Elevation of gun.	Velocity of ball.	Range.	Time of flight.
oz.		feet.	feet.	
2	15°	860	4100	9″
4	15	1230	5100	12
8	15	1640	6000	14½
12	15	1680	6700	15½
2	45	860	5100	21

"But as we are not yet provided with a ſufficient number and variety of experiments on which to eſtabliſh true rules for practical gunnery, independent of the parabolic theory, we muſt content ourſelves with the data of ſome one certain experimented range and time of flight at a given angle of elevation; and then by help of theſe and the rules of the parabolic theory, determine the like circumſtances for other elevations that are not greatly different from the former."

III. *Deſcents on Inclined Planes, and Curve Surfaces.*

258. PROP. *The force which accelerates or retards the motion of a body upon an inclined plane is to the force of gravity, as the height of the plane to its length, or as the ſine of the plane's elevation to radius.*

Let AD (fig. 6. pl. XII.) be the plane, of which the height is BD, and the length AD; and let C be the place of a body upon the plane. From C draw the vertical CG to repreſent g the accelerating or retarding force of gravity, according as the body is ſuppoſed to be moving downwards or upwards, and complete the parallelogram CEGF: now the force CG may obviouſly be reſolved into the two CF, CE, of which the former being in the direction perpendicular to the plane is extinguiſhed by its reaction, but the latter being parallel to the plane has its entire effect, and is that which either directly co-operates with or oppoſes the motion of the body along the plane. But the triangles ADB, GCE, are ſimilar, the ſides of the latter being reſpectively perpendicular to the ſides of the former: therefore, GC : CE : : DA : DB : : Rad. : ſin. DAB. And, if we ſuppoſe the body at any other point C′, the reſolution of the forces will again furniſh the ſame reſult. Whence the propoſition is manifeſt.

COR. 1. Since the accelerating or retarding force, on the ſame plane, is in a given ratio to that of gravity, which is a uniform force, it alſo is a uniform force.

COR. 2. Hence the laws before laid down for accelerated and retarded motions hold good for thoſe on inclined planes: thus, in deſcending along the ſame plane, or planes equally inclined, the velocities acquired are as the ſpaces deſcended from quieſcence; the ſpaces deſcended are as the ſquares of the velocities, or as the ſquares of the times; and, if a body be thrown up an inclined plane, with the velocity it acquired in deſcending, it will loſe all its motion, and will aſcend to the height from which it deſcended in an equal time, and will repaſs any point of the plane with the ſame velocity as it paſſed it in deſcending.

COR. 3. If h repreſent the height and l the length of the plane, the accelerating or retarding force will be repreſented by $\frac{h}{l}\times g$; or, if the angle DAB be denoted by e, that force will be repreſented by $g.\ \text{ſin.}\ e.$ Hence if we ſubſtitute this for g, in the values of s and v, in art. 243. they will become $s=g.\ \text{ſin.}\ e.\ \frac{1}{2}t^2$, and $v=g.\ \text{ſin.}\ e.\ t=\sqrt{2g\ s.\ \text{sin.}\ e}$: and the equations for t will undergo ſimilar modifications. Theſe will enable us to deduce ſome uſeful theorems as further corollaries.

Cor. 4. *If two bodies deſcend from the higheſt point* D *of the plane at the ſame inſtant, one deſcending along the plane* DA, *the other through the vertical* DB, *their contemporeous poſitions* C *and* I *will always be ſhewn by the interſections of a line as* CI *with the two planes, this line being conſtantly perpendicular to* DA. This will appear by comparing the equation $s=g.\ \text{ſin.}\ e.\ \frac{1}{2}t^2.$ for the motion down the plane, with the correſponding equation $s'=\frac{1}{2}g\,t'^2$ expreſſing the vertical motion: for, if we divide one of theſe equations by the other, we have $\frac{s}{s'}=\frac{g.\ \text{sin.}\ e.\ \frac{1}{2}t^2}{\frac{1}{2}g\,t'^2}$ which becauſe $t=t'$, gives $s=s'\ \text{ſin.}\ e$; whence $s' : s :: 1 : \text{ſin.}\ e$, that is, DI : DC : : Rad. : ſin. DAB, which indicates that DI is the hypothenuſe of a right-angled triangle, whoſe angle I is equal to the elevation of the plane.

Cor. 5. *If the diameter of a circle be perpendicular to the horizon, and chords be drawn from either extremity, the times of deſcent down all the chords will be equal, and each equal to the time of free deſcent through the vertical diameter.* Thus in fig. 6. pl. XII. the angle at C being a right angle, the time of running through AC on the inclined plane is (by the preceding corol.) equal to the time of falling through AB; and, ſince D and E are right angles, the ſame will be true of the deſcents through AD, and AE: alſo, if AI be drawn parallel to CB, the time of deſcent through AI will, in like manner, be equal to that through AB; and becauſe AI=CB, and have equal elevations, they will be deſcribed in equal times: and the ſame may be ſhewn of DB, EB, &c.

Cor. 6. If, from any point A in a vertical plane, any number of lines AC, AD, AE, AF, AG, &c. be drawn in that plane, and bodies be let fall from quieſcence at A along theſe lines, then at the end of any equal times whatever theſe bodies will all be found in the circumference of a circle paſſing through A (fig. 7. pl. XII.) Thus, when the body falling vertically is at E, the others will be at C, D, F, G, reſpectively: again, when this body is at e, the others will be at c, d, f, g, in the circumference of a circle touching the former in A; and, when this body is at B, the others will be at c', d', f', g', in a third circle touching the other two in A: and ſo on. All which is manifeſt from Cor. 5.

COR. 7. And if the lines diverging from A are not confined to one plane, but directed all around, then bodies deſcending along theſe lines will at every ſeparate inſtant of time be all found in the ſurface of ſome one of a ſeries of ſpheres, which touch one another in the ſuperior point A.

COR. 8. *If any line* CH (fig. 5. pl. XII.) *be drawn parallel to the horizon, and if two bodies move from quieſcence at* D, *the one along the inclined plane* DA, *the other through the vertical* DB, *the velocity acquired by the one at* C *will be equal to that acquired by the other at* H; *thoſe velocities being eſtimated by the directions of their reſpective motions.* For the equation compriſing the conditions of velocity of the body moving along DA is $v = \sqrt{2gs \,.\, \sin. e}$ (Cor. 3.), and the correſponding equation for the vertical deſcent is $v' = \sqrt{2gs'}$. But in the preſent inſtance, $v = v'$, therefore $2gs \,.\,$ ſin $e = 2gs'$, or $s \,.\,$ ſin. $e = s'$; conſequently $s' : s :: $ ſin. $e : 1$, that is, DH : DC : ſin. DAB : rad.

Hence CH is parallel to the horizontal line AB.

COR. 9. Hence, ſince the ſame might be ſhewn of the correſponding point C′ of any other plane A D, *if ſeveral bodies are let fall from one point* D, *without any impulſion, and run over different planes, they will all have acquired the ſame velocity when they arrive at any horizontal plane.*

COR. 10. *If two or more bodies have equal velocities at any equal altitudes* C, H, C′, *their velocities will be equal at all other equal altitudes,* A, B, A′.

COR. 11. *Hence, alſo, the velocities acquired by deſcending down any planes are as the ſquare roots of their heights* DB. Thus, the velocity at A, being the ſame as the velocity at B, is as $\sqrt{\text{DB}}$.

COR. 12. *The time of deſcent along* DA *is to the time of deſcent through the vertical* DB *as* DA *to* DB. Here DA $= s$, DB $= s'$, and the equations are $s = g$, ſin. $e \,.\, \frac{1}{2} t^2$, and $s' = \frac{1}{2} g t^2$; whence, if we convert theſe into an analogy, we have $t^2 : t'^2 :: \frac{s}{\sin. e} : s' :: \frac{\text{DA}}{\sin. e} : \text{DB}$. But, in the triangle DAB, $\frac{\text{DB}}{\text{DA}} =$ ſin. e; therefore, $t^2 : t'^2 :: \text{DA}^2 : \text{DB}^2$, or $t : t' :: \text{DA} : \text{DB}$.

COR. 13. As the above proportion obtains whatever is the inclination of the plane, it reſults, that *the times employed to paſs over various inclined planes of the ſame height are reſpectively as the lengths of the planes.*

259. PROP. *If a body deſcend along any number of contiguous planes, it will ultimately acquire the ſame velocity as would have*

been acquired by falling perpendicularly through the height of the ſyſtem; ſuppoſing no velocity loſt on paſſing from one plane to another.

Let AB, BC, CD, (fig. 8. pl. XII.) be the contiguous planes: produce DC, CB, the two lower till they meet AF parallel to the horizon in F and E reſpectively. Then, by Cor. 9. of the preceding prop. the velocity at B is the ſame whether the body deſcend through AB or EB: and therefore the velocity at C will be the ſame whether the body fall through ABC or EC, which is alſo, by the ſame corol. equal to the velocity acquired by falling through FC. Conſequently the velocity will be the ſame at D, on this hypotheſis, whether the body paſſed over the ſyſtem ABCD, or through the ſingle plane FD, or, laſtly, through the vertical FG.

Cor. Hence, comparing this prop. with corol. 2. and 11. of the preceding, it follows that the velocities acquired in falling down any ſyſtems of planes are as the ſquare roots of their vertical altitudes; and that, if the body be projected from D up the planes with the velocity acquired, it will aſcend through this or any other ſyſtem of planes (no part of which is *reclining*) to the ſame perpendicular height from D.

260. Prop. *When a body moves over a ſyſtem of planes, the velocity loſt in paſſing from any plane to the ſucceeding one is to the velocity it had then acquired as the verſed ſine of the angle made by the planes to radius.*

Conceive that the body after moving over the plane AB (fig. 9.) has ſuch a velocity as would carry it over the ſpace BF in the next ſecond, or unit of time, if the motion were not obſtructed: from F draw the line FD perpendicular to the ſecond plane BC, and complete the parallelogram DE, then will the velocity BF be reſolvable into the two BE, BD; of which the former is evidently deſtroyed by the reſiſtance of the plane, and of conſequence the velocity BF will be reduced to the velocity BD on the plane BC. Now, with centre B and radius BF deſcribe the arc FI, ſo ſhall DI=BF−BD= the velocity loſt at the angle B; and it is obvious that DI : BF : : vers. ſin. DBF : rad. As in the propoſition.

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261. When a body falls freely by the continual ſolicitation of gravity, every particle in it is equally accelerated, or every particle deſcends towards the horizon with the ſame velocity, and, therefore, no rotation will be given to the body. The ſame may be ſaid when a body deſcends along a perfectly ſmooth inclined plane, if that part of the force which is in a direction perpendicular to the plane be ſupported; that is, if a perpendicular to the plane drawn from the centre of gravity

of the body pierce the plane in a point which is in contact with the body. But if this part of the force be not ſuſtained by the plane, the body will partly roll and partly ſlide, till this force is ſuſtained; and then the body will ſlide wholly. When the lateral motion is entirely prevented by the adheſion of the plane, the body will keep at reſt if the vertical from its centre of gravity fall within its baſe; but if it fall below that baſe the body will roll. If the adheſion be not ſufficient to prevent all lateral motion, the body will partly roll and partly ſlide. In all theſe caſes the ſpace deſcribed, the time of motion, &c. muſt be aſcertained by means of other principles; the principal, &c. of which will be explained in the chapter on Rotation (art. 315.).

262. Prop. *The times of deſcribing two ſyſtems of inclined planes* ABCD, *a b c d* (fig. 8. pl. XII.), *whoſe number, inclinations, and ratio of their lengths, are the ſame, are to each other as the ſquare roots of the lengths of the ſyſtems.*

Becauſe the planes are equally inclined to the direction of the force, the time of running through AB is to that through *ab* as $\sqrt{AB}$ to $\sqrt{ab}$. (art. 258. cor. 11.) And if bodies fall down EC, *ec*, then will time in EC : time in *ec* : : $\sqrt{EC}$: $\sqrt{ec}$: : $\sqrt{AB}$: $\sqrt{ab}$; alſo, time in EB : time in *eb* : : $\sqrt{EB}$: $\sqrt{eb}$: : $\sqrt{AB}$: $\sqrt{ab}$; therefore, time in EC : time in *ec* : : time in EB : time in *eb*; and hence, the remainder, or time in BC : remainder, or time in *bc* : : $\sqrt{AB}$: $\sqrt{ab}$, by Euc. V. 19. And ſince (by hyp.) no motion is loſt in paſſing from one plane to another, the times of paſſing over BC, and *bc*, are the ſame, whether the bodies begin to move at A and *a*, or at E and *e*: ſo that when the bodies deſcend along ABC, *abc*, it will be time in BC : time in *bc* : : $\sqrt{AB}$: $\sqrt{ab}$. In the ſame manner it may be ſhewn that time in CD : time in *cd* : : $\sqrt{AB}$: $\sqrt{ab}$. Hence, time in AB : time in *ab* : : time in BC : time in *bc* : : time in CD : time in *cd*; and conſequently the time in AB + BC + CD : time in *ab* + *bc* + *cd* : : time in AB : time in *ab* : : $\sqrt{AB}$: $\sqrt{ab}$: : $\sqrt{AB+BC+CD}$: $\sqrt{ab+bc+cd}$.

Cor. 1. If the lengths of the planes, and their angles of inclination ABE, BCF, be diminiſhed indefinitely, the limits, to which theſe ſyſtems approximate, are ſimilar curves, ſimilarly poſited, in which (as will be ſeen in the next prop.) no velocity is loſt: hence *the whole times of deſcent through theſe curves will be as the ſquare roots of their lengths.*

Cor. 2. *The times of deſcents along ſimilar circular arcs, ſimilarly ſituated, are as the ſquare roots of the arcs, or as the ſquare roots of the radii of their reſpective circles.*

263. Prop. *If a body fall from quieſcence down a curve ſurface which is perfectly ſmooth, the velocity acquired is equal to that*

which would be acquired in falling through the ſame perpendicular height.

For, by art. 260. when a body paſſes from one plane AB (fig. 9.) to another BC, the whole velocity : the velocity loſt : : radius : verſed ſine of DBF. Conſequently, when the angle FBD is diminiſhed *ſine limite*, the velocity loſt is diminiſhed *ſine limite:* and if the lengths of the planes as well as their angles of inclinations be indefinitely diminiſhed, the ſyſtem approximates to a curve as its limit, in which there is no velocity loſt: wherefore, the whole velocity acquired will be equal to that which a body would acquire in falling through the ſame vertical altitude. Art. 259.

Or, if we apply the fluxional method to the known properties of curves in general, the ſame thing may be ſhewn thus. Suppoſe the body moves from quieſcence at C, (fig. 10. pl. XII.) and that at the end of the time t the conſtant action of gravity has brought it to the point M, in the ſame vertical plane YAX. On the horizontal axis AX let $AP=X$, and on the vertical AY let $AB=h$, the right ordinate to the point M, that is, $PM=Y$, and the ſpace deſcribed $CM=s$; then will the velocity of the moveable in the direction of the element of the curve at M be $\frac{\dot{s}}{\dot{t}}$. Gravity acting upon the body at M in the direction MP, its force muſt be decompoſed into two others, the one perpendicular to the ſurface in M will be annihilated by its reaction, the other will be in the direction of the tangent at M: and as the coſine of the angle which this tangent makes with AX is $\frac{\dot{y}}{\dot{s}}$, or rather $-\frac{\dot{y}}{\dot{s}}$, ſince s increaſes as y decreaſes, the component of the gravitating force in the direction of the tangent will be $-g\frac{\dot{y}}{\dot{s}}$; thus we have $\frac{\dot{v}}{\dot{t}}=-g\frac{\dot{y}}{\dot{s}}$. From this we find $\frac{\dot{s}}{\dot{t}}\dot{v}=-gy$, or $v\dot{v}=-g\dot{y}$, becauſe $v=\frac{\dot{s}}{\dot{t}}$ (art. 232. I.). Taking the fluent we have $\frac{1}{2}v^2=-gy+C$. Now, at the point C, $v=0$, and $y=h$, therefore $C=gh$, and conſequently $v^2=2gh-2gy=2g(h-y)=2g\times BN$. Whence it follows, ſince BN here $=s$ in art. 243. that the body has in M the ſame velocity in the direction of the tangent that it would have had at N after falling freely through BN. Thus alſo it appears that the theorem exhibited in cor. 9. art. 258. is only a particular caſe of the principle juſt deduced.

Cor. 1. *If a body be projected up a curve (having no point of contrary flexure) the perpendicular height to which it will riſe is* *equal to that through which it muſt fall to acquire the velocity of projection.* For the body will always have its aſcending motion retarded by a force whoſe intenſity is equal to that which would accelerate the motion when deſcending.

COR. 2. If MZ*m* (fig. 10. pl. XII.) be a curve, whoſe loweſt point is Z, and the parts MZ, Z*m*, equal and ſimilar; a body by falling through MZ will acquire ſuch a velocity as will carry it to *m*: and, ſince the velocities at all equal altitudes in the deſcent and aſcent are equal, the whole time of the aſcent will be equal to that of the deſcent.

COR. 3. *The circumſtances of the laſt corollary will obtain in like manner, if the body be retained in the curve by a ſtring which is in every point perpendicular to it*: for the ſtring in that caſe will ſuſtain the part of the weight which in the former is ſuſtained by the curve.

264. PROP. *The velocities acquired in deſcending through different arcs of the ſame circle are as the chords of the arcs, the tangent to the loweſt point of each arc being horizontal.*

Let a body fall through the arc AD (fig. 11. pl. XII.), it will (art. 263.) have the ſame velocity at the point D as if it had fallen through the vertical FD, AF being horizontal: and a body falling through the arc BD will, for the ſame reaſon, have a velocity at D equal to that which would have been acquired by falling freely through ED. Now if bodies fall ſucceſſively from reſt at F and E, the velocities acquired on arriving at D will be (art. 243.) as $\surd$DF and $\surd$DE. But, by a well-known property of the circle, $\surd DF : \surd DE :: AD : BD$; and conſequently the velocities acquired by falling through any arcs ABD, BD, of the circle terminating in D the lower extremity of the vertical diameter, are as the correſponding chords AD, BD.

COR. 1. If we would give to a body on arriving at D a velocity double, triple, &c. the velocity it has at D, after falling through the arc BD, we have only to ſet off from D the chord DA double, triple, &c. the chord BD, and ABD will be the arc, after falling through which the body will have the required velocity.

COR. 2. If we wiſh the body on arriving at D to have any propoſed velocity, five feet per ſecond, for example: then find by the proper theorem (art. 243.) the height *s* due to that velocity; and having taken on the vertical CD a line DF equal to this height, from a point C above F attach a thread of the length DC to the body, and draw it aſide from the vertical till it cut the horizontal line AF in A; from this place let the body fall, and it will have at D the propoſed velocity.

265. PROP. *The accelerating force of a body, gravitating in a*

circular arc, is to its abſolute weight, as the ſine of its angular diſtance from the loweſt point of the circle, is to the radius.

Let M (fig. 12. pl. XII.) be the body deſcending along the arc MA, whoſe loweſt point is A and centre C: let ME be the ſine of the arc MA, and let MT be a tangent to the circle at M, meeting the vertical *a*A produced in T. Then will the accelerating force of the body at M be the ſame as if it were placed on the tangent MT, conſidered as an inclined plane; and will, therefore, be to its abſolute weight as ET to MT (art. 258.); or, becauſe of the ſimilar triangles MET, CEM, as EM to CM, or, as the ſine of the arc AM (the meaſure of the angular diſtance MCA) is to radius.

Cor. 1. In the ſame circle the accelerating force is every where as the ſine of the arc MA; but becauſe the ſine and arc approximate to the ratio of equality while the arc decreaſes, the ratio of the ſines to each other will approximate to, and be ultimately equal to, the ratio of their correſponding arcs to each other. Therefore, ultimately, the accelerating force is as the arc MA, the diſtance of the body from the loweſt point A, meaſured along the circle in which the body moves.

Cor. 2. Hence the times of deſcent down unequal arcs MA, *o*A, approximate to equality, while thoſe arcs decreaſe *in infinitum*, and ultimately thoſe times will be equal. For, when the accelerating force varies as the diſtances (art. 237.) the times of deſcent will be equal from whatever place the body begins to deſcend.

266. Prop. *Things being as in the laſt propoſition, draw* MA *the chord of the arc* M*o*A; *then the accelerating force of the body upon the arc at* M *will be to the accelerating force of the body placed any where upon the chord* MA *ultimately as* 2 *to* 1.

For, draw the diameter *a*CA, join *a*M, CM, let A*t* be a tangent to the circle at A, and MT a tangent to the circle at M, thoſe tangents meeting at *t*. Then, conſidering the chord MA and tangent M*t* as inclined planes whoſe altitudes are equal, the accelerating force on the plane M*t* (equal to that on the arc at M) is to the accelerating force on the plane MA, as $\frac{1}{Mt}$ to $\frac{1}{MA}$, or as MA to M*t*; or, by reaſon of the ſimilar triangles MA*t*, M*a*C, as M*a* to MC, that is, ultimately, as 2 is to 1.

Cor. *The time of deſcent through the arc* M*o*A *is to the time of deſcent through the chord* MA *ultimately as* 3·141593 *to* 4, *or as the circumference of a circle is to four diameters.* For, ultimately, the length of the arc and chord are equal, and ultimately the accelerating force of the body on the arc is every where as its diſtance from the loweſt point A (art. 265. cor. 1.); but the accelerating force of the body on the chord is uniform, and equal to half the firſt force on the arc at M by the prop.; therefore, (art. 237. cor.) the time of the deſcent through the arc is to the time of deſcent through the chord as the circumference is to four diameters.

COR. 2. Suppoſing the periphery of a circle to coincide with the perimeter of a polygon whoſe ſides are increaſed in number, and diminiſhed in magnitude *in infinitum*, and that the leaſt poſſible arc of a circle coincides accurately with its chord (which is the language of *indiviſibles)*, it would follow that the time of a body's deſcent in ſuch an arc would be *equal* to the time of deſcent down the chord: and ſuch is the concluſion of Dr. *Keill* in his xvth Lecture, and before him of M. *Parent*. But it appears, from the preceding corollary, that in fact theſe times are ſo far from being equal, as Keill and Parent maintain, that the time of deſcent down the arc is leſs than that down the chord nearly in the ratio of 3 to 4. And from the method of prime and ultimate ratios applied to this caſe, it is evident, that while the arc and chord approximate to equality, the times of deſcending along them do not approximate; for, by the doctrine of limits, no part of a curve, how ſmall ſoever, can ever be ſafely taken for its chord; but even when they ſo far approach each other that their lengths may be conſidered as equal, the curve ſtill remains a curve; its inclination is different from that of the chord; the accelerating force along the curve perpetually varies, and that in proportion to the diſtance of the body from the loweſt point, while the accelerating force along the chord remains conſtant; and therefore the times of deſcribing theſe ſpaces are unequal, even ſuppoſing the lengths could be the ſame.

Principle of M. D'Alembert.

267. Before we entirely quit the ſubject of motions along inclined planes, we ſhall briefly notice a general principle which M. D'Alembert firſt preſented in his *Dynimique*, pa. 73, and which is mentioned here becauſe we mean to ſhew its application to a part of the ſubject now under diſcuſſion; not meaning to follow it univerſally. His propoſition is this:

PROP. "*In whatever manner ſeveral bodies change their actual motions, if we conceive that the motion which each body would have in the ſucceeding inſtant, if it were quite free, is decompoſed into two others, of which one is the motion which it really takes in conſequence of their mutual actions, the ſecond muſt be ſuch, that if each body were impelled by this force alone (that is, by the force which would produce this ſecond motion), all the bodies would remain in equilibrio.*"

This is evident: for if theſe ſecond conſtituent forces are not ſuch as would put the ſyſtem in equilibrio, the other conſtituent motions could not be thoſe which the bodies really take in conſequence of their mutual action, but would be changed by the firſt.

The uſe of this propoſition will appear from the following examples.

I. Let there be three bodies B, B′, B″, and let the forces F, F′, F″, act upon them, ſo as to give them the velocities v, v', v'', in any directions whatever, producing the quantities of motion B v, B′ v', B″ v'', which we may call F, F′, F″, becauſe the momenta are the proper meaſures of the moving forces. Let us further ſuppoſe that by ſtriking each other, or being any way connected with each other, they cannot take theſe motions F, F′, F′, but really take the motions f, f', f''. It is obvious that we may conſider the motion F impreſſed upon the body B to be compoſed of the motion f which it really takes, and of another motion ϕ. In like manner F′ may be reſolved into f' which it actually takes, and another ϕ'; and again F″ into f'' and ϕ''. The motions will be the ſame whether B be acted upon with the force F, or the conſtituent forces f and ϕ; whether B′ be acted upon by F′, or by f' and ϕ'; and B″ by the force F″, or the component forces f'' and ϕ''. Now, by the ſuppoſition, the bodies actually take the motions f, f', f'': therefore the motions ϕ, ϕ', ϕ'', muſt be ſuch as will not derange the motions f, f', f'': that is to ſay, if the bodies had only the motions ϕ, ϕ', ϕ'', impreſſed upon them they would deſtroy each other, and the ſyſtem would remain at reſt.

II. *Motions upon double inclined planes.* Let AC, CB (fig. 7. pl. VI.), repreſent two inclined planes of equal altitudes, ſet back to back at DC, the angles of elevation being DAC $= e$, and DBC $= e$; and let the two weights W, W′, united by a thread WCW′ paſſing over the pulley C, act the one upon the other. Then, calling w, w', their maſſes, or their weights, let us determine the circumſtances of their motion.

At the end of the time t, w will have a velocity v, and gravity would impreſs upon it in the inſtant $\dot{t}$ following a new velocity $= g$. ſin. e . $\dot{t}$ (art. 258. cor. 3.), provided the weight w were then entirely free: but, by the diſpoſition of the ſyſtem, $\dot{v}$ will be the velocity which obtains in reality. Then eſtimating the ſpaces in the direction CW, as the body W′ moves with an equal velocity but in a contrary ſenſe, it is obvious that the decompoſition may be made as follows. At the end of the time $t + \dot{t}$, we have for the velocity impreſſed upon

$$W \ldots\ldots v + g.\,\text{sin.}\,e.\,\dot{t}, \text{ where } \begin{cases} v+\dot{v} \ldots \text{effective velocity from C towards A.} \\ g.\,\text{sin.}\,e.\,\dot{t}-\dot{v} \ldots \text{velocity destroyed.} \end{cases}$$

$$W' \ldots -v + g.\,\text{sin.}\,e'.\,\dot{t}, \text{ where } \begin{cases} -v-\dot{v} \ldots \text{effective velocity from C towards B.} \\ \dot{v}+g.\,\text{sin.}\,e'.\,\dot{t} \ldots \text{velocity destroyed.} \end{cases}$$

If, therefore, gravity impreſſes, during the time $\dot{t}$, upon the maſſes w, w', the reſpective velocities $g.\,\text{ſin.}\,e.\,\dot{t}-\dot{v}$, and $g.\,\text{ſin.}\,e'.\,\dot{t}+\dot{v}$, the ſyſtem will be in equilibrio. The quantities of motion being therefore equal, we have

$$w.\,g.\,\text{ſin.}\,e.\,\dot{t}-w\,\dot{v}=w'.\,g.\,\text{ſin.}\,e'.\,\dot{t}+w'\,\dot{v}.$$

Whence we deduce for the effective accelerating force

$$\text{(I.)} \ldots \phi=\frac{\dot{v}}{\dot{t}}=\frac{w\,\text{sin.}\,e-w'\,\text{sin.}\,e'}{w+w'}\times g.$$

Thus it appears that the motion is uniformly varied: and we readily find

$$\text{(II.)} \ldots v=\frac{w\,\text{sin.}\,e-w'\,\text{sin.}\,e'}{w+w'}\,g.\,t.$$

$$\text{(III.)} \ldots s=\frac{w\,\text{sin.}\,e-w'\,\text{sin.}\,e'}{w+w'}.\,\tfrac{1}{2}\,g\,t^2.$$

Such are the equations from which the conditions of the motion are determined. If the two planes are vertical, then is $e=e'=\frac{1}{2}\pi$; by means of which we may find the values that are applicable in the ſubſequent example.

III. *Motion on the fixed pulley.* The weights P and W are united by the thread PAABW running over the pulley A (fig 2. pl. VI.); the force of gravity g which ſolicits them cannot produce its entire effect, becauſe its action upon the body $P=w$ counteracts its effect upon the other body $W=w'$. At the end of the time $t+\dot{t}$, the velocity of w, in the direction from A to P, will be $v+\dot{v}$, inſtead of $v+g\,\dot{t}$, which it would have had if the weight had become free at the end of the time t: thus $g\,\dot{t}-\dot{v}$ will be the velocity deſtroyed. In like manner, w' having the ſame velocity but in a contrary direction, will have the velocity $-(v+\dot{v})$ inſtead of $-(v+g\,\dot{t})$, which it would have had, if at the end of the time t the connection of the two bodies ceaſed; ſo that $g\dot{t}+\dot{v}$ will be the velocity deſtroyed. Then, as in the preceding example, making the momenta equal, we find for the effective accelerating force

$$\text{(1.)} \ldots \phi=\frac{\dot{v}}{\dot{t}}=\frac{w-w'}{w+w'}\times g.$$

This being an expreſſion for a conſtant accelerating force, we conclude, as in the former inſtance, that the motion is uniformly varied, and that the circumſtances of motion and of velocity may be found by the equations

$$(2.) \ . \ . \ . \ v=\frac{w-w'}{w+w'}gt \ . \ . \ . \ (3.) \ . \ . \ . \ s=\frac{w-w'}{2w+2w'}gt^2.$$

In this inveſtigation we have neglected the inertia of the ſtring and the pulley.

From formulæ ſimilar to theſe Mr. Atwood deduced (in his valuable treatiſe on the Motion of Bodies) an ingenious method of verifying the received theorems relative to the effects of gravity on falling and riſing bodies. He contrived a machine, in which two weights as w and w' in our theorems, or P and W in the figure laſt referred to, are connected by a thread which runs over a fixed pulley A: the laſt given theorems for ϕ, s, and v, ſhew that by varying the weights w, w', we may change the value of the fraction $\frac{w-w'}{w+w'}$, and thus reduce the accelerating force, and its conſequent velocity and ſpace in a given time, in ſuch a ratio with reſpect to thoſe ariſing from gravity, that they ſhall become eaſily meaſureable, and the theory verified. Mr. Atwood's apparatus has its ſeveral parts ſo adapted as to be ſuſceptible of great preciſion: but as this is not the place to enter into detail, the reader may turn to the article *Atwood's machine* in the ſecond volume, where a more full account is given of its nature and uſe.

In finding the equations (2.) (3.) above, we ſuppoſed that each body began to move from quieſcence at a point taken for the origin of the ſpaces denoted by s. But ſuppoſe this is not the caſe, and that we impreſs upon w an initial velocity of V carrying it downward. This velocity muſt be parted between the two maſſes w, w', according to the ſame law as if w ſtruck the body w' at reſt with the velocity V: thus, the velocity common to the two weights would be $\frac{wV}{w+w'}$. We have this value for the velocity at the end of the time $t=0$; therefore

$$(4.) \ . \ . \ . \ v=\frac{wV+(w-w')gt}{w+w'}.$$

From this we readily obtain s in terms of t: and the time may be determined by the equation $wV=(w-w')gt$.

IV. *Motion on the Axis in Peritrochio.* Let the weight P (fig. 4. pl. V.) be denoted by w, and the weight W by w'; to determine the circumſtances of the motion. Let the radius of the wheel$=$R, that of the axle$=r$: the velocity of w at the end of the time t will be $v+\dot{u}$, inſtead of $v+g\dot{t}$, as in the former caſes; that of w' will manifeſtly be to that of w in the ratio of R to r; it will, therefore, be $-\frac{r}{R}(v+\dot{v})$ inſtead of being $-\frac{r}{R}(v+g\dot{t})$: the velocities loſt are therefore $g\dot{t}-\dot{v}$ and $g\dot{t}-\frac{r}{R}v$.

Then making the momenta equal with reſpect to the axe EF, we find

$$\text{(i.)} \ldots \phi=\frac{\dot{v}}{\dot{t}}=\frac{R^2w-Rrw'}{R^2w+r^2w'}\times g.$$

Conſequently, the motion is uniformly accelerated in this caſe likewiſe. So that we readily deduce the equations for the motion of w or P.

$$\text{(ii.)} \ldots v=\frac{R^2w-Rrw'}{R^2w+r^2w'}gt. \ldots \text{(iii.)} \ldots s=\frac{R^2w-Rrw'}{R^2w+r^2w'}\cdot\tfrac{1}{2}gt^2.$$

And the velocity and ſpace, with reſpect to w', will be

$$\text{(iv.)} \ldots V=\frac{Rrw-r^2w'}{R^2w+r^4w'}gt. \ldots \text{(v.)} \ldots S=\frac{Rrw-r^2w'}{R^2w+r^2w'}\cdot\tfrac{1}{2}gt^2.$$

Thus much is here offered on the application of M. D'Alembert's principle: the further uſe of ſome of theſe theorems will appear, when we treat of the maximum of machines in motion, Chap. VI.

IV. *On the Simple Pendulum, the Cycloïdal Pendulum, and the Curve of Swifteſt Deſcent.*

268. We have already ſeen (art. 263. cor. 2.) that a heavy body after having deſcended through an arc MZ (fig. 10. pl. XII.) of any curve CZm, will, abſtracting from the effects of friction and the air's reſiſtance, mount up the oppoſite branch Zm till it arrives at the point m, whoſe diſtance from the horizontal line AX is equal to MP, the diſtance of the point M from the ſame; and that the time of deſcribing Zm will be equal to that of deſcribing MZ. Having at the point m loſt all its motion, it will begin to deſcend again through mZ, and at Z will have acquired a velocity ſuch as will cauſe it to riſe to the point M where its motion firſt commenced, where again it will be in a ſtate of quieſcence; and from this it will move a ſecond time through MZm, and back again through mZM; and ſo on continually. Such will be the motions whether the body run over a curve ſurface, in conſequence of the joint effects of gravity and the reaction of the ſurface; or whether it be made to de-

ſcribe the curve in conſequence of being faſtened to a ſtring CM (fig. 12.) whoſe centre C is fixed. If we conſider the thread CM as inextenſible, and without weight, and the body at M as a point with reſpect to the length of the thread, then is the ſyſtem known by the name of the *Simple Pendulum*. The motion of ſuch a pendulum in one direction from a ſtate of reſt, till it begins to return in an oppoſite direction, is called a *vibration* or an *oſcillation;* and the time employed in ſuch motion the *time of vibration*, or *of an oſcillation*. Thus, if the pendulum move from quieſcence at M, the time it occupies in paſſing through the arc MA m is the time of vibration; and this, it is manifeſt, is *double the time employed in paſſing over* MA. This being admitted it would be eaſy to apply ſome of the theorems in arts. 264. . . 266. to the vibrations of pendulums in circular arcs; as they would then aſſume the form below.

I. If a pendulum vibrate in the arc of a circle, the velocity of the ball, at its loweſt point, will be as the chord of the arc which it deſcribes in its deſcent.

II. The force which accelerates a pendulum is to the force of gravity, as the ſine of its angular diſtance from the loweſt point to radius.

III. The times of vibrations in very ſmall circular arcs are very nearly equal.

IV. The time of vibration in an indefinitely ſmall circular arc is to the time in which a body would fall through half the length of the pendulum, as the periphery of a circle to its diameter.

V. The time of vibration is as the ſquare root of the length of the pendulum; the force of gravity remaining the ſame.

And from theſe the whole doctrine of pendulums vibrating in circular arcs might readily be deduced. But, as the ſuppoſition that vibrations in ſmall arcs of equal circles are all performed in equal times is not quite correct, it may be better to draw the chief properties of pendulums from a diſtinct inveſtigation, by means of which we may alſo determine the magnitude of the error which can occur in vibrations through any aſſigned arc. This may be accompliſhed thus.

269. Prop. *To determine the time of vibration of a pendulum in any circular arc.*

Let MA m (fig. 12. pl. XII.) be the propoſed arc; and let the radius or length of the pendulum CM $=l$, AE $=b$, A$p=x$, $po=y$, the variable arc $=s$, the accelerating force or force of gravity $=\varphi$, and the velocity due to the height Ep, or (art. 263.) the velocity acquired by falling through Mo, put $=v=\sqrt{2\varphi(b-x)}$ (art. 243.). Now it is known, that $v=\frac{\dot{s}}{\dot{t}}$ (art. 232. I.); and comparing theſe two values of v we find $\dot{t}=\int\frac{\dot{s}}{\sqrt{2\varphi(b-x)}}$. But the known property of the circle gives for the value of the element of the curve $\dot{s}=\frac{-l\dot{x}}{\sqrt{2lx-xx}}$, which is taken negatively, becauſe the arc s diminiſhes as the time augments: this value of $\dot{s}$ introduced into the preceding equation tr..nsforms it to

$$\dot{t}=\frac{-l\dot{x}}{\sqrt{(2lx-xx).\,2\varphi(b-x)}}=\frac{-\dot{x}}{\sqrt{bx-xx}}\times\frac{l}{\sqrt{2\varphi(2l-x)}}=\frac{\sqrt{l}}{2\sqrt{\varphi}}\times\left\{1-\frac{x}{2l}\right.^{-\frac{1}{2}}\times\frac{-\dot{x}}{\sqrt{bx-xx}}=\tfrac{1}{2}\sqrt{\frac{l}{\varphi}}\times\frac{-x}{\sqrt{bx-xx}}\times\left(1+\tfrac{1}{2}\cdot\frac{x}{2l}+\frac{1.3}{2.4}\cdot\frac{x^2}{4l^2}+\frac{1.3.5}{2.4.6}\cdot\frac{x^3}{8a^3}+\frac{1.3.5.7}{2.4.6.8}\cdot\frac{x^4}{16a^4}+\&c.\right),$$

the factor $\left(1-\frac{x}{2l}\right)^{-\frac{1}{2}}$ being expanded by the binomial theorem.

To obtain the time of deſcent from M to A, we muſt find the fluent of each term in the ſeries, in ſuch manner that they may vaniſh when $x=b$. Now we ſee at once, that the variable factor of all the terms of this ſeries will be of the form $\frac{-x^{n}\dot{x}}{\sqrt{bx-xx}}$; that is to ſay, theſe factors will be $\frac{-\dot{x}}{\sqrt{bx-xx}}$, $\frac{-x\dot{x}}{\sqrt{bx-xx}}$, $\frac{-x^2\dot{x}}{\sqrt{bx-xx}}$, &c. Whence it appears that, taking π to 1, the ratio of the circumference of a circle to its diameter, the fluents of each of the terms, taken between $x=b$, and $x=0$, are π, $\pi\cdot\frac{b}{2}$, $\pi\cdot\frac{1.3.b^2}{2.4}$, $\pi\cdot\frac{1.3.5.b^3}{2.4.6}$, &c. Subſtituting theſe values in that of t, it will become

$$t=\tfrac{1}{2}\pi\sqrt{\frac{l}{\varphi}}\left(1+\frac{1^2}{2^2}\cdot\frac{b}{2l}+\frac{1^2.3^2}{2^2.4^2}\cdot\frac{b^2}{4l^2}+\frac{1^2.3^2.5^2}{2^2.4^2.6^2}\cdot\frac{b^3}{8l^3}+\&c.\right)$$

This, therefore, is the time employed by the body in deſcending from M to A: but with the velocity acquired at A the body would proceed along the equal branch A m of the curve, and would have all its velocity extinguiſhed at m, after a time from A equal to the time of deſcent from M to A: conſequently, the time of a complete oſcillation will be double the former; that is,

$$t=\pi\sqrt{\frac{l}{\varphi}}\times\left(1+\frac{1^2}{2^2}\cdot\frac{b}{2l}+\frac{1^2.3^2}{2^2.4^2}\cdot\frac{b^2}{4l^2}+\frac{1^2.3^2.5^2}{2^2.4^2.6^2}\cdot\frac{b^3}{8l^3}+\&c.\right)$$

The relation $\frac{b}{l}$ or $\frac{AE}{AC}$ expreſſes the verſed-ſine of an arc of the ſame angular value as AM, having unity for its radius. And as the verſed ſine of ſmall angles is exceedingly minute, the ſeries $1+\frac{1^2}{2^2}\cdot\frac{b}{2l}+\frac{1^2.3^2}{2^2.4^2}\cdot\frac{b^2}{4l^2}+$ &c. will converge very rapidly. So that in almoſt every caſe the two firſt terms will ſuffice, or t nearly $=\pi\sqrt{\frac{l}{\varphi}}\times\left(1+\frac{b}{8l}\right)$; therefore the times of vibrations in different arcs are as $8l+b$, or as 8 times the radius + the verſed ſine of the arc. Or, when the arc is very ſmall, all the terms but the firſt may be neglected, and we ſhall have for the duration of the oſcillation,

$$t=\pi\sqrt{\frac{l}{\varphi}}.$$

The quantity b does not enter into this value of t: conſequently when the vibrations are made through very ſmall arcs, their duration may be conſidered as not depending on the magnitude of the arc; and the oſcillations may be regarded as iſochronal.

270. To determine the numerical error which may be committed, by aſſuming the iſochroniſm of vibrations in ſmall arcs, let us ſuppoſe the arc $\mathrm{MA}=5^\circ$; the length of the oſcillation of a ſecond pendulum would be expreſſed by $1=\pi\sqrt{\frac{l}{\varphi}}$ or $l=\frac{\varphi}{\pi^2}$ on the ſuppoſition of the arc being extremely ſmall. The verſed ſine of 5° is $\cdot0038053=\frac{b}{l}$: and $\frac{1^2}{2^2}\cdot\frac{b}{2l}=\cdot0004757$. As to the third term, it is leſs than ·0000001. We have therefore for the time employed by the pendulum in moving through an arc of twice 5°, $1+(1\times\cdot0004757)$; and the time of oſcillating through an arc of 10° differs not from an oſcillation in an infinitely ſmall arc more than ·0004757. This multiplied into 86400, the ſeconds in 24 hours, gives nearly $41\frac{1}{10}$ ſeconds. So that a pendulum of the ſame length as that which vibrates in ſeconds, through infinitely ſmall arcs, would loſe about 41^{s} per day, if it deſcribed arcs of 5° on each ſide the vertical CA. If the arcs deſcribed on each ſide of the vertical were only 1°, of which the verſed ſine is ·0001523, it would be found by a ſimilar proceſs that the daily retardation would be about $1\frac{2}{3}^{s}$. And for a half degree on each ſide it would be about $\frac{2}{3}$ of a ſecond.

Or, if D denote the degrees the pendulum deſcribes on each ſide of the vertical, the time it would be retarded in a ſecond would be nearly expreſſed by $\frac{D^2}{51150}$, and conſequently the time loſt in 24 hours would be $24 \times 60 \times 60 \times \frac{D^2}{51150} = \frac{5}{3} D^2$ nearly. In like manner the time loſt in 24 hours by deſcribing δ degrees on each ſide of the vertical would be $\frac{5}{3}\delta^2$ nearly. Conſequently, if a ſecond pendulum keeps true time in one of theſe arcs, the ſeconds loſt or gained per day by vibrating in the other will be $\frac{5}{3}(D^2 - \delta^2)$. Thus, for example, if a ſecond pendulum meaſure true time in an arc of 3° on each ſide of the vertical, it will loſe $11\frac{2}{3}$ ſeconds per day by vibrating through 4° on each ſide, and nearly 45 ſeconds per day by deſcribing 6° on each ſide.

Thus then it appears that the vibrations in very ſmall circular arcs may be regarded as ſenſibly iſochronal: and whenever great accuracy is required, the neceſſary correction may be eaſily applied, as above. Reaſoning by analogy, we ſee alſo that the oſcillations in very ſmall arcs of any curve whatever are always ſenſibly iſochronal; becauſe an arc of any curve will, for a ſmall ſpace, coincide with its circle of curvature. The equation $t = \pi\sqrt{\frac{l}{\varphi}}$ will furniſh ſome other important conſequences, which we ſhall now proceed to deduce.

271. I. *The durations of the vibrations of pendulums are reſpectively as the ſquare roots of their length.* For, let l, l', be the lengths of the pendulums, t, t', the times of vibration; then, we have $t = \pi\sqrt{\frac{l}{\varphi}}$, and $t' = \pi\sqrt{\frac{l'}{\varphi}}$: conſequently,

$$t : t' :: \pi\sqrt{\frac{l}{\varphi}} : \pi\sqrt{\frac{l'}{\varphi}} :: \sqrt{l} : \sqrt{l'}.$$

II. *The time of an oſcillation is to the time in which a heavy body would fall through half the length of the pendulum, as the periphery of a circle to its diameter.* For in the former caſe the time is denoted by $t = \pi\sqrt{\frac{l}{\varphi}}$; and in the latter the theorem $t = \sqrt{\frac{2s}{g}}$ (art. 243.) when accommodated to the preſent purpoſe will be $t' = \sqrt{\frac{l}{\varphi}}$. Conſequently $t : t' :: \pi : 1$.

III. *If two pendulums of different lengths are ſolicited by different gravitating forces, the times of oſcillations are as the ſquare roots of the lengths of the pendulums directly, and the ſquare roots of the quantities expreſſing the gravitating forces inverſely.* For $t = \pi\sqrt{\frac{l}{\varphi}}$, and $t' = \sqrt{\frac{l'}{g}}$, give $t : t' :: \sqrt{\frac{l}{\varphi}} : \sqrt{\frac{l'}{g}}$.

IV. *The lengths of two pendulums are reſpectively in the inverſe ratio of the ſquares of the number of oſcillations made in the ſame time.* In the time T let the pendulum make n oſcillations, then is $\frac{T}{n}$ = the time of one oſcillation: conſequently $t = \frac{T}{n} = \pi\sqrt{\frac{l}{\phi}}$, and $l = \frac{\phi T^2}{\pi^2 n^2}$. In like manner, if l' be the length of another pendulum, n' its number of oſcillations in the ſame time T, the force of gravity φ being ſuppoſed the ſame (as is always aſſumed unleſs otherwiſe expreſſed); we ſhall have $l' = \frac{\phi T^2}{\pi^2 n'^2}$. Whence

$$l : l' :: \frac{1}{n^2} : \frac{1}{n'^2} :: n'^2 : n^2.$$

By means of this theorem, the length of the ſeconds pendulum may be readily aſcertained experimentally: for, if we take a pendulum of any determinate length l', and count the number of vibrations n' which it makes in a given time, half an hour, for example, or 1800 ſeconds, then will the length l of the ſeconds pendulum be found by this analogy, $(1800)^2 ; n'^2 :: l : l$. By this method, among others, has the length of the ſeconds pendulum in the latitude of London been found $= 39\frac{1}{8}$ inches. It muſt be recollected though, that the length is not the ſame at all parts of the earth's ſurface; the difference ariſing from the inequality of diſtance from the centre, and the conſequent variations in the centrifugal force.

Another good method of aſcertaining the length of a pendulum to vibrate in a certain time is that which was firſt propoſed by Mr. *Hatton*, and afterwards executed by Mr. *Whitehurſt*. It conſiſts in the application of a moveable point of ſuſpenſion to the ſame pendulum; which thus gives the abſolute effects of two pendulums, the difference of whoſe lengths is known, being the interval between the points of ſuſpenſion in the two caſes: and the ratio of their lengths is alſo known from obſerving the number of vibrations performed in a given time. Whence, there being two equations and two unknown quantities, the actual lengths of the pendulums themſelves are eaſily deduced. Thus, we ſhall have $l : l' :: n'^2 : n^2$, and $l - l' = d$, the meaſured diſtance between the points of ſuſpenſion: conſequently $l = \frac{d n'^2}{n'^2 - n^2}$ and $l' = \frac{d n^2}{n'^2 - n^2}$.

V. *If a clock keep true time very nearly, the variation in the length of the pendulum neceſſary to correct the error will be equal to twice the product of the length of the pendulum and the error in time divided by the time of obſervation in which that error is accumulated.*

Let the time of obſervation be $=\mathrm{T}$, the time gained or loſt by the pendulum $=t'$, the length of the pendulum $=l$, and the ſpace which the pendulum muſt be lengthened or ſhortened $=\lambda$: then we have $l : l\pm\lambda :: (\mathrm{T}\pm t')^2 : \mathrm{T}^2$; and from this is found $\lambda=\dfrac{l\,t'\,(\mp 2\,\mathrm{T}-t')}{(\mathrm{T}\pm t')^2}$. Neglecting t', which is very ſmall with reſpect to $2\,\mathrm{T}$, the equation becomes $\lambda=\dfrac{\pm\,2\,l\,t'}{\mathrm{T}}$; which is the ſame as the rule.

COR. If the pendulum be one that ſhould beat ſeconds, and t' the daily variation be given in minutes, and n be the number of threads in an inch of the ſcrew which raiſes and depreſſes the bob of the pendulum, then $\lambda=\dfrac{\pm 2\times 39\frac{1}{8}\times n\,t'}{24\times 60}\doteq{}\cdot05434\,n\,t'=\frac{2}{37}\,n\,t'$ nearly, for the number of threads which the bob muſt be raiſed or lowered, to make the pendulum vibrate truly.

VI. *The length of a ſeconds pendulum being known in any place, the ſpace through which a heavy body would fall freely in the ſame time and place may be readily found.* For the equation $t=\pi\sqrt{\dfrac{l}{\varphi}}$, becomes $1=\pi\sqrt{\dfrac{39\frac{1}{8}}{\varphi}}$, whence we have $\varphi=\pi^2 l=\pi^2\times 39\frac{1}{8}=386\cdot14$ inches $=32\frac{1}{6}$ feet nearly, the appropriate meaſure of the force of gravity; correſponding with the aſſumed value of g in art. 242. And half this, or $16\frac{1}{12}$ feet, is the ſpace deſcended by a heavy body in the firſt ſecond from quieſcence.

Or the ſame concluſion may be obtained rather differently from No. II. of the preſent article: for $\pi : 1 :: 1^s : \dfrac{1}{\pi}$ the time in which a heavy body would fall through a vertical line of $19\frac{9}{16}$ inches, and by the laws of falling bodies the ſpaces deſcribed are as the ſquares of the times; therefore $\dfrac{1}{\pi^2} : 1^s :: 19\frac{9}{16} : 19\frac{9}{16}\times\pi^2=193\cdot07$ inches $=16\frac{1}{12}$ feet nearly, the ſame as before.

It is obvious alſo, that by the reverſe of this method we may find the length of a ſeconds pendulum, having given the ſpace fallen through by a heavy body in the firſt ſecond of time from quieſcence. And thus the theory and experiments mutually aſſiſt each other, in determining data of ſuch importance as theſe, in many other branches of ſcience.

272. DEFS. If the circumference of a circle be rolled along a

right line, until any point O of that circumference which was in contact with the line come in contact with it again, that point P will deſcribe a curve called a *cycloid.* The line on which the circle rolls is the *baſe* of the cycloid; and the diameter of that circle which is perpendicular to the baſe when the circle touches its middle point, is the *axis* of the cycloid.

The circle which rolls along the right line is called the *generating circle.*

Cor. The baſe of the cycloid is equal to the circumference of the generating circle.

273. Prop. *If a line* OFB *be drawn from any point* O *in the cycloid, parallel to its baſe, and meet the generating circle deſcribed upon the axis in* F, *the circular arc* F*e*V *is equal to the right line* OF.

Let the generating circle touch the baſe in D (fig. 1. pl. XIII.) when the generating point is at O. DE perpendicular to AC and equal to CV is the diameter of the circle DOE. Draw the chords OE, FV. Then, ſince $DE = CV$, and $DG = CB$, the remainder $GE = BV$: conſequently $OG = \sqrt{DG.GE} = FB = \sqrt{CB.BV}$. Add FG to each of theſe equals, then will $OF = GB$. Alſo $OE = \sqrt{ED.EG} = VF = \sqrt{VG.VB}$; therefore arc $E\,i\,O =$ arc $V\,e\,F$: and ſince every point in $O\,n\,D$ has been ſucceſſively in contact with AD, while the point O moved from A to its preſent poſition, $O\,n\,D = AD$, and $EOD = AC$: hence $E\,i\,O = DC$; and conſequently $V\,e\,F = E\,i\,O = DC = GB = OF$.

Cor. Becauſe OF is always equal to the arc $V\,e\,F$ or $E\,i\,O$, their cotemporary increments or decrements are equal; that is, the initial motions of the point O which traces out the cycloidal arc, the one parallel to the baſe AC, the other in the direction of the circle or its tangent at O, are equal to each other.

274. Prop. *If an ordinate* OB *be drawn* (fig. 1. pl. XIII.) *from any point* O *of the cycloidal arc, interſecting the circle* CFV *in* F, *then will the tangent* OE *of the cycloid be parallel to the chord* FV *of the generating circle.*

Draw the tangent $T\,t$ to the circle DOE in O, and produce BO, DO, to b and d; then are the initial motions of the point O in the directions $O\,t$, OG, and equal to each other, by corol. to the foregoing propoſition. And if the parallelogram of motions be conſtructed upon the equal ſides coinciding with $O\,t$, OG, its diagonal, will, by the compoſition of motion, be in the direction of the tangent: therefore, the contiguous ſides of the parallelogram being equal, it will be either a rhombus or a ſquare, and its diagonal will biſect the angle $t\,O\,G$; that is, OE the tangent in O will biſect the angle $t\,O\,G$, or make $t\,O\,E = EOG$. But $GOD = ODT$, becauſe OG and TD are parallel; and $ODT = TOD$, becauſe the tangents TO, TD are equal; alſo $TOD =$

tOd, being vertical or oppoſite angles; therefore $GOD = tOd$. Then adding equals to equals, $DOG + GOE = dOt + tOE$, or $dOE = DOE$; conſequently OE is at right angles to Dd, and meets the diameter DE in its extremity E. Again, CD is equal and parallel to FO, therefore DO is equal and parallel to CF; and hence, ſince $DOE = CFV =$ a right angle, OE is parallel to FV.

Cor. 1. Since DO is perpendicular to OE, the tangent to the cycloid at O, it coincides in direction with the radius of curvature at O.

Cor. 2. A tangent to the cycloid at the vertex V is perpendicular to the axis VC, and parallel to the baſe AC.

The method of determining the tangent to the cycloid by the compoſition of motion was firſt given by Roberval, in his "*Obſervations on the Compoſition of Motion, and on the Tangents of Curve Lines,*" publiſhed in the Memoirs of the Academy of Sciences, 1666. The ſame method is again applied to the caſe of cycloids and epicycloids, by Mr. Weſt, in his *Mathematics*, publiſhed in 1762. Afterwards by Dr. Parkinſon, in his *Mechanics*, publiſhed in 1785. And again by Mr. Ludlam, in his *Eſſays*, 1787. Neither of theſe gentlemen ſeem aware that their method is the ſame in principle as Roberval's.

275. Prop. *The involute of a ſemicycloid* AOU (fig. 13. pl. XII.) *is an equal ſemicycloid* UPV *in an oppoſite direction, the extremity of the baſe of the latter being in contact with the vertex of the former.*

From any point O draw OB parallel to AC, cutting the generating circle in F, and join FU. Draw OP a tangent to the cycloid in O, and at E, the point where it cuts the line UW drawn from U parallel to CA, let fall ED perpendicular to UW, and equal to CU. On ED as a diameter deſcribe a circle interſecting the tangent OP in ſome point P. Then (art. 274.) OE is equal and parallel to FU; and (art. 273.) OF is equal to the arc FeU. The circles CFU, DPE are equal, as are likewiſe the angles FUE and UEP; the chords FU, EP, therefore, are equal, and cut off equal arches. Becauſe OFUE is a parallelogram, UE is equal to FO, or equal to FeU, or equal to EnP. But if the circle EPD had been placed on the line UW at U, and had rolled from U to E, the arch diſengaged would have been equal to UE, and the point which was in contact with U would now be in P, in the periphery of a ſemicycloid UPV, equal to AOU, having the line UN equal and parallel to AC for its baſe, and NV equal and parallel to CU for its axis: and ſince the ſame may be ſhewn to obtain with reſpect to any other point in AOU, the cycloid UPV is the involute of AOU, as in the propoſition.

Cor. 1. The arch OU of the cycloid is equal to twice the correſponding chord FU of the generating circle: for this arch is equal to the evolved line OEP; and it has been ſhewn that OE=EP; conſequently OP=OU=2 OE=2 FU.

Cor. 2. The arch of a ſemicycloid is equal to twice the diameter of its generating circle; and the whole cycloidal arch equal to four times the diameter of the generating circle.

Cor. 3. The deſcription of the cycloid UPV by the evolution of the cycloid AOU furniſhes a ſimple method of determining the area of any cycloid. For ſince OP is always parallel to FU, the former OP will ſweep over the whole ſpace AOUPV, while UF ſweeps over the whole ſurface of the ſemicircle UFC; and ſince OP is always double the ſimultaneous UF, the ſpace AOUPV will be quadruple of the ſemicircle UFC. Alſo, the ſpace deſcribed by OE in any inſtant is manifeſtly one-fourth of the ſpace deſcribed in the ſame interval by the double line OP; therefore, the ſpace AOUN is equal to the ſemicircle UFC: each of them is one-third of the area NUPV, or its equal AOUC; and the ſpace ACFUOA is two-thirds of the ſemicycloid AOUC.

Cor. 4. From this propoſition it will be eaſy to make a pendulum oſcillate in a given cycloid, as UVW, whoſe baſe is parallel to the horizon. Thus, produce the axis VN till VA= 2 VN; through A draw a line AC parallel to UW, and $=\frac{1}{2}$UW; on AC as a ſemi-baſe with axis CU deſcribe a ſemicycloid AOU; and in like manner deſcribe another ſemicycloid AW turned the contrary way: then a ſtring whoſe length is AOU or AV, being fixed by one end at A, and in its motion coming gradually into contact with one or other of the cycloidal cheeks AOU, AW, will deſcribe the cycloid UVW by its other extremity.

276. Prop. *To find the time of a pendulum's oſcillation, in the arc of a cycloid.*

Let UVW (fig. 13. pl. XII.) be the cycloid in which the body is to oſcillate: then by the preceding prop. and corols. VN=NA, and AOU=AW=AV=$\frac{1}{2}$ UVW. The accelerating force in any point T of the curve is the ſame as if the body were placed upon the tangent T*t*: and by the nature of the inclined plane, the force of gravity : force in direction T*t* :: T*t* : TG :: QV : VH (by ſin. triangles) :: VN : VQ. In like manner, taking any other point S in the curve, the force of gravity : force in the curve :: VR : VI :: VN : VR. Conſequently the accelerating force of a body placed on different points T, S, of the curve, varies as the correſponding chords VQ, VR, of the generating circle, or as the portions VT, VS, of the curve, meaſuring from the vertex; theſe portions VT, VS, being the doubles of the correſponding chords VQ, VR, by cor. 1. of the laſt article. This problem, therefore, is analogous to that diſcuſſed in arts. 237. 238. reſpecting a body which is attracted by a force which varies directly as the diſtance from the loweſt point V: conſequently, in this caſe as well as that the times of deſcent from any points on the arc where the body begins to move from quieſcence to the loweſt point V are equal; and, as equal forces acting in oppoſite directions will deſtroy equal quantities of motion in equal times, the times of aſcent from V along the other branch of the curve will all be equal likewiſe: hence the times of oſcillation in a cycloid, whether through greater or leſs arches, are equal.

The expreſſion $\frac{1}{2}\pi\sqrt{\frac{d}{mf}}$ in art. 237. when adapted to the preſent caſe becomes $\frac{1}{2}\pi\sqrt{\frac{l}{32\frac{1}{6}}}$, l being equal to WTV or AV; this denotes the time of deſcent through half the cycloid: hence, in the caſe of vibration, $t=\pi\sqrt{\frac{l}{32\frac{1}{6}}}$ for the time of oſcillation in a pendulum whoſe radius of curvature at its vertex is equal to l. This expreſſion agrees with that given in art. 269. for pendulums vibrating in very ſmall circular arcs; conſequently, the chief inferences made from that equation will equally apply to the cycloidal pendulum.

COR. *The time of deſcent through* WV *is to the time of deſcent through* SV *(after having paſſed over* WS*) as the ſemicircumference* NRV *to the arc* RQV. For, on the radius AV deſcribe the quadrant AWZ, make VK = 2 VR, and draw KL parallel to UW. Then is $\frac{1}{2}$ VR : $\frac{1}{2}$ VN : : 2 VR : 2 VN : : VK : VA. But VK is the ſine of the arc LZ to the radius VA, and $\frac{1}{2}$ VR is the ſine of half the arc VQR to the radius $\frac{1}{2}$ VN; conſequently LZ and $\frac{1}{2}$ VQR are ſimilar portions of their reſpective circumferences: and therefore ſince the radii of the two circles are as 4 to 1, the arc LZ equal 2 RQV. Now, AV being equal to WSV, and VK = 2 VR = VTS, and the times of deſcent through AV, KV, being as the arcs ALZ, LZ (art. 237.) we ſhall have, time in WSV : time in SV : : ALZ : LZ : : 2 NRV : 2 RQV : : NRV : RQV.

COR. 2. *The time of deſcribing any arc* VTS *after having fallen through* UPV, *is to the time of deſcribing* UPV *as the arc* VQR *to the ſemicircular arc* VRN. This is evident, becauſe the time of deſcribing VS after paſſing over UPV is manifeſtly the ſame as the time of deſcribing SV, after having run through WS.

The iſochroniſm of vibrations in cycloidal arcs is demonſtrated upon the ſuppoſition that the whole maſs of the pendulum is concentrated in a point; a ſuppoſition which cannot actually take place in any vibrating body; and when the pendulum is of finite magnitude there is no point given in poſition which determines the length of the pendulum; for, that which is called the centre of oſcillation will not occupy the ſame place in the given body when deſcribing different parts of the track it moves through, but will be continually moved in reſpect of the pendulum itſelf during its vibration. This circumſtance has prevented any general determination of the time of vibration in a cycloidal arc except in the imaginary caſe here conſidered: the property of iſochroniſm, however, obtaining here, has occaſioned the name of *Tautochrones* to be applied to cycloids.

Many other reaſons have induced the artiſts to abandon the uſe of the cycloidal pendulum, although it was commonly adopted for ſome time after its firſt invention by Huygens. The principal are, the difficulty with which the metallic cheeks are bent into the true cycloidal form; the improbability of their long retaining it, ſuppoſing it once given; and the changes of which the pendulum is ſuſceptible in conſequence of the expanſion and contraction by heat and cold. Theſe ſources of error are ſuch as the peculiar property of the cycloid cannot obviate: and, as the variations from iſochroniſm in very ſmall circular arcs is very trifling, the cycloidal pendulum is now wholly diſuſed in practice.

Curve of ſwifteſt Deſcent.

277. At firſt view it might be imagined that as a right line is the ſhorteſt path from one point to another, ſo it ſhould be the line of quickeſt deſcent from one point to another not ſituated in the ſame horizontal line: but it has been already ſeen (art. 266. cor.) that the times of deſcent through arcs of circles in certain poſitions are leſs than the times of running down the chords. And there does not appear any reaſon why other curves may not be found through which bodies ſhall paſs from one to the other of two given points in leſs time than they would paſs in circular arcs. The general problem was firſt propoſed in 1697, by John Bernoulli, under the title of the *Brachyſtochronon*, or "that curve along the concave ſide of which if a heavy body deſcend it will paſs in the leaſt time poſſible from one point to another, the two points not being in the ſame vertical line." The problem was truly anſwered the ſame year by Leibnitz, Newton, L'Hoſpital, and James Bernoulli. The problem has been reconſidered more recently by Venturi, who, beſides arriving at the ſame concluſions as the original inveſtigators, has determined that there is a minimum of time of deſcent even in a circular arc: for an arc of a circle which does not exceed 60 degrees

is a curve of ſpeedier deſcent than any other curve which can be drawn within the ſame arc; and the arc of 90 degrees is a curve of ſpeedier deſcent than any other curve which can be drawn without the ſame arc. And many other theorems might be found, particularly if we were to aſſume different hypotheſes of gravity: but all we ſhall attempt here is to give the ſolution of the problem upon the common ſuppoſition of gravity being a conſtant force acting in parallel lines.

PROP. *To determine the curve along which a body, ſolicited by gravity, will paſs from one given point* A *to another point* B, *not in the ſame vertical line, in the ſhorteſt time poſſible.*

Let AC (fig. 2. pl. XIII.) be parallel and BE perpendicular to the horizon, interſecting each other in E, and let PM be any ordinate to the curve parallel to EB. Let AP $=x$, PM $=y$, AM $=z$; then the velocity at M will (arts. 243. 263.) be expreſſed by $y^{\frac{1}{2}}$, and conſequently the fluxion of the time of deſcent through AM will be truly defined by $\frac{\dot{z}}{y^{\frac{1}{2}}}$ or its equal $y^{-\frac{1}{2}} \times (\dot{x}\dot{x} + \dot{y}\dot{y})^{\frac{1}{2}}$. Here, therefore, the fluent of $y^{-\frac{1}{2}} \times (\dot{x}\dot{x} + \dot{y}\dot{y})^{\frac{1}{2}}$ is to be a *minimum* when that of $\dot{x}$ obtains a given value AE. Whence we muſt have $y^{-\frac{1}{2}}\dot{x} \times (\dot{x}\dot{x} + \dot{y}\dot{y})^{-\frac{1}{2}} =$ a conſtant quantity; which, in order that the terms may be homologous, may be denoted by $a^{-\frac{1}{2}}$, or $\frac{1}{\sqrt{a}}$. Then $a^{\frac{1}{2}}\dot{x} = y^{\frac{1}{2}} \times (\dot{x}\dot{x} + \dot{y}\dot{y})^{\frac{1}{2}}$; therefore $\dot{x} = \frac{y^{\frac{1}{2}}\dot{y}}{\sqrt{a-y}} = \frac{y\dot{y}}{\sqrt{ay-y^2}}$; and $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2} = \frac{a^{\frac{1}{2}}\dot{y}}{\sqrt{a-y}}$; conſequently $z = 2a - 2a^{\frac{1}{2}} . \sqrt{a-y}$. Hence when $y=a$, $z=2a$; that is, if theſe values of y and z be repreſented by CV and the arch VMA, the latter will be double the former; which, as we have ſhewn (art. 275. cor. 2.), is a property of the cycloid whoſe vertex is V, and diameter of its generating circle CV.

When the point B falls on the other ſide the vertex with reſpect to A, as it muſt do when EB is leſs with reſpect to AE than VC to AC, or than the diameter of a circle to half its circumference, the proceſs will be the ſame, and it will terminate in a ſimilar concluſion. The cycloid therefore is the curve required.

COR. 1. If the celerity be ſuppoſed as any function F of the quantity y, the problem may be reſolved in the ſame manner: in that caſe the equation of the curve will be $\frac{\dot{x} \times (\dot{x}^2 + \dot{y}^2)^{-\frac{1}{2}}}{F} = \frac{1}{a}$.

Cor. 2. From the above inveſtigation a very ſimple conſtruction may be deduced. Thus, A and P (fig. 3. pl. XIII.) being the given points: through A draw the horizontal line AB, on part of which as a baſe deſcribe any inverted cycloid A*pb*; join A, P, by the line AP cutting the curve A*pb* in *p*; join *pb*, and parallel to it draw PB meeting AB in B: ſo ſhall AB be the baſe of the inverted cycloid AVB, through which the body will paſs from A to P in the ſhorteſt time poſſible. This conſtruction is founded upon the property that cycloids are ſimilar curves, having only one conſtant quantity entering their equation, namely, the diameter of the generating circle. We ſee, alſo, that for any two given points A, P, there can be only one cycloid which will anſwer the conditions of the problem of the brachyſtochronon.

It ſhould be remarked, that this ſimple conſtruction was firſt given by Newton in the Philoſophical Tranſactions, No. 224.

278. Prop. *If* AVP *be the line of ſwifteſt paſſage from* A *to* P (fig. 4. pl. XIII.), *and* AP *the right line joining the points* A *and* P; *then, if* PS *be drawn perpendicular to the curve in* P, *and* AS *let fall perpendicularly from the point* A *upon* PS, *the time in which a body drawn by gravity deſcribes the right line* AP, *is to the time in which it would paſs from* A *to* P *along the curve* AVP, *as* AP *to* AS.

Through P draw PN parallel to the axis of the cycloid, and PT parallel to the baſe BA, meeting the axis in H, and a circle deſcribed on the diameter CV in F and Q, and, laſtly, meeting the cycloid in T. Draw the chord CF, which (art. 274.) will be parallel to PS. Whence PM=FC, and MC=PF=(art. 273.) FV: and conſequently AM=CQVF. By art. 276. the time in arc ATV : time in CV : : ſemicircumference : diameter; and by art. 276. cor. 2. the time of deſcribing VP (after having paſſed over AV) : time in ATV : : VF : VQC. Therefore time in AVP : time in CV : : arc CQVF : diam. CV. But, time in CV : time in NP or CH : : CV : CF; therefore, by equality, time in AVP : time in NP : : arc CQVF : chord CF : : AM : MP. Again, time of fall through NP : time in right line AP : : NP : AP; therefore, the ratio of the time along AVP to the time in AP is compoſed of the ratios of AM to MP, and of NP to AP; that is, time along AVP : time in AP : : AM . NP : MP . PA. But AM . NP = MP . AS, each being the double of the triangle AMP. Conſequently, the time in which a body falling from reſt runs through the curve of the cycloid AVP, is to the time in which it would paſs over the right line AP, as MP . AS to MP . PA, or as AS to AP. And in the ſame manner (mutat. mutan.) the demonſtration proceeds when the point P is between A and V.

Cor. 1. Since AP the hypothenuſe of the right-angled triangle

ASP is always greater than AS, the time of deſcent through the right line AP is always greater than the time of paſſing through the arc AP.

COR. 2. When P coincides with V, PS and AS coincide with VC and AC reſpectively; and then, time in arc AV to time in chord AV, as $\frac{1}{2}\pi$ to $\sqrt{1+\frac{1}{4}\pi^2}$, or as 1·5708 to 1·8621 nearly.

CHAPTER III.

On Central Forces.

279. SINCE a body, when once put into motion, will (art. 21.), unleſs prevented by obſtacles, perſevere in that ſtate with the ſame velocity and the ſame direction, it follows that a body cannot deſcribe a curve line unleſs it is ſubjected to the action of a conſtant force, or meets with obſtacles after obſtacles which change at every inſtant the direction of its motion. If the force which acts on a moving body according to any direction different from that in which it is moving, acts at finite intervals of time, and communicates at each interval a determinate velocity, the body will deſcribe a polygon. But if the body has received at firſt a finite velocity, and the force which deflects it from its path acts continually or without interruption, the body will then deſcribe a curve line: ſuch is the effect of the conſtant ſolicitation of gravity; and ſuch alſo that of the reſiſtance of fluids.

A body which is moving in a curvilinear track may be conſidered, at any inſtant whatever, as if it were moving along the tangent of that point of the curve at which it is found; and if the force which deflects the body at any inſtant ceaſes to act, the body will perſevere in its motion according to the direction of that agent.

DEFS. 1. The *centre of attraction or of force* is the point towards which any body is ſolicited or impelled.

2. *Deflecting force* is the force which tends to bend the courſe of a body at every inſtant.

3. *Centripetal force* is the force which tends conſtantly to ſolicit or to impel a body towards a certain fixed point or centre.

4. *Centrifugal force* is that by which it would recede from ſuch centre, were it not prevented by the centripetal force.

5. Theſe two forces are called jointly CENTRAL FORCES.

COR. The centrifugal and centripetal forces, being correlatives in circular motions, may be repreſented by the ſame line.

6. *Projectile force* is that with which the body would run out in a tangent to its path, if there were no centripetal force to prevent it.

Cor. The centripetal and projectile forces are heterogeneous, and cannot be compared. For the action of the one is inceſſant, but that of the other is impulſive.

7. The path deſcribed by a body acted upon by a centripetal force is often called its *trajectory*, or its *orbit*.

8. *Radius vector* is a line drawn from the centre to which the force is referred, or in which it is ſuppoſed to act, to any point in the trajectory wherein the body is found.

9. *Angular velocity* is the velocity with which the angle is deſcribed which is contained between any two poſitions of the radius vector; it is meaſured by an arc of a circle whoſe radius is a unit of diſtance, and comprehended between two poſitions of the radius vector at the interval of a unit of time.

10. When a body moves round a centre in an orbit or trajectory which returns into itſelf, the time employed by the body after paſſing a certain point before it returns to that point again is called the *periodical time*.

280. Prop. *If from two points* A *and* D, *equally remote from the centre of attraction* C (fig. 5. pl. XIII.) *two bodies move with equal velocities, the one along the right line* AC, *the other in a curve line* DBQ, *their velocities at all equal diſtances from the centre will be equal.*

Let DK in the tangent of the curve at the point D be the ſpace which would be deſcribed in an evaneſcent portion of time with the velocity at D, FG the arc of a circle whoſe centre is C, and GK its tangent; and while AF would be deſcribed with the velocity at A, let FH be added to it by the attractive force. Draw the arc HI, and IL a tangent to it at I, meeting DK produced in L: draw alſo KB parallel to DC, and LB perpendicular to DL. Then, $DG : DK :: GI : KL :: KL : KB$, by ſimilar triangles; therefore, $GI : KB :: DG^2 : DK^2$; and conſequently KB will be the ſpace deſcribed by the attractive force, while DK would be deſcribed with the velocity at D; for the force may be conſidered as uniform during the deſcription of the evaneſcent increments, and the ſpaces deſcribed by the action of a uniform force are as the ſquares of the times (art. 227.). Hence, the reſultant will be DB, which is ultimately equal to DL; and the whole velocity will be increaſed in the ratio of DL to DK, or of DI to DG, or of AH to AF: conſequently, ſince H, I, and L, are ultimately equidiſtant from C, the velocities in EA and ED are always equally increaſed at equal diſtances, and will, therefore, always remain equal at equal diſtances.

Cor. The ſame thing may in ſimilar circumſtances be ſhewn to obtain with reſpect to the velocities in EC, and any other curve $E'D'Q'$; and conſequently it will hold with regard to the two curves: therefore, generally, *two or more bodies being attracted toward a given centre with equal forces at equal diſtances, if their velocities be once equal at equal diſtances, they will remain always equal at equal diſtances, whatever be their directions.*

281. Prop. *A body revolving in an orbit, by the joint effects of a projectile and a centripetal force, deſcribes by its radius vector equal areas in equal times, and in unequal times areas proportional to the times.*

Let ABCD, &c. (fig. 7. pl. XIII.) be part of an orbit deſcribed by a body, which is ſolicited by a centripetal force towards the point φ. If the projectile force alone acted, the body would move uniformly in a right line as AV: but if the centripetal force be ſuppoſed to act by ſeparate impulſes or ſolicitations, as at B, C, D, &c. when the body receives the impulſe at B it will be drawn out of its courſe towards φ, and (art. 217.) will deſcribe the diagonal BC in the ſame time as the projectile force alone would have made it deſcribe BC. After equal intervals ſimilar effects will take place at C, D, &c. Since AB, BC, &c. are the lines deſcribed in equal times by the body, the areas deſcribed in equal times by the radius vector will be AφB, BφC, &c. Now AB, B*c*, expreſſing ſpaces paſſed over by a uniform motion, are equal baſes of the triangles AφB, Bφ*c*, which being terminated by the ſame point φ, have likewiſe equal altitudes, and are therefore equal. And becauſe the body at B is by the joint action of the projectile and centripetal forces carried forwards in the diagonal BC of the parallelogram G*c*, the oppoſite ſides GB, C*c*, are parallel, and C*c* is parallel to Bφ. But Bφ is the common baſe of the two triangles BφC, Bφ*c*; and theſe triangles are between the ſame parallels; therefore they are equal. Conſequently AφB, which has been proved equal to Bφ*c*, is likewiſe equal to BφC; that is, the areas deſcribed in equal times are equal. And, by compoſition, any ſums of theſe areas are to each other as the times in which they are deſcribed: that is, the areas are univerſally as the times.

Let the number of theſe triangles be augmented, and their breadths diminiſhed indefinitely, the centripetal force being now ſuppoſed to act continually: then will the ultimate perimeter ABCD, &c. be a curve line, which is always concave towards φ the centre of force; and, the above reaſoning being ſtill applicable to thoſe triangles whoſe breadths are indefinitely diminiſhed, the areas will be as the times.

It does not neceſſarily follow, that the centripetal force ſhould cauſe the body always to *approach* the centre of force; it may continue to recede from it, notwithſtanding its being drawn by a force reſiding there: but this property muſt always belong to its motion, that the trajectory which it deſcribes be concave towards the centre of force.

COR. 1. *The plane in which the trajectory lies paſſes through the centre of force.* For the diagonal BC of the parallelogram cG is in the ſame plane as its ſides; therefore BC, Bϕ, are in the ſame plane: in the ſame manner Bd, CD, Cϕ, are in one plane, being the ſame as the former; and ſo on.

COR. 2. *The projectile velocity of a body revolving in a curve about an immoveable centre of force, is reciprocally as the perpendicular let fall from that centre upon the tangent to that point of the orbit where the body is when its velocity is eſtimated.* For the area of any of the triangles AϕB, BϕC, being conſtant, the baſe which repreſents the velocity is reciprocally as the perpendicular demitted upon it from ϕ.

COR. 3. *The angular velocity at the centre of force is reciprocally as the ſquare of the body's diſtance from that centre.* For, if the ſmall triangles ϕCD, ϕBA (fig. 8. pl. XIII.) are equal, they are deſcribed in equal times: and 2 area $C\phi D = \phi C \times CQ$; alſo 2 area $\phi BA = \phi B \times BP$; therefore $\phi C \times CQ = \phi B \times BP$. But, angle $C\phi D$: angle $A\phi B :: CQ : cq :: \phi C \times CQ : \phi C \times cq :: \phi B \times BP : \phi C \times cq ::$ area $\phi BA ::$ area $\phi cq :: \phi B^2 : \phi C^2$ or ϕC^2.

COR. 4. If AB, BC, and DE, EF (fig. 7.), the arcs deſcribed in equal times, be completed into the parallelograms AC, DF, the centripetal forces at B and E will be to each other in the ultimate ratio of the diagonals BG, EZ, when thoſe arcs are indefinitely diminiſhed. For the motions of the body BC, EF, are compounded of the motions Bc, BG, and Ef, EZ; but BG and EZ are equal to Cc and Ef, which, as appears from this propoſition, are generated by the impulſes of the centripetal force in B and E, and are therefore proportional to thoſe impulſes.

COR. 5. *The forces with which bodies are drawn into curvilinear orbits are to each other as double the verſed ſines* $\frac{1}{2}$ GB, $\frac{1}{2}$ ZE, *of the indefinitely ſmall arcs* ABC, DEF, *deſcribed in equal times; and theſe verſed ſines converge to the centre* ϕ, *and biſect the chords when thoſe arcs are diminiſhed indefinitely:* for ſuch verſed ſines are half the diagonals of parallelograms; BG, EZ, being biſected by the diagonals AC, FD.

COR. 6. Converſely, *if the radius vector of a body deſcribe areas proportional to the times about any point* ϕ, *it is urged towards that point by the centripetal force.* For if ϕ be not the centre of force, let it be ſome other point, as S: then, by the prop. the body will ſo move that its radius vector will deſcribe alſo areas proportional to the times, about the centre S; which is impoſſible; for it is manifeſt a body cannot univerſally deſcribe areas proportional to the times about two different centres in the ſame plane.

282. Prop. *To determine the ratio of the forces by which bodies tending to the centres of given circles are made to revolve in their peripheries.*

Let AMa (fig. 12. pl. XII.) be the circle in which one of the bodies moves round the centre of force C, and let the indefinitely little arch A o be the diſtance it moves over in a given or conſtant particle of time; then, cor. 5. of the preceding prop. the centripetal force at o will be meaſured by twice Ap. And by the nature of the circle, the chord and arc A o will be ultimately equal in length; whence $\mathrm{A\,o}^2 = \mathrm{A}a \times \mathrm{A}p = \mathrm{AC} \times 2\,\mathrm{A}p$; conſequently, $2\,\mathrm{A}p = \frac{\mathrm{A\,o}^2}{\mathrm{AC}}$. And the ſame may be ſhewn with reſpect to the motion in any other circle, So that, if R, and r, denote the radii of two circles, F, and f, the reſpective central forces, V and v, the velocities with which the bodies move in their peripheries, we ſhall have $\mathrm{F} : f :: \frac{\mathrm{V}^2}{\mathrm{R}} : \frac{v^2}{r}$; therefore, *the forces are as the ſquares of the velocities directly, and as the radii inverſely.*

Cor. 1. *In a circle the velocity is uniform, if the centre of force coincide with the centre of the circle.* For the radius which is the perpendicular to the tangent (cor. 2. art. 281.) is a conſtant quantity.

Cor. 2. Becauſe $\mathrm{F} : f :: \frac{\mathrm{V}^2}{\mathrm{R}} : \frac{v^2}{r}$, it follows that

$$\mathrm{V} : v :: \sqrt{\mathrm{RF}} : \sqrt{rf}, \text{ and}$$

$$\mathrm{R} : r :: \frac{\mathrm{V}^2}{\mathrm{F}} : \frac{v^2}{\mathrm{F}}.$$

Cor. 3. Comparing the analogy $\mathrm{V} : v :: \sqrt{\mathrm{RF}} : \sqrt{rf}$, with the expreſſion $s \propto \frac{v^2}{2f}$ in uniformly accelerated motions, it follows that *the velocity is every where equal to that which a body would acquire in falling by the ſame uniform force through half the radius.*

Cor. 4. If the ratio of the periodic times be denoted by that of P to p, then the ratio of the velocities being as $\frac{\mathrm{R}}{p}$ to $\frac{r}{p}$, we ſhall have by equality $\sqrt{\mathrm{RF}} : \sqrt{rf} :: \frac{\mathrm{R}}{\mathrm{P}} : \frac{r}{p}$; whence alſo

$$F : f :: \frac{R}{P^2} : \frac{r}{p^2}, \text{ and}$$

$$R : r :: FP^2 : fp^2.$$

COR. 5. A few other properties of circular motion under different hypotheſes may be thrown together; thus,

If $P : p :: 1 : 1$, then $F : f :: R : r$
If $V^2 : v^2 :: r : R$, then $F : f :: r^2 : R^2 :: V^4 : v^4$.
alſo $P : p :: R^3 : r^3$.
If $F : f :: 1 : 1$, then $P : p :: \sqrt{R} : \sqrt{r}$.
If $V : v :: r : R$, then $F : f :: r^3 : R^3 :: V^3 : v^3$.

COR. 6. If the meaſure of the force, or the velocity which would be uniformly generated in a unit of time, be expounded by any power r^n of the radius AC (fig. 12. pl. XII.), then the diſtance through which a body would freely deſcend in the ſame time by the conſtant operation of that force will (art. 243.) be expreſſed by $\frac{1}{2} r^n$. Hence, as the diſtances deſcended by means of the ſame force uniformly continued are as the ſquares of the times, it is evident if the time of moving through Ao be denoted by t, that the diſtance Ap deſcended in that time will be denoted by $\frac{t^2}{1^2} \times \frac{1}{2} r^n$: ſo that we ſhall have $Ao = (\sqrt{2Ap.AC}) = \frac{t}{1} \times r^{\frac{n+1}{2}}$; which being the diſtance deſcribed by the revolving body in the time t, it follows that the ſpace paſſed over in the given time 1, will be equal to $r^{\frac{n+1}{2}}$.

COR. 7. Hence, to find the periodic time we have $r^{\frac{n+1}{2}} : \pi \times 2r$ (the whole periphery) $:: 1 : 2\pi r \div r^{\frac{n+1}{2}} = 2\pi^{\frac{1-n}{2}}$, the true meaſure of the periodic time.

COR. 8. Hence alſo it follows that if n be expounded by $1, 0, -1, -2, -3$ ſucceſſively, the velocity correſponding will be as r, $r^{\frac{1}{2}}$, 1, $r^{-\frac{1}{2}}$, and r^{-1}; and the time of revolution as 1, $r^{\frac{1}{2}}$, r, $r^{\frac{3}{2}}$, and r^2, reſpectively.

SCHOLIUM.

283. From the preceding propoſition and its corollaries the velocity and periodic time of a body revolving in a circle, at any given diſtance from the earth's centre, by means of its own gravity, may be deduced. Thus, let the radius of the earth ($=21000000$ feet, nearly) be denoted by r, and the ſpace through which a heavy body falls at the ſurface ($=16\frac{1}{12}$ feet) by $\frac{1}{2}g$, the force of gravity at the ſurface being denoted by g;

then will the velocity per ſecond in a circle at the ſurface (cor. 2.) be $= \sqrt{gr} = 26000$ feet nearly; and the time of revolution $= \frac{2\pi r}{\sqrt{gr}} = \pi\sqrt{\frac{4r}{g}} = 5075$ ſeconds. Let R be put for the radius of any other circle deſcribed by a projectile about the earth's centre: then, becauſe the force of gravitation above the ſurface varies inverſely as the ſquare of the diſtance, we have, by cor. 8. $r^{-\frac{1}{2}} : R^{-\frac{1}{2}} ::$ 26000 feet (velocity per ſecond at the ſurface) : $26000\sqrt{\frac{r}{R}}$, the velocity in the circle whoſe radius is R. And $r^{\frac{3}{2}} : R^{\frac{3}{2}} :: 5075^s$ (the periodic time at the ſurface) : $5075\sqrt{\frac{R^3}{r^3}}$, the periodic time in the circle whoſe radius is R.

For example, if R be aſſumed equal to 60 r, the diſtance of the moon from the earth, the expreſſion for the velocity will become $3356\frac{1}{2}$ feet per ſecond; and that for the periodic time will become 2360035^s or $27\frac{3}{10}$ days, nearly.

284. Thus alſo the ratio of the forces of gravitation of the moon towards the ſun and the earth may be eſtimated. For $365\frac{1}{4}$ days being the periodic time of the earth and moon about the ſun, and 27·3 days the periodic time of the moon about the earth; alſo 60 being the diſtance of the moon from the earth in terms of the earth's radius, and 23920 her mean diſtance from the ſun in the ſame meaſure, we have, by cor. 4. $\frac{23920}{365{\cdot}25^2} :: \frac{0}{27{\cdot}3^2} :: F : f :: 2\frac{2}{9} : 1$ nearly; that is, the moon's gravitation towards the ſun is to her gravitation towards the earth as $2\frac{2}{9}$ to 1 nearly.

285. Again, from the ſame principles the centrifugal force of a body at the equator ariſing from the rotation of the earth is derived. For the propoſition and corollaries apply to centrifugal forces as well as centripetal ones; the terms, as before obſerved, being correlatives (when thoſe two alone keep the body in its orbit). And we have juſt found (art. 283.) that the time of revolution is 5075^s when the centrifugal force would become equal to the gravity; alſo (cor. 4.) it appears that the forces in circles having the ſame radii are reciprocally as the ſquares of the periodic times: hence, therefore, ſince the earth's rotation is performed in $23^h.\ 56^m.$ or $86160^s.$, we have $86160^2 : 5075^2 ::$ the force of gravity : the centrifugal force of a body at the equator ariſing from the earth's rotation :: $1 : \frac{1}{289}$ nearly.

286. Since the time of revolution of a body under the equa-

tor EQ (fig. 6. pl. XIII.), and in any parallel of latitude BG, is equal, the centrifugal forces (cor. 5.) are as the diſtances CE, AB, from the axis of motion, or, as radius CB, to the coſine AB of the latitude. But in any latitude, as at B, the centrifugal force is not (as under the equator) oppoſite to the whole gravity, but only a part of it, which alſo is to the whole as the coſine of the latitude to radius. For, produce AB the direction of the centrifugal force to D, and CB the direction of gravity, till it meet a perpendicular let fall upon it from D at F; then BD repreſenting the whole centrifugal force at B, BF will repreſent that part of it which is directly oppoſed to gravity; but BD : BF : : BC : AB : : rad. : coſin. BE. Therefore, combining theſe two ratios, it follows, that *the diminution of gravity at the ſurface of the earth ariſing from the centrifugal force varies as the ſquare of the coſine of the latitude.*

The law juſt ſtated for the diminution of gravity is on the ſuppoſition of the earth's ſphericity: but as the polar axis of the earth is rather ſhorter than the equatoreal, the above theorem will not be quite exact: it would, however, take us too far from the ſubject of central forces to conſider that particular here; we ſhall therefore merely add a general theorem for the relations of gravity under different latitudes, which is this. Let γ, the gravity under the equator, G that at either pole, g that under any latitude λ, then $g = (1 + {\cdot}0052848 \text{ ſine}^2\lambda)\,\gamma$; and $\therefore G = 1.0052848\,\gamma$.

287. But now to determine more univerſally the ratio of the force of a body revolving in any given circle to its gravity; we have already given $\pi\sqrt{\frac{4r}{g}}$, for the periodic time at the ſurface of the earth, when the gravity and centrifugal force are equal: if, therefore, the time of revolution in any circle whoſe radius is ϱ feet be denoted by t ſeconds, it will follow, from cor. 4. of the prop. that $\frac{r}{\pi^2 . \frac{4r}{g}} : \frac{\varrho}{t^2}$: : gravity of body : its centrif. force in that circle; which is as unity to $\frac{4\pi^2\varrho}{g t^2}$, or as 1 to $1{\cdot}2274 \times \frac{\varrho}{t^2}$ very nearly. Thus, if the length of a ſling by which a ſtone is whirled about be 2 feet, and the time of revolution half a ſecond, the force by which the ſtone endeavours to fly off will be to its weight as $1{\cdot}2274 \times \frac{2}{{\cdot}25}$ to 1, or as $9{\cdot}8192$ to unity.

The circumference of the circle whoſe radius is ϱ being $2\pi\varrho$, and t the periodic time, the velocity in the circle will be $\frac{2\pi\varrho}{t}$; and if this be put for v in the general expreſſion $v=\sqrt{2gs}$ (art. 243.) we ſhall have $\frac{2\pi\varrho}{t}=\sqrt{2gs}$, whence $s=\frac{2\pi^2\varrho^2}{gt^2}$, the ſpace a heavy body muſt deſcend freely to acquire the velocity in the circle; but it is manifeſt that $\varrho:\frac{4\pi^2\varrho^2}{gt^2}::1:\frac{4\pi^2\varrho}{gt^2}$, the preceding expreſſion for the centrifugal force; ſo that our theorem agrees with the comprehenſive one firſt given by the Marquis de l'Hopital, namely, *find from what height the body muſt have fallen to acquire the velocity in the circle; then, as the radius of the circle to double that height, ſo is the weight of the body to its centrifugal force.*

288. From the general proportion in the preceding article, the centrifugal force and periodic time of a pendulum deſcribing a conical ſurface may readily be found. Thus, let CA (fig. 9. pl. XIII.) the length of the pendulum, be denoted by l; CS, the altitude of the cone, by a; the ſemidiameter AS of the baſe by ϱ, and the periodic time by t: then, becauſe the body is retained in the circle by three different forces, viz. the centrifugal force $\frac{4\pi^2\varrho}{gt^2}$, in the direction SA, the force of gravity, or the weight, 1, in a direction parallel to CS, and the force of the thread AC, compounded of the two former; it follows that CS : CA or as $a:b::$ weight of body at A : force upon the thread at A; alſo, as $1:\frac{4\pi^2\varrho}{gt^2}::$ CS : SA $::a:\varrho$. Whence $gt^2=4a\pi^2$, and $t=2\pi\sqrt{\frac{a}{g}}=1{\cdot}10784\sqrt{a}$. Conſequently, *the periodic time varies as the ſquare root of the altitude of the conic pendulum, let the radius of its baſe be what it may.*

Becauſe $\frac{1}{2}gt^2$ or its equal $2a\pi^2$, repreſents the ſpace a heavy body will deſcend by its own gravity in t ſeconds (art. 243.); and becauſe $1^2:\pi^2::2a:2a\pi^2=\frac{1}{2}gt^2$; it hence appears, that *as the ſquare of the diameter of any circle is to the ſquare of its periphery, ſo is twice the altitude of the cone to the diſtance a heavy body will freely deſcend in the time of a complete gyration of the conical pendulum.*

Hence alſo, if CS be to CA as a ſquare inſcribed in a circle to the ſquare of its circumference, or *if the angle* CAS *be nearly*

$2^\circ\ 53'\ \frac{5}{6}$, *the periodic time of the pendulum will be equal to the time of free deſcent through* CS *.

289. PROP. *To determine the law of centripetal force tending to a given point* C, *by which a body may deſcribe a given curve* APQ.

Draw PT a tangent to the curve at any point P (fig. 10. pl. XIII.), and from C, the propoſed centre of force, demit the perpendicular CT: let PO, the radius of curvature at the point P, be denoted by R, the diſtance or radius vector CP by ρ, and the perpendicular CT upon the tangent by p, the velocity in the curve being denoted by v. Then, becauſe the centripetal forces in circles are as the ſquares of the velocities directly and the radii inverſely (art. 282.), it follows that the force tending to the point O, by which the body might be retained in its orbit at P, which muſt manifeſtly be the ſame as the force in the circle whoſe radius is PO, will be defined by $\frac{v^2}{R}$, or by $\frac{1}{p^2 R}$, ſince v is inverſely as p, by cor. 2. art. 281. Wherefore, by the reſolution of forces, we ſhall have CT $(=p)$: CP $(=\varrho)$: : $\frac{1}{p^2 R}$ (force in direction PO) : $\frac{\varrho}{p^3 R}$, force in the direction PC.—Now, the general expreſſion for the radius of curvature is $R = \frac{\varrho \dot{\varrho}}{\dot{p}}$ which value of R ſubſtituted for it in the preceding expreſſion for F, the centripetal force towards C, will transform it to this; $F = \frac{\dot{p}}{p^3 \dot{\varrho}}$, an equation expreſſing the law required.

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* When a body is made to deſcribe a circle by being fixed to one extremity of a ſtring (or of an inflexible bar), while the other extremity is attached to an immoveable point, or by moving along the concave ſuperficies of a poliſhed ſphere or cylinder; in both caſes, whatever be the proportion in which the centrifugal force is increaſed by increaſing the velocity of the projection, the reaction of the ſtring or of the ſurface will always be increaſed in the ſame proportion, ſo that the body will deſcribe the ſame circle with different degrees of velocity. But when the centripetal force and the diſtance from the centre are given, the velocity is given (art. 282. cor. 3.), being that which would be acquired by falling down half the diſtance. If, therefore, the velocity be increaſed whilſt the centripetal force continues the ſame, the centrifugal force being increaſed in the duplicate ratio of the velocity, it will be greater than the centripetal; therefore in the time that the body would have deſcribed any diſtance MT in the tangent (fig. 12. pl. XII.) it will be drawn to a greater diſtance than A from the centre, and will have deſcribed a curve exterior to the circle MA *m*. For a like reaſon, if the velocity be diminiſhed, the centrifugal force becoming leſs than the centripetal, the body will deſcribe a curve interior to the circle: but if the centripetal force be at the ſame time increaſed or diminiſhed in the ſame proportion, the body will ſtill be retained at the ſame diſtance from the centre, and deſcribe the circle MA *m a*.

Hence it is manifeſt that when a body deſcribes any orbit exterior or interior to that of the circle, the tangent being perpendicular to the radius vector, the centrifugal force of the body in its orbit is equal to the centripetal force with which the body would deſcribe a circle at the ſame diſtance, and with the ſame velocity in the direction perpendicular to the radius vector. The ſame will be true if the direction be not perpendicular to the radius vector: for in this caſe, if the motion be reſolved into two, one in the direction of the radius vector, and the other perpendicular to it, the latter is the only part which will increaſe or diminiſh the centrifugal force.

In this laſt caſe the body is retained in its orbit by three forces; the centripetal and centrifugal forces, and that part of the motion in the tangent which is in the direction of the radius vector.—NEWTON *on Ultimate Ratios*.

Another expreſſion, which will be ſometimes uſeful, may be found by taking the value of the radius of curvature in terms of the arc and its rectangular co-ordinates, that is, $R = \frac{\dot{z}^3}{-\dot{x}\ddot{y}}$; for this introduced into the expreſſion $F = \frac{\varrho}{p^3 R}$ will convert it to $F = \frac{\mp \varrho \dot{x} \ddot{y}}{p^3 \dot{z}^3}$.

Cor. 1. If the point C be ſo remote that all right lines drawn from thence to the curve may be conſidered as parallel to each other, then making Pr perpendicular to Cp (Pp being an evaneſcent portion of the curve), the force will be as $\frac{-\varrho \dot{x} \ddot{y}}{(CP \times Pr)^3}$, or barely as $\frac{-\dot{x} \ddot{y}}{(Pr)^3}$, ſince $\varrho = CP$ may in this caſe be rejected.

This expreſſion being general in all caſes where the force acts in parallel directions, it hence follows that the force which always acting in the direction of the ordinate PM would retain the body in its orbit is every-where as $\frac{-\ddot{y}}{\dot{x}^2}$; becauſe in this caſe PC coincides with PM, and Pr becomes $= \dot{x}$.

Cor. 2. Since the force tending to the point C is univerſally as $\frac{CP}{CT^3 \times PO}$ or $\frac{\varrho}{p^3 R}$, the force to any other point c will of conſequence be as $\frac{CP}{ct^3 \times PO}$. Therefore, the forces to different centres C and c, about which equal areas are deſcribed in the ſame time, are to each other in the inverſe ratio of $\frac{CT^3}{CP}$ to $\frac{ct^3}{cP}$.

Cor. 3. Hence alſo the ratio of the velocity at P to that by which a body might revolve in a circle about the centre C at the diſtance CP is eaſily obtained. For, ſince the velocity at P is that by which the body would revolve in a circle about the centre O, and the forces tending to the centres O and C are to each other as $p = CT$ and $\varrho = CQ$; it therefore follows, if the ratio ſought be aſſumed as v to u, that $\frac{v^2}{PO} : \frac{u^2}{PC} :: p : \varrho$ (art. 282.). Whence alſo $v^2 : u^2 :: p \times PO$ $(= p\,R) : \varrho \times PC$ $(= \varrho^2)$; and conſequently $v : u :: \sqrt{\frac{pR}{\varrho\varrho}} : 1 :: \sqrt{\frac{p\dot{\varrho}}{\varrho\dot{p}}} : 1 :: \sqrt{\frac{\dot{\varrho}}{\varrho}} : \sqrt{\frac{\dot{p}}{p}}$ becauſe $R = \frac{\varrho\dot{\varrho}}{\dot{p}}$.

Cor. 4. Finally, the law of centripetal force being given, the nature of the trajectory may hence be found: for, ſince the force F is univerſally defined by $\frac{\dot{p}}{p^3\dot{\varrho}}$, it is manifeſt that the fluent of $F\dot{\varrho} = \frac{-1}{2p^2}$, which, when F is given in terms of ϱ, will become known; and then, the relation between p and ϱ being given, the curve itſelf is known.

We now ſhew the application of this propoſition by an example or two.

290. Ex. I. *Let it be required to find the force tending to the centre of an ellipſe when a body revolves in its periphery.*

Let the ſemitranſverſe C′A (fig. 11. pl. XIII.), be denoted by a, the ſemiconjugate axe C′E by b. the radius vector C′P′ by ϱ, and its ſemiconjugate C′R by n: then, by the nature of the ellipſe, (Emerſon's Con. I. 34.) $\varrho\varrho + nn = aa + bb$, whence $n = \sqrt{a^2+b^2-\varrho^2}$: again (ibid. I. 37.) $n\ (= \sqrt{a^2+b^2-\varrho^2}) : b :: a : p$ $(= c' t) = \frac{ab}{\sqrt{a^2+b^2-\varrho^2}}$; therefore $\dot{p} = \frac{ab\varrho\dot{\varrho}}{(a^2+b^2-\varrho^2)^{\frac{3}{2}}}$. Conſequently, $\frac{\dot{p}}{p^3\dot{\varrho}} = \frac{ab\varrho\dot{\varrho}}{(a^2+b^2-\varrho^2)^{\frac{1}{2}}} \times \frac{(a^2+b^2-\varrho^2)^{\frac{3}{2}}}{a^3b^3\dot{\varrho}} = \frac{\varrho}{a^2b^2}$. So that the force tending to the centre of the ellipſe is directly as the radius vector.

Ex. II. *To find the law of the centripetal force, by which a body* *tending to the focus* C *is made to revolve in the periphery of an ellipſe.*

From the other focus F (fig. 11.) draw FS parallel to CT, meeting the tangent TP at right angles in S; join FP: then put AB$=2a$, C′D$=b$, as before, and the latus-rectum or parameter $\left(\frac{2b^2}{a}\right)=\lambda$. Denoting CP, and CT, as before, we have FP$=$AB$-$CP$=2a-\varrho$; whence, by reaſon of the ſimilar triangles CPT, FPS, it will be $\varrho : p :: 2a-\varrho :$ FS$=\frac{p(2a-\varrho)}{\varrho}$. But, by the nature of the curve, FS$\times$CT$=$C′D^2: whence, $\frac{p^2.(2a-\rho)}{\rho}=b^2$; and conſequently $\frac{1}{p^2}=\frac{2a}{b^2\rho}-\frac{1}{b^2}$. The fluxion of the latter expreſſion is $-\frac{2\dot{p}}{p^3}=-\frac{2a\dot{\varrho}}{b^2\varrho^2}$. So that we have (art. 289.) $\frac{\dot{p}}{p^3\dot{\varrho}}=\frac{a}{b^2}\times\frac{1}{\varrho^2}=\frac{2}{\lambda\varrho^2}$: and (art. 289. cor. 3.) $\sqrt{\frac{p\dot{\varrho}}{\varrho\dot{p}}}=\sqrt{\frac{2.(2a-\varrho)}{2a}}=\sqrt{\frac{\text{FP}}{\text{AC}'}}$. Hence it appears, that the centripetal force is in this caſe reciprocally as the ſquare of the diſtance ϱ or CP; and that the velocity at P is to that by which the body might deſcribe a circle at the diſtance CP, everywhere in the ratio of $\sqrt{}$FP to $\sqrt{}$AC′.

Otherwiſe thus: Let another body deſcending in a right-line begin to fall with the ſame velocity, then among the general equations for variable emotions we have (art. 232. III.) $\phi=\frac{v\dot{v}}{\dot{s}}$, where s is equivalent to ϱ in the preſent caſe; and $\dot{\varrho}$ being conſidered as negative, the equation becomes $\phi=-\frac{v\dot{v}}{\dot{\varrho}}$; the ſame being likewiſe true in the curve by art. 280. Now $v=\frac{1}{p}$ (art. 211. cor. 2.) and its fluxion $\dot{v}=\frac{-\dot{p}}{p^2}$, therefore $\phi=\frac{1}{p}.\frac{p}{p^2\dot{\varrho}}=\frac{\dot{p}}{p^3\dot{\varrho}}$: which is the ſame expreſſion as at art. 289. but deduced from art. 280. The reſt may then be determined from the properties of the ellipſe, as above.

Ex. III. *Required the law which would cauſe the body to move in a hyperbola, the force tending to the focus.*

In this inſtance, proceeding as before, we ſhall have $\frac{p^2.(2a+\varrho)}{\varrho} = b^2$, inſtead of $\frac{p^2.(2a-\varrho)}{\varrho} = b^2$, as in the ellipſe. From this we obtain $\frac{1}{p^2} = \frac{2a}{b^2\varrho} + \frac{1}{b^2}$; whence there will reſult $\frac{\dot{p}}{p^3\dot{\varrho}} = \frac{2}{\lambda\varrho^2}$, the very ſame as in the ellipſe.

Ex. IV. *To find the law which, when the force tends to the focus, would cauſe the body to move in a parabola.*

Here the equation will be $\frac{2a+0}{\varrho}.p^2 = b^2$, or $\frac{p^2}{\varrho} = \frac{b^2}{2a} = \frac{1}{4}\lambda$; and the force again as $\frac{2}{\lambda\varrho^2}$. But the meaſure of the velocity will in this inſtance become $\sqrt{\frac{p\dot{\varrho}}{\varrho\dot{p}}} = \sqrt{\frac{2.(2a-0)}{2a}} = \sqrt{2}$: conſequently the velocity in a parabola is to that by which the body might deſcribe a circle at the ſame diſtance from the centre of force in the conſtant ratio of $\sqrt{2}$ to unity.

291. PROP. *To determine the ratio of the velocities of bodies revolving in different orbits, about either the ſame or different centres; the orbits themſelves and the forces tending to the centres being given.*

Let APD (fig. 11. pl. XIII.) be any orbit which a body deſcribes about the centre of force C; let the force itſelf at the principal vertex A be denoted by F; let r denote the ſemiparameter, or the radius of curvature at A, and let CT be perpendicular to the tangent TP. Then (art. 282. cor. 2.) the velocity at A is always as $\sqrt{Fr}$; and (art. 281. cor. 2.) we have CT : CA : : $\sqrt{Fr}$ (the velocity at A) : $\frac{CA}{CT}\sqrt{Fr}$, the velocity at P. This anſwers, however the values of AC, r, and F, may vary.

COR. 1. If the centripetal force be as the ſquare of the diſtance inverſely, or $F \propto \frac{1}{AC^2}$, the velocity at P will become $\frac{AC}{CT}\sqrt{\frac{r}{AC^2}}$ or $\frac{\sqrt{r}}{CT}$. Conſequently *the velocities of bodies revolving in different orbits about a common centre are directly as the ſquare roots of the parameters, and reciprocally as the perpendiculars from the centre of force to the tangents to the curve at the points where the bodies are.*

Cor. 2. If the velocity at P be denoted by Pp, and Cp be drawn, then, ſince $Pp \propto \frac{\sqrt{r}}{CT}$, it follows that $\sqrt{r} \propto Pp \,.\, CT$, or as the triangle CP$p$. Hence, *the areas deſcribed about a common centre of force, by the radius vector, in a given time, are in the ſubduplicate ratio of the parameters.*

Cor. 3. And, ſince the area of the curve APDB, when an ellipſe, is known to be as $AC' \,.\, CD$, or as $AC' \,.\, \sqrt{r\,AC'}$; whence, if this be applied to $\sqrt{r}$, expreſſing, by the laſt cor. the area deſcribed in a given part of time, we ſhall obtain $AC' \,.\, \sqrt{AC'}$ or $AC'^{\frac{3}{2}}$ for the relative meaſure of the time of a complete revolution. Therefore, it appears, *that the periodic times, let the ſpecies of the ellipſes be what they will, are in the ſeſquiplicate ratio of the principal axes.*

This, of conſequence, obtains when the ellipſe becomes a circle: agreeing with what has been previouſly ſhewn, under the fourth caſe of cor. 8. art. 282.

Cor. 4. Hence, therefore, it follows *that the periodic time in an ellipſe is the ſame as in a circle whoſe diameter is equal to the tranſverſe axis, or radius equal to the mean diſtance* CD.

SCHOLIUM.

It appears from this propoſition and corollaries that the periodic time of a planet under the influence of a force varying inverſely as the ſquare of the diſtance depends *on its mean diſtance alone*, and will be the ſame whether the planet deſcribe a circle or an ellipſe having any degree of excentricity whatever. Now ſuppoſe the ſhorter axis DE of the ellipſe ADBE (fig. 11. pl. XIII.) to diminiſh continually, the longer axis AB remaining the ſame: then, as the extremity of the invariable line DC moves from D towards C′, the extremity C will move towards A, ſo that when D coincides with C′, C will coincide with A, and the ellipſe will be tranformed into a ſtraight line AB, the length of which is equal to 2 CD. In all the ſucceſſive ellipſes produced by this gradual diminution of C′D the periodic time remains unchanged, if the force acting at C continues unchanged. Juſt before the perfect coincidence of D with C′ the ellipſe may be conceived as undiſtinguiſhable from the line AB; and the revolution in ſuch ellipſe undiſtinguiſhable from the aſcent of the body along the right line from A to B, and the ſubſequent deſcent in an equal time, from B to A. Conſequently a body ſolicited by ſuch a central force will deſcend from B to A in half the time of the revolution in the ellipſe ADBE; and the time of deſcending through any diſtance DC (ſuppoſing the projectile force extinguiſhed) is half the periodic time of a body revolving at *half* that diſtance from the ſun. Hence we ſee that *the ſquares of the times of fall-*

ing to the ſun, or other centre of force, are as the cubes of the diſtances from it: agreeing with the concluſion in art. 236. To find this time of deſcent in particular inſtances, multiply half the periodic time by the ſquare root of the cube of $\frac{1}{2}$, or the whole periodic time by $\frac{1}{2}\sqrt{\frac{1}{8}}$, or by $\frac{1}{8}\sqrt{2}$, or by ·1767767. This proceſs gives us 4 *d.* 20 *h.* for the time of deſcent of the moon to the earth. Mercury to the ſun 15 *d.* 13 *h.* Venus, 39 *d.* 17 *h.* The earth, 64 *d.* 13¾ *h.* Mars, 121 *d.* 10½ *h.* Jupiter, 765 *d.* 21⅔ *h.* Saturn, 1901 *d.* 22½ *h.* Herſchell, 5433 *d.* 17 *h.* And the ſame rule will apply to the deſcent of the ſatellites of Jupiter, Saturn, &c. to their reſpective primaries. See Thorp's Newton, vol. I. pa. 193. Whiſton's Mathematical Philoſophy, pa. 173.

292. PROP. *The centripetal force tending to a given point* C (fig. 11.) *being inverſely as the ſquare of the diſtance, and the direction and velocity of a body at any point* P *being given; to determine the path in which the body moves, and, if it returns, the periodic time.*

It is manifeſt from art. 290. that the trajectory is a conic ſection, of which the point C is one of the foci. Let the other focus be F. Upon the tangent demit the perpendiculars CT, FS, and draw CP, FP. Let $AC'=a$, $CQ=\varrho$, as before; the ſine of the angle of direction $CPT=m$, to radius 1; and let the given velocity at P be to that by which the body might revolve in a circle about the centre at the diſtance CP, in any given ratio of n to unity. Then, by art. 290. $n:1::\sqrt{PF}:\sqrt{AC'}$; therefore, $n^2:1::PF\ (=2a-\varrho):AC'=a$: whence we find $AC'=\frac{\varrho}{2-n^2}$. Again, ſince $CT=m\,.\,CP$, and $FS=m\,.\,FP$ (the angles CPT, FPS being equal), we have $C'D^2=CT\,.\,FS=m^2\,.\,CP.FP=\frac{m^2n^2\varrho^2}{2-n^2}$, ſo that the ſemiconjugate axis C′D is likewiſe given.

Laſtly, by cor. 3. of the preceding article, it will be $CN^{\frac{3}{2}}:CP^{\frac{3}{2}}::T$ (the periodic time in any given circle whoſe radius is CN) : $T.CP^{\frac{3}{2}}\div CN^{\frac{3}{2}}$, the required time of revolution when the orbit is an ellipſe; that is, when n^2 is leſs than 2. For, when $n^2=2$, the axis of the curve (expreſſed by $\frac{2\varrho}{2-n^2}$) becomes infinite, and the orbit degenerates into a parabola: and if n^2 exceeds 2, the axis becomes negative, and the curve is a hyperbola, whoſe principal diameters are $\frac{2\varrho}{n^2-2}$ and $\frac{2mn\varrho}{\sqrt{(n^2-2)}}$ reſpectively.

Cor. 1. Becauſe neither the value of AC′ nor the expreſſion for the periodic time is affected with m, it follows that the principal axis and the periodic time will remain invariable, if the velocity at P be the ſame, whatever the direction at that point may be.

Cor. 2. We may readily apply this propoſition to the caſe of bodies projected from the ſurface of the earth: for it has already been ſhewn (art. 283.) that a body projected with a velocity of 26000 feet, or 4·92424 miles, per ſecond, would deſcribe a circle at its ſurface: and by this prop. $1 : \sqrt{2} ::$ 26000 feet : 36769 feet or 6·96393 miles, velocity of projection when a parabola would be the trajectory. Hence appears the truth of what is often remarked in popular treatiſes of aſtronomy; that if a body were projected from the earth's ſurface with a velocity exceeding 7 miles per ſecond, it would (if not reſiſted by the air) deſcribe a hyperbola; if with a velocity of rather leſs than 7 miles per ſecond, it would deſcribe a parabola; and if the initial velocity were between 7 and 5 miles per ſecond, the body would deſcribe an ellipſis: if the velocity be leſs than 5 miles (or 4·92424) per ſecond, the body would not deſcribe a complete ellipſe, as its periphery would in part, if not entirely, fall within the circumference of the earth, and of courſe the motion of the body would be ſtopped at the firſt point of interſection of the two curves.

293. Prop. *The attraction of a corpuſcle to a ſphere is juſt the ſame as if all the matter of the ſphere were collected into its centre: the force being ſuppoſed to vary inverſely as the ſquare of the diſtance.*

In order to prove the truth of this propoſition, we muſt firſt inveſtigate the force by which a corpuſcle would be attracted towards a circular plane. Let O, therefore, be the centre of the circle ABCD (fig. 12. pl. XIII.) to which the corpuſcle P is attracted. Suppoſe $a\,b\,c\,d$ a ſmaller circle in the ſame plane as ABCD, and having the ſame centre, and by ſuppoſition the attraction of P to any particle c will be as $\frac{1}{Pc^2}$. Put $PO = d$, $Pc = x$, the number 3·14159 being as heretofore repreſented by π: then $Oc^2 = x^2 - d^2$, and $\pi(x^2 - d^2) =$ area of circle $a\,b\,c\,d$. The fluxion of that area is therefore $= 2\pi x\dot{x}$; and, by the reſolution of forces, $x : d :: \frac{1}{x^2} : d x^{-3}$, the attraction of P to c in the direction PO. Hence the fluxion of the whole force is truly defined by $2\pi x\dot{x} \times d x^{-3}$ or its equal $2\pi x^{-2}\dot{x}$; and the force itſelf by $\int 2\pi x^{-2}\dot{x}$: which when properly corrected becomes

$-\frac{2\pi d}{x}+\frac{2\pi d}{d}=2\pi\left(1-\frac{d}{x}\right)$. So that when $x=\mathrm{PC}$ the attraction to the whole circle becomes $2\pi\left(1-\frac{\mathrm{PO}}{\mathrm{PC}}\right)$, or the attraction varies as $1-\frac{\mathrm{PO}}{\mathrm{PC}}$.

Now, let ABCD (fig. 13.) be a ſphere, and P a corpuſcle attracted towards it. Draw PAOC through the centre O; and let BID be the diameter of a ſection a B c D of the ſphere perpendicular to PC. Put the radius $\mathrm{AO}=r$, $\mathrm{PO}=d$, $\mathrm{AP}=d-r=c$, $\mathrm{PI}=y$ and $\mathrm{PB}=\mathrm{PD}=c+x$: then will $\mathrm{AI}=y-c$, $\mathrm{CI}=2r-y+c$, and conſequently $\mathrm{AI}\,.\,\mathrm{CI}=\mathrm{BI}^2=(y-c)\,.\,(2r-y+c)=\mathrm{PB}^2-\mathrm{PI}^2=(c+x)^2-y^2$. From this equation there ariſes $y=\frac{2rc+2c^2+2cx+x^2}{2r+2c}=\frac{2dc+2cx+x^2}{2d}$, becauſe $d=r+c$. Whence it follows, that $2\pi\left(1-\frac{\mathrm{PI}}{\mathrm{PB}}\right)$ the value of the attraction towards the circle a B c D is equal to $2\pi\left(1-\frac{2dc+2cx+x^2}{2d\,.\,(c+x)}\right)=\frac{2\pi(2rx-x^2)}{2d(c+x)}$: this multiplied by $\frac{c\dot{x}+x\dot{x}}{d}=\dot{y}$ gives $\pi\,\frac{(2ax\dot{x}-x^2\dot{x})}{d^2}$ for the fluxion of the required force. The fluent of this is $\frac{\pi\left(rx^2-\frac{1}{3}x^3\right)}{d^2}$, which expreſſes the attraction of the ſegment ABD. And this, when B and D coincide with C and $x=2r$, becomes $\frac{4\pi r^3}{3d^2}$, for the meaſure of the attraction of the whole ſphere, which therefore varies as $\frac{r^3}{d^2}$. Now if the denſity δ of the ſphere ſhould vary, the attraction muſt *(cæt. par.)* vary as δ: ſo that for all ſpheres the attraction varies as $\frac{\delta r^3}{d^2}$. But the quantity of matter Q varies as δr^3; and conſequently the attraction varies as $\frac{Q}{d^2}$. Therefore, if the ſpheres were evaneſcent in magnitude, or the ſame quantity of matter condenſed into the centres, the attraction would be the ſame. Q. E. D.

COR. 1. When $d=r$, the value of the attraction $\frac{4\pi\delta r^3}{3d^2}$, becomes $\frac{4}{3}\pi\delta r$. So that *at the ſurface of the ſphere the attraction is directly as the radius.*

Cor. 2. *If the moleculæ of two ſpheres* S, S′, *attract each other by a force varying inverſely as the ſquare of the diſtance, the attraction is the ſame as if the whole quantity of matter in each ſphere were collected into its reſpective centre.*

Cor. 3. Hence what has been proved reſpecting the attraction of two corpuſcles when the force varies inverſely as the ſquare of the diſtance, holds true for two ſpheres, the particles of which attract each other according to the ſame laws. Conſequently, *if the moleculæ of two ſpheres act upon each other according to that law, one ſphere will deſcribe a conic ſection about the other in one of its foci.*

SCHOLIUM.

294. Since it is known that the ſun and all the planets are nearly ſpherical, and that they revolve in ellipſes in one of the foci of which the ſun is always very nearly; the preceding propoſitions, therefore, furniſh Aſtronomers with juſt reaſon for concluding, that *each planet is attracted to the ſun by a force which is directly as the quantity of matter in the central body, and inverſely as the ſquare of the diſtance of their centres: and that the conſtituent particles alſo attract each other by a force which varies in like manner.* But we need not ſtop here. It is one of the conſtant laws of nature (art. 21.) that a body cannot act upon another, without being ſubject to an equal contrary reaction. Thus, the planets and the comets being drawn towards the ſun, likewiſe attract the ſun towards them by the ſame law: thus alſo the ſatellites are attracted towards the planets, and the planets contrarily towards the ſatellites; and both theſe again towards the ſun. This attractive property is, therefore, common to the ſun, planets, their ſatellites, and to comets; and, of conſequence, we may regard the mutual gravitation of the celeſtial bodies as a property generally obtaining throughout the univerſe. This conſideration will lead to a few more general propoſitions which will immediately follow: we may juſt obſerve previouſly, that what is already done will enable the ſtudent to underſtand the principles on which the quantity of matter and denſity of the planets are commonly aſcertained. For the effects of attraction at different diſtances being as ſtated at the beginning of this ſcholium, and the quantity of matter being jointly as the magnitude and denſity of a body; if therefore the effects of the attraction of different bodies are known, and their diameters, we can find their denſities, and thence their quantities of matter. Now to find the denſities, let δ repreſent the denſity of the central body, r its radius, Q its quantity of

matter, T the periodic time of the revolving body, d its mean diſtance from the central body, and s the natural ſine of the angle under which r appears at the diſtance d. Then $Q \propto \delta r^3$, while $T^2 \propto \frac{d^3}{Q}$, and this again, $\propto \frac{d^3}{\delta r^3}$: hence $\delta \propto \frac{d^3}{r^3 T^2}$. But $s = \frac{r}{d}$, or $\frac{1}{s} = \frac{d}{r}$; whence $\frac{1}{s^3} = \frac{d^3}{r^3}$, and conſequently $\delta \propto \frac{1}{s^3 T^2}$. From this general expreſſion the comparative denſities of moſt of the planets have been determined. But this brief ſketch of method muſt ſuffice: for this is not a place to enter more into the detail.

295. PROP. *The common centre of gravity of two bodies is not affected by their mutual attractions.*

Since a body B attracting another body B′ (fig. 14. pl. XIII.) exerts its influence equally upon every particle of B′, the acceleration of B′ to B is the ſame whatever the quantity of matter in B′ may be, and will vary as the quantity of matter in B: the magnitudes of the body being ſuppoſed indefinitely ſmall with reſpect to their diſtance. And in like manner the acceleration of B towards B′, from the attraction of the latter, is in proportion to *its* quantity of matter. Let therefore B and B′ attract each other, G being their centre of gravity; then the acceleration of B′ towards B from B's action : acceleration of B towards B′ from B′s action : : B : B′ : : GB′ : GB (art. 110). Hence, the ſpaces Be, B′e', which the bodies paſs over in indefinitely ſmall portions of time, in conſequence of their mutual attractions, will be as GB′, and GB. Conſequently the remainders Ge', Ge, muſt be in the ſame ratio; and, therefore, G continues to be the common centre of gravity of the two bodies.

COR. 1. *If while the bodies act on each other they be projected from* B′ *and* B *in oppoſite and parallel directions, with velocities proportional to their diſtances from the centre of gravity, they will deſcribe ſimilar figures about that centre.*

For let B′d and Bd be the ſpaces which the bodies urged by the projectile force would deſcribe while their attractions would have carried them to e' and e; then, completing the parallelograms $e'd'$, $e\,d$, the bodies at the end of that time will be found at b' and b. Now the ſpaces deſcribed being as the velocities Bd' or eb' : Bd or eb : : Ge' : Ge; hence the angle BGb' = BGb, and conſequently b'Gb is a right line: alſo Gb' : Gb : : GB′ : GB; ſo that G is the common centre of gravity of the bodies in their new ſituation b', b; and the ſame may be ſhewn after a ſecond and after a third interval of time, and ſo on.

Cor. 2. If we conceive each body to be acted upon, at the ſame time, by equal accelerative forces in the ſame direction, the *relative* motions of the two bodies will not be altered, and they will ſtill continue to deſcribe ſimilar figures about G which is then in motion. And by varying the motion of the ſyſtem, the abſolute initial velocities of B and B' may be varied *ad libitum*. *Hence, if* B *and* B' *be projected with any velocities, they will continue to revolve about their centre of gravity, and deſcribe ſimilar figures about it; and the centre of gravity, not being diſturbed by their mutual attractions, will continue to move on uniformly in a right line.*

Cor. 3. *If the attractions are inverſely as the ſquares of the diſtances, the bodies will, with regard to their common centre of gravity, deſcribe ſimilar conic ſections about that point as a focus.*

296. Prop. *The periodic time of two bodies* B, B', *attracting each other with any forces revolving about their common centre of gravity* G, *is to the periodical time of one of the bodies* B' *revolving about the other unmoved, and deſcribing a ſimilar figure, in the ſubduplicate ratio of the other body* B *to the ſum of the bodies* $B+B'$.

Let $B'D'$ be the orbit deſcribed about the centre C (fig. 4.), and $B'D''$ that deſcribed about B. Draw the common tangent $B'R''$, and taking $B'D'$, $B'D'$, indefinitely ſmall, draw $GD'R'$, and parallel to it $BD'R''$; then will $B'D'$, $B'D'$, be ſimilar parts of the ſimilar curves. The times in which the bodies are drawn from the tangent $B'R''$ through the ſpaces $R'D'$, $R''D''$, with the ſame force, will be as $\sqrt{D'R'}$ and $\sqrt{D''R''}$ (art. 232.), that is, becauſe of the ſimilar figures $GB'R'D'$, $GB'R''D''$, as $\sqrt{GB'}$ to $\sqrt{BB'}$; or again, ſince GB' is to BB' as B to $B+B'$ (art. 110.) the times will be as $\sqrt{B}$ to $\sqrt{B+B'}$. But time in $R'D'$ = time in $B'D'$, and time in $B'D''$ = time in $R''D''$: alſo the whole periodic times vary as the times of deſcribing ſimilar portions. Therefore period. time in wh. curve $B'D'$: period. time in wh. curve $B'D''$ :: $\sqrt{B} : \sqrt{B+B}$.

Cor. *Bodies revolving about their common centre of gravity deſcribe by their radii vectores areas proportional to the times of deſcription.*

297. Prop. *If two bodies* B *and* B', *attracting each other mutually with forces reciprocally proportional to the ſquare of their diſtances, revolve about their common centre of gravity* G; *then will the principal axis of the ellipſe which the body* B *deſcribes about the other* B' *by this motion, be to the principal axis of the ellipſe which the ſame body* B *might deſcribe about the other* B' *quieſcent in the ſame periodic time, as* $\sqrt[3]{B+B'}$, to $\sqrt[3]{B'}$.

Conceive a body B'' to be placed at G, whoſe attraction upon B ſhall be equal to that of B': then, as the attraction varies as the quantity of matter directly and ſquare of the diſtance inverſely, we have $\frac{B''}{BG^2}=\frac{B'}{B'B^2}$, and therefore $B''=B'\times\frac{BG^2}{B'B^2}$. Now the ſquares of the periodic times of bodies revolving about the focus of an ellipſe vary as the cubes of the major axes directly, and the abſolute forces inverſely; therefore, if the periodic time be given, the major axis muſt vary as the cube root of the abſolute force. Conſequently, major axis of ellipſe deſcribed by B about B″: major axis of ellipſe deſcribed by B about B′ at reſt in the ſame periodic time $: : B'^{\frac{1}{3}}\times\frac{BG^{\frac{2}{3}}}{B'B^{\frac{2}{3}}} : B'^{\frac{1}{3}} : : BG^{\frac{2}{3}} : B'B^{\frac{2}{3}}$. Again, by ſimilar figures, the major axis of B about B′ at reſt : major axis of B about B″ : : BB′ : BG. Therefore, *componendo*, major axis of ellipſe deſcribed by B when both bodies revolve about their centre of gravity : major axis of ellipſe deſcribed by B about B′ in quieſcence in the ſame time $: : B'B^{\frac{1}{3}} : BG^{\frac{1}{3}} : : \sqrt[3]{B+B} : \sqrt[3]{B'}$.

COR. *If two bodies attracting each other move about their common centre of gravity their motions will be the ſame as if they did not attract each other, but were both attracted with the ſame forces by a third body placed in the centre of gravity.*

298. PROP. *If a body projected in a given direction be conſtantly drawn towards two fixed points which are not both in the ſame plane with the initial direction, it will deſcribe equal ſolids in equal times about the right line joining the ſaid points: and the converſe.*

Let the time be divided into any number of equal parts, and in the firſt moment let the body deſcribe the line AQ (fig. 1. pl. XIV.); in the ſecond, if not prevented, it would proceed to deſcribe the line QT equal to AQ; but at Q it is acted on by centripetal forces tending to the centres C and B: let theſe forces be expreſſed by the lines QS, QV, which would be deſcribed in conſequence of their individual energy, while the body would be carried by the projectile motion from Q to T. Complete the parallelopiped whoſe ſides are the lines QS, QV, QT, the body by the joint action of theſe forces will deſcribe QD the diagonal of the parallelopiped (arts. 64. 217.). But the ſolid QCBA=QCBT, becauſe they have the ſame baſe QCB, and the ſame altitude (for the line TA cuts the plane QBA in Q, and TG=QA); alſo the ſolid QCBT=QCBD, becauſe they ſtand on the ſame baſe, and are between the ſame parallel planes QCB, DT. In the ſame manner it is evident that equal ſolids will be deſcribed in other equal moments of time, round the ſame points. If, therefore, the number of intervals of time, and of the right lines AQ, QD, be indefinitely increaſed, the path of the body will ultimately become a curve, and the body impelled by continued forces will deſcribe round the points C, B, equal ſolids in equal times; and in any times ſolids proportional to the times.

Converſely, if theſe ſolids QCBA, QCBD, deſcribed in equal times, be equal, the line TD will be parallel to the plane QCB; and therefore the body will be urged by forces QS, QV, tending to the points C, B.

Cor. *The orbit* AQD *&c. is not contained in one plane, except in ſome particular caſes.* For, that the orbit may not deviate from any plane, the forces on both ſides of it ought to be equal and ſimilarly ſituated.

SCHOLIUM.

229. By means of the law juſt laid down, philoſophers explain the motion of the moon and ſatellites, which are attracted towards two principal centres. But it would be incompatible with the objects of this work to enlarge here upon theſe and other particulars, connected with the doctrine of attraction and centripetal forces. The ſtudent who wiſhes to obtain a profound acquaintance with ſuch matters ſhould conſult Newton's *Principia*, lib. I. Clairaut's *Théorie de la Lune*; Euler's *Theoria Motuum Lunæ*; D'Alembert's *Recherches ſur differens Points du Syſtème du Monde*; Simpſon's *Eſſays and Tracts*; Friſi's *Coſmographia and De Gravitate Univerſali Corporum*; La Place's *Mecanique Celeſte*; and Vince's *Syſtem of Aſtronomy*, vol. 2.

CHAPTER IV.

On the Rotation of Bodies about fixed Axes, and in free Space; with Theorems relative to the Centres of Oſcillation, Gyration, Percuſſion, Spontaneous Rotation, &c.

300. In demonſtrating the chief propoſitions relative to rectilinear motion, whether uniform or variable, accelerated or retarded, we have either conſidered the bodies as phyſical points, or we have imagined that the impelling force has been impreſſed, and the reſiſtance exerted, in the direction of a right line, paſſing through the centre of gravity of the body moved; and, therefore, that every particle of ſuch body muſt partake of the ſame degree of velocity as that with which the centre of gravity moves. But in numerous inſtances which occur to the mechaniſt a body or ſyſtem of bodies is ſo ſituated that when any force or any number of forces are impreſſed upon it, it cannot take any other motion than one of rotation about a fixed axis, which may either paſs through the body or ſyſtem, or be at an extremity of it: ſo that the velocity of the conſtituent moleculæ of the ſyſtem ſhall be greater or leſs according to the greater or leſs diſtance of any individual particle from the axis about which the motion is performed. And in ſuch caſes it is neceſſary to call to our aid other conſiderations than what were required when diſcuſſing the properties of acceleration and retardation.

Again, with reſpect to the motion of bodies at liberty to move freely by the action of any force impreſſed, if any ſuch body receives an impulſion in any direction which does not paſs through the centre of gravity, the motion which will enſue is a rotatory one: for if at the ſame moment a body is impelled according to any direction AB (not paſſing through the centre of gravity), an equal and oppoſite force is exerted upon the body, in a parallel direction, CG, paſſing through the centre of gravity, the centre of gravity will manifeſtly be kept at reſt: nevertheleſs, it is clear that the other parts of the body will not be in a ſtate of quieſcence; becauſe the two forces, though equal, are not directly oppoſite: ſo that the only motion that the body can have, its centre of gravity being at reſt, is evidently a motion of rotation about that centre. Now, the fixed axis round which a body revolves is preſſed by the impelling force while it generates rotatory motion; but the axis being (by hyp.) immoveable, reacts equally againſt that preſſure, and when it paſſes through the centre of gravity would, as above ſtated, cauſe each particle to move with the ſame velocity, and in the direction of the force. If, then, the force which preſſes againſt a fixed axis in given circumſtances be aſcertained, the motion of the body in free ſpace when the axis is removed will be known: for the latter motion will conſiſt of the rotatory motion about the axis paſſing through the centre of gravity conſidered as fixed, compounded with the motion of the centre of gravity cauſed by the force now at liberty to impel that centre, the fixed axis which paſſed through it being removed.

When a ſolid body receives an impulſe on any one point, or that point is urged in any way by a moving force, it cannot move unleſs the other points with which it is connected by the force of coheſion move alſo (except the force of impulſion is ſufficient to overcome that of coheſion; a caſe which is not meant to be conſidered here). And whatever is the motion of any particle, that particle muſt be conceived as urged by a force preciſely competent to the production of that motion, by acting immediately on the particle itſelf. The particle immediately impelled by the external force is either preſſed towards its neighbouring particles, or is drawn from them; and by this endeavour to change its place the connecting forces are excited, or brought into action. We are but little acquainted with the nature of theſe connecting forces: but this is not of much conſequence in a mechanical point of view; for the fact that the forces by which the moleculæ of bodies act on each other are equal is ſufficient for our preſent purpoſe.

In the propoſitions we are about to enter upon the attention will be chiefly called to two objects, viz. The moving force by which the revolving motion is generated; and the inertia of the different parts of the ſyſtem moved. And both the effects of the moving force and of the inertia of the particles depend upon their diſtance from the axis of motion, all other things being the ſame: if both theſe be aſcertained the abſolute acceleration will be determined, and conſequently the abſolute velocity generated in a given time.

Theſe preliminary obſervations may be terminated by remarking, that, ſince in rotatory motions all the particles of a body are ſuppoſed to turn together without changing their relative poſitions; therefore, when a body has made a complete rotation, each particle has deſcribed the circumference of a circle, and the whole paths of the various particles will be in the ratio of their reſpective circumferences, or of their radii; and the ſame is true of any portion of a rotation: ſo that the velocities of the different particles are proportional to their diſtances from the axis of rotation, and are not adequate meaſures of the *rotatory velocity of the body;* its proper meaſure is the angle deſcribed by any radius vector of the body in a unit of time.

301. PROP. *If a body revolve about an axis, the particles of which that body is compoſed reſiſt, by their inertia, the communication of motion to any given point, with forces which are as the particles themſelves, and the ſquares of their diſtances from the axis of motion jointly.*

Let a force ϕ be applied at any point A (fig. 2. pl. XIV.) in order to communicate motion to a ſyſtem of particles p, p', p'', &c. revolving at determinate diſtances round the centre of motion C. Let A be ſuch a quantity of matter as will, if concentrated in A, have the ſame effect in reſiſting the communication of motion to that point, by its inertia, when any particle p is removed from the ſituation P, as that particle would have, revolving at the diſtance PC. Now the effect of the given force ϕ acting at the point A (in a direction ϕ A perpendicular to CA), to move a body at that point, is to its effect to move a body at P inverſely as thoſe diſtances; or as PC to AC, by the nature of the lever: and if theſe bodies be moved with equal angular velocities, their diſtances from the axis being then (art. 300.) as the ſpaces deſcribed in a given time, the moving forces are inverſely as the ſpaces deſcribed. But taking for a moment the notation of art. 228. we there have $F \propto \frac{BV^2}{S}$, or $FS \propto BV^2$; in which, if $F \propto \frac{1}{S}$, then will $B \propto \frac{1}{V^2}$: conſequently, the quantities of matter muſt be inverſely as the ſquares of the velocities; or, in the preſent caſe, inverſely as the ſquares of the diſtances from the axis; that is, $A : p :: PC^2 : AC^2$; whence we have $A = \frac{p \cdot PC^2}{AC^2}$, which indicates that the reſiſtance of the particle p at the diſtance PC is equivalent to the reſiſtance of the maſs $\frac{p \cdot PC^2}{AC^2}$, at the diſtance AS. In like manner, taking another particle p', at the diſtance P′C, and a correſponding quantity of matter A′ concentrated into the ſame point A, we ſhall have the reſiſtance of the particle p' at its diſtance equal to the reſiſtance of the maſs $\frac{p' \cdot P'C^2}{AC^2}$ at the diſtance AC. And the ſame may be ſhewn of other particles p'', p''', &c. Conſequently, if we uſe, as hitherto, the character $\int$ to denote the whole fluent, or ſum of all the ſeparate reſiſtances, we ſhall have the reſiſtance of the whole revolving body expreſſed by $\int \frac{p\,.\,PC^2}{AC^2}$.

Cor. 1. *The force which accelerates the point* A *of any body revolving on an axis, to which point that force* φ *is applied, is equal to the product of the force into the ſquare of the diſtance* AC, *divided by the ſums of the products of all the moleculæ into the ſquares of their reſpective diſtances from* C, *the centre of motion.*

For the maſs moved is $\int \frac{p\,.\,CP^2}{AC^2}$, and the moving force is φ: but the accelerating force is equivalent to the quotient of the moving force by the maſs, and is therefore repreſented by $\int \frac{\varphi\,.\,AC^2}{p\,.\,PC^2}$.

Cor. 2. *The angular velocity of a ſyſtem, generated in a given time, by any force* φ *at* A, *perpendicular to* AC, *is proportional to the rectangle of the force into the diſtance at which it acts, divided by the ſums of the products of all the moleculæ into the ſquares of their reſpective diſtances.*

For the abſolute velocity of the point A is as the accelerating force, and the angular velocity is as the abſolute velocity directly and the diſtance reciprocally; therefore the angular velocity is as $\int \frac{\varphi\,.\,AC^2}{p\,.\,PC^2} \times \frac{1}{AC}$, or as $\varphi\,.\,AC \div \int p\,.\,PC^2$.

Cor. 3. *The angular motion of any ſyſtem, generated by a uniform force, will be a motion uniformly accelerated.*

This is evident, becauſe the accelerating force is in a conſtant ratio to the uniform force φ.

Cor. 4. What has been here ſhewn with reſpect to moleculæ ſituated on a right line paſſing through a centre of motion will hold equally with regard to a body or ſyſtem moving upon an axis: for all the particles of ſuch body may be conceived to be transferred to the plane in which the axis of ſuſpenſion CP performs its motion, by an orthographical projection, the lines of transference being all parallel to the axis of motion; this ſuppoſition will, it is obvious, neither affect the place of the centre of gravity (with regard to the axis of motion) nor the angular motion of the body.

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302. The preceding inveſtigation has been conducted upon the ſuppoſition that the forces were each applied in a direction

perpendicular to the line drawn from each point of application to its reſpective centre of motion: but the ſame concluſion would reſult if the moving forces were applied in any direction, provided that, inſtead of theſe forces, we had taken their equivalents when reduced to the perpendicular direction. As this propoſition and its corollaries are, however, of great importance in all that depends upon rotatory motion; and as the ſame concluſions, when drawn from different methods, ſtrike the mind with greater force; we ſhall here conſider the matter in another way.

Let an axis of rotation paſs through C (fig. 4. pl. XIV.) perpendicular to the plane of the figure, and let a body fixed firmly to this axis be acted upon by ſeveral accelerating forces, we are to enquire into the circumſtances of the motion produced. Suppoſe, at firſt, that a particle p, ſituated at P in the figure, is urged about the fixed axis by a force φ, applied in the direction PD : that force tends to impreſs upon the particle p a certain velocity in the direction PD, yet, in conſequence of the mutual coheſion between the different moleculæ of the body, and the connection of the whole with the fixed axis, the velocity can only be produced actually in the initial direction P d perpendicular to CP. Drawing, therefore, Pd perpendicular to CP, the force φ muſt be decompoſed into two others, in the directions P d, PC, of which the one P e will be extinguiſhed by the reſiſtance of the axis, and the other P d has place; where, of conſequence, we have P $d = \varphi$ coſ. DP d (art. 59.). If, therefore, we denote the diſtance AP of p from the axis of rotation by r, and the perpendicular diſtance DC of the force from the axis by δ, we have, in the triangle CPD, ſin. DPC = cos. DP $d = \frac{\delta}{r}$; and, conſequently, P $d = \varphi \frac{\delta}{r}$. This expreſſion would repreſent the effective accelerating force of the particle p, if that particle were alone; but the connection of this particle with the others, and the operation of the forces acting on the latter, change this effect: if, then, the arc a be run over at the end of the time t by a particle at a unit of diſtance from the fixed axis, $a\,r$ will be the arc deſcribed in the ſame time by the particle p; ſo that the velocity, and the effective accelerating force, (art. 232. I. VI.) of this latter will be $r \frac{\dot{a}}{\dot{t}}$, and $r \frac{\ddot{a}}{\dot{t}^2}$ reſpectively.

Now comparing this force with the former, and obſerving that they are both directed in the ſame right line, we may have recourſe to the principle of D'Alembert, ſtated in art. 267.

The accelerating force $\phi\frac{\delta}{r}$ which is impreſſed in the direction of the initial motion, muſt be decompoſed into

$r\frac{\ddot{a}}{t^2}$ . . . effective accelerating force,

$\phi\frac{\delta}{r} - r\frac{\ddot{a}}{t^2}$ . . . $\left\{\text{accel. force deſtroyed by the action of the other powers in the ſyſtem.}\right.$

In like manner we may proceed to inveſtigate the effects of other forces ϕ', ϕ'', &c. acting upon the moleculæ p', p'', &c. (whoſe diſtances are r', r'', &c.) in the directions P′D′, P″D″, &c., and at the diſtances δ', δ'', &c. the expreſſions being the ſame with the letters accented ſimilarly: the whole, therefore, will be in equilibrio when impreſſed by the moving forces

$$\frac{\delta}{r}\phi p - \frac{\ddot{a}}{t^2} r p,\ \frac{\delta'}{r'}\phi' p' - \frac{\ddot{a}}{t^2} r' p',\ \frac{\delta''}{r''}\phi'' p'' - \frac{\ddot{a}}{t^2} r'' p'',\ \&c.$$

Now as the ſyſtem is attached to a fixed axis, only one equation (art. 98.) is neceſſary to expreſs the ſtate of equilibrium. And, if we ſuppoſe that the forces ϕ, ϕ', ϕ'', &c. are parallel to the plane of the figure, or perpendicular to the plane of rotation (and they may all be reſolved to ſuch planes by an obvious proceſs), it will be merely requiſite to make the ſum of the moments of the powers (art. 60.) with reſpect to the fixed axis equal to nothing. Here, that of the firſt force will be $\delta\phi p - \frac{\ddot{a}}{t^2} r^2 p$, and the moments of the other forces will be expreſſed in the ſame manner, adding the accents. Thus, then, we ſhall have

$$\delta\phi p + \delta'\phi' p' + \&c. = (r^2 p + r'^2 p' + \&c.)\frac{\ddot{a}}{t^2}.$$

Or taking the character $\int$ as before, and denoting the angular velocity by U, we ſhall have $\int\phi\delta p = \frac{\ddot{a}}{t^2}\int r^2 p$: whence, $\frac{\ddot{a}}{t^2} = \frac{\dot{U}}{t} = \int\frac{\phi\delta p}{r^2 p}$.

Hence, if the quantity $\int r^2 p$, which is the ſum of the products of the ſeveral moleculæ into the ſquares of their reſpective diſtances from the axis, be called the *momentum of inertia*, and if $\frac{\dot{U}}{t}$ be called the *angular accelerating force*, the equation juſt given may be thus ſtated in words at length:

‘*The angular accelerating force is the quotient of the ſum of the* *momenta of the moving forces, or* (art. 60.) *of their reſultant, divided by the momentum of inertia.*

303. From the above equation we may readily obtain an expreſſion for the angular velocity. For (art. 232. II.) $\varphi = \frac{\dot{v}}{\dot{t}}$, &c. give $\int \varphi \, \delta p = \frac{\dot{v}}{\dot{t}} \delta p = \frac{\dot{v}}{\dot{t}} \delta p + \frac{\dot{v}'}{\dot{t}} \delta' p' +$ &c. whence we have $\dot{U} = \frac{\dot{v} \, \delta p + \dot{v}' \, \delta' p' + \&c.}{r^2 p + r'^2 p' + \&c.}$; and, taking the fluents,

$$U = \frac{v \, \delta p + v' \, \delta' p' + \&c.}{r^2 p + r'^2 p' + \&c.} = \int \frac{v \, \delta p}{r^2 p}.$$

From which it appears that cor. 2. art. 301. is not confined to a ſingle force; but may be extended to as many forces as we pleaſe, and applied to the body in any directions whatever.

304. As to the part P e of the force φ, which operates as a preſſure upon the fixed axis, and is entirely deſtroyed by its reaction, it may be eaſily determined. For P $e = \varphi$ cos. DPC, and DP $= \sqrt{r^2 - \delta^2}$; therefore, cos. DPC $= \frac{\text{DP}}{\text{PC}} = \sqrt{1 - \left(\frac{d}{r}\right)^2}$ conſequently,

$$\text{P}\,e = \varphi \sqrt{\left[1 - \left(\frac{d}{r}\right)\right]}.$$

And the ſame may be ſhewn with regard to the effect of any other forces φ', φ'', &c. upon the axis of motion.

305. DEF. The *centre of oſcillation* is that point in the axis of ſuſpenſion of a vibrating body in which, if all the matter of the ſyſtem were collected, any force applied there would generate the ſame angular velocity in a given time as the ſame force at the centre of gravity, the parts of the ſyſtem revolving in their reſpective places.

Or, ſince the force of gravity upon the whole body may be conſidered as a ſingle force (equivalent to the weight of the body) applied at its centre of gravity, the *centre of oſcillation* is that point in a vibrating body into which, if the whole were concentrated and attached to the ſame axis of motion, it would then vibrate in the ſame time the body does in its natural ſtate.

COR. From the firſt definition it follows that the centre of oſcillation is ſituated in a right line paſſing through the centre of gravity, and perpendicular to the axis of motion *.

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* Since, according to the former of theſe definitions, the centre of oſcillation is a definite point, and according to the latter it is variable; it might, perhaps, be better to ſay, that a body has a *centre* of oſcillation when it is attached to a centre of motion, and has a *line* of oſcillation when it moves upon an axis; the *line* of oſcillation in the latter caſe being parallel to the axis of motion. But this is merely hinted by the way, and not introduced into the text.

306. Prop. *If a body vibrate about an axis by the force of gravity, the diſtance of the centre of oſcillation from that axis is equal to the quotient ariſing from dividing the ſum of the products of each particle into the ſquare of its diſtance from the axis, by the product of the mass or body into the diſtance of its centre of gravity from the axis.*

Let AB (fig. 3. pl. XIV.) be a plane paſſing through the centre of gravity G of the body perpendicular to the axis of vibration on which the body is orthographically projected; C the centre of motion of that plane, or of the whole body reduced to that plane; O the centre of oſcillation in the line CG produced, and p, p', &c. the conſtituent moleculæ of the body thus projected. Through C draw CX parallel to the horizon, and upon that line demit the verticals or perpendiculars pd, $p'd'$, Gg, Oo, &c. Now, ſince the angular velocity of the ſeveral particles is not changed by this projection, we have $p \cdot pC^2 + p' \cdot p'C^2 + \&c. = \int p \cdot pC^2$, the momentum of inertia of the whole body B, while $B \cdot OC^2$ will expreſs the momentum of inertia of an equal body concentrated into the point O. And ſince the force of gravity φ acts in the parallel directions pd, $p'd'$, &c. we ſhall have $\varphi \cdot p \cdot dC + \varphi \cdot p' \cdot d'C + \&c.$ for the effect of gravity to turn the body about C, by the nature of parallel forces, and this is equal to $\varphi \cdot B \cdot Cg$, by the nature of the centre of gravity (art. 108.). And if B were concentrated in the point O, then would $\varphi \cdot B \cdot Co$ be the accelerating force of gravity to turn the body about C. But, that the ſame angular velocity may be generated in both caſes, the quotients of the accelerating forces with reſpect to C, by the momenta of inertia, muſt be equal (art. 301. cor. 2. 303.), that is, $\frac{\varphi \cdot B \cdot Cg}{\int p \cdot pC^2} = \frac{\varphi \cdot B \cdot Co}{B \cdot OC^2}$, whence $OC^2 = \frac{\varphi \cdot B \cdot Co \cdot \int p \cdot pC^2}{\varphi \cdot B \cdot B \cdot Cg}$. Now, the triangles CGg, COo being ſimilar, we have $\frac{CO}{CG} = \frac{Co}{Cg}$: hence, ſubſtituting the former fraction for the latter in the preceding value of OC^2, it becomes $OC^2 = \frac{CO}{CG} \times \frac{\varphi \cdot B \cdot \int p \cdot pC^2}{\varphi \cdot B \cdot B} = \frac{CO}{CG} \cdot \frac{\int p \cdot pC^2}{B}$: conſequently

$$CO = \frac{\int p \cdot pC^2}{B \cdot CG} = \frac{\int pr^2}{B \cdot CG}.$$

This expreſſion, being independent of the line CX, will continue the ſame for all inclinations of the like CO; ſo that the centre of oſcillation O, thus determined, is a *fixed point*, agreeably to the definition.

Cor. 1. *Hence any body* AB, *ſuſpended at* C, *and vibrating,* *may, ſo far as regards the time of vibration, be conſidered as a ſimple pendulum whoſe length is* CO.

COR. 2. *That* CO *may be the equivalent pendulum, and* O *the centre of oſcillation,* O *muſt be in the line* CG, otherwiſe it would not reſt in the ſame poſition with the body, when no force was keeping it out of its vertical poſition.

COR. 3. *If a body be turned about its centre of gravity in a direction perpendicular to the axis of motion, the place of the centre of oſcillation will remain unaltered.* For the quantity $\int p\,r^2$ will be not at all affected by ſuch a motion of the body.

COR. 4. The diſtance of the centre of gravity from that of oſcillation (if the plane of the body's motion remain unaltered) will be reciprocally as the diſtance of the former from the point of ſuſpenſion. *If, therefore, that diſtance be found when the point of ſuſpenſion is in the vertex, or ſo ſituated that the operation may become the moſt ſimple, the value thereof in any other propoſed poſition of that point will likewiſe be given by one ſingle proportion.*

COR. 5. The product of the diſtances of the centre of gravity, and that of oſcillation, from the axis of motion, is manifeſtly a conſtant quantity for the ſame plane of vibration. *If, therefore, the centre of oſcillation be made the point of ſuſpenſion, the point of ſuſpenſion will become the centre of oſcillation.*

COR. 6. *Hence alſo, if upon the plane of vibration paſſing through the centre of gravity of any body two concentric circles be deſcribed, having the common centre* G, *and radii* GC, GO, *the body ſuſpended from any point in the periphery of either circle will perform an oſcillation in the ſame time.*

Some other corollaries depending upon the relation ſubſiſting between the centres of oſcillation and of gyration will be given after we have treated of that centre: when alſo we ſhall apply the propoſitions to the determination of theſe centres in various bodies. Previous to this, however, we may here ſhew how

307. *To find the diſtance of the centre of oſcillation from the point of ſuſpenſion experimentally.*

Suſpend the body propoſed very freely by the given point, and make it vibrate in ſmall arcs; then count the number n of vibrations it makes in a minute, by a good ſtop-watch; ſo ſhall the diſtance CO of the centre of oſcillation and point of ſuſpenſion be denoted by $\frac{140850}{n^2}$. For the length of the ſimple ſecond pendulum being $39\frac{1}{8}$ inches; and the lengths of pendulums being reciprocally as the ſquare of the number of vibrations made in a given time; therefore $n^2 : 60^2 :: 39\frac{1}{8} : \frac{60^2 \times 39\frac{1}{8}}{n^2} =$

$\frac{140850}{n^2}$, the length of the ſimple pendulum which oſcillates n times in a minute, or the diſtance CO in the compound pendulum.

Or, two-thirds of the length of a thin cylindrical rod, ſuſpended at one end, and vibrating in the ſame time as the body, will give CO; the reaſon of which will ſoon appear.

308. Prop. *In a compound pendulum conſiſting of ſeveral bodies revolving about a common axis the centre of oſcillation may be determined by this proceſs: add together the ſeveral products of each maſs into the diſtances of the reſpective centres of oſcillation and gravity from the axis of motion, and divide the ſum by the product of the whole ſyſtem into the diſtance of the common centre of gravity from the axis of motion; the diſtance of the centre of oſcillation from the ſame axis will be repreſented by that quotient.*

For it appears from art. 306. that with reſpect to any body B in the compound maſs, $\int p r^2 = B . CO . CG$; and again, with regard to another body B′ in the ſame ſyſtem $\int p' r'^2 = B' . CO' . CG'$, and ſo on: conſequently $S . \int . p r^2 = \int B . CO . CG$; that is, the ſum of all the $p r^2$ in the whole pendulum is equal to the ſum of all the products in each part of the pendulum, which ariſe from multiplying each part into the product of the diſtances of its centres of gravity and of oſcillation from the common axis of motion. If, therefore, this latter ſum be divided by the product of the compound mass into the diſtance of the common centre of gravity from the axis of motion, the quotient (art. 306.) will be the diſtance of the centre of oſcillation from the ſame axis: for $\frac{\int . \int p r^2}{\int B . CO . CG} \propto \frac{\int p r^2}{B . CO . CG}$; whence $CO = \frac{\int p r^2}{B . CG}$, as well with reſpect to the compound maſs as to any of its conſtituent parts.

309. Def. The *centre of gyration* is that point in which, if all the matter contained in a revolving ſyſtem were collected, the ſame angular velocity will be generated in the ſame time by a given force acting at any place as would be generated by the ſame force acting ſimilarly in the body or ſyſtem itſelf.

When the axis of motion paſſes through the centre of gravity, then is this centre called the *principal* centre of gyration.

310. Prop. *If the ſum of the products formed by multiplying each particle of a ſyſtem into the ſquare of its diſtance from the axis of motion, that is, if the momentum of inertia be divided by the whole maſs, the ſquare root of the quotient will be the diſtance of the centre of gyration from the axis of motion.*

For, if CR be the diſtance from the axis of motion to the centre of gyration, the expreſſion for the angular motion $\frac{\phi \cdot AC}{\int p \cdot PC^2}$ or $\frac{\phi \cdot AC}{\int p r^2}$ (art. 301. cor. 2.) will be transformed to $\frac{\phi \cdot AC}{B \cdot CR^2}$; and theſe by the def. muſt be equal. Conſequently, $B \cdot CR^2 = \int p\, r^2$, and $CR = \sqrt{\frac{\int p r^2}{B}}$.

COR. 1. *The diſtance of the centre of gyration from the axis of motion is a mean proportional between the diſtances of the centres of oſcillation and gravity, from the ſame axis.*

For $CR^2 = \frac{\int p r^2}{B}$, while $CO = \frac{\int p r^2}{B \cdot CG}$; therefore $CO \cdot CG = \frac{\int p r^2}{B} = CR^2$, and CO : CR : : CR : CG. (Fig. 2, pl. XIV.)

COR. 2. *The diſtance between the centre of gravity and principal centre of gyration is a mean proportional between the diſtances of the centres of motion and oſcillation, from the centre of gravity.*

For $PC^2 = PG^2 + GC^2 \pm 2\, PG \cdot GC$; therefore, by art. 307, we have $OC = \int p \cdot \frac{PG^2 + GC^2 \pm 2\, PG \cdot GC}{B \cdot CG}$. But from the nature of the centre of gravity, $\int p \cdot PG$ on each ſide of G are equal, and, becauſe 2 CG is conſtant, $\int \pm 2\, PG \cdot GC = O$. Alſo, becauſe $\int p \cdot GC^2 = B \cdot GC^2$, we have $OC = GC + \int \frac{p \cdot PG^2}{B \cdot GC}$, and $OG = \int \frac{p \cdot PG^2}{B \cdot GC}$; whence this corollary is manifeſt.

COR. 3. *The time in which a body vibrates will be the leaſt poſſible when the axis of motion paſſes through the principal centre of gyration.*

For if D and d repreſent the diſtances from the centre of gravity to the point of ſuſpenſion, and to the centre of gyration, it will be D : D+d : : D+d : $\frac{(D+d)^2}{D}$, the diſtance from the point of ſuſpenſion to the centre of oſcillation (cor. 1.), and this latter expreſſion will obviouſly be a minimum when $D = d$.

COR. 4. If we wiſh to know what quantity of matter M muſt be placed at any other diſtance AC from C, ſo that the inertia may remain the ſame, we muſt have $AC^2 \cdot M = CR^2 \cdot B$, whence $M = \frac{B \cdot CR^2}{AC^2}$.

Cor. 5. And, for a like reaſon, if any number of bodies B, B′, B″, be put in motion about a common axis paſſing through C, by a force acting at A, the ſyſtem will have the ſame angular velocity, if, inſtead of thoſe bodies placed at the diſtances CB, CB′ CB″, there be ſubſtituted bodies equal to $\frac{CB^2}{CA^2}B, \frac{CB'^2}{CA^2}B', \frac{CB''^2}{CA^2}B''$, all concentrated into the point A.

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311. Having now demonſtrated the chief properties of the centres of oſcillation and gyration, we may apply them to a few examples of finding the poſition of thoſe points in bodies of given ſhape. And, firſt, let us ſpeak of

The Centre of Oſcillation.

I. Let CP (fig. 2, pl. XIV.) be a right line, or very ſlender cylinder, moving freely about its extreme point C as a centre of ſuſpenſion.

Then, if CA, conſidered as variable, be denoted by x, the force of $\dot{x}$ particles at A will be defined by $x^2\dot{x}$: its fluent, therefore, that is, $\frac{1}{3}x^3$, denotes the momentum of inertia of all the particles in CA; and this, when CA becomes equal to CP, will be $\frac{1}{3}CP^3$. In this caſe, too, the body $B=CP$, and $CG=\frac{1}{2}CP$: conſequently (art. 306.) $\frac{\int pr^2}{B.CG}=\frac{\frac{1}{3}CP^3}{\frac{1}{2}CP.CP}=\frac{\frac{1}{3}CP^3}{\frac{1}{2}CP^2}=\frac{2}{3}CP=CO$, the diſtance of the centre of oſcillation from the point of ſuſpenſion.

II. Let the point of ſuſpenſion of the ſlender cylinder CP be S, to find the centre of oſcillation.

Put $CP=v$, $CS=y$, $PS=v-y$, and $SO=x$, then $(v-y)^2\times\dot{v}$ will expreſs the fluxion of all the pr^2 in CP, while $(v-y).\dot{v}$ repreſents the fluxion of the product of B . CG. Hence $x=\int\frac{(v-y)^2\dot{v}}{(v-y)\dot{v}}=\frac{2v^2-6vy+6y^2}{3v-6y}$, where y and v may be taken in any ratio to each other. If $y=0$, or C and S coincide, then $x=\frac{2}{3}v=\frac{2}{3}CP$, as in the preceding example. If $y=\frac{1}{3}v$, then alſo $x=\frac{2}{3}v=\frac{2}{3}CP$; in which caſe O and P will coincide. So that, *whether a very ſlender cylinder be ſuſpended at one end, or at* $\frac{1}{3}$ *of its length, its vibrations will be performed at the ſame time.* (See art. 306, cor. 5.) When $y=\frac{1}{2}v$, then $x=\infty$, or the centre of oſcillation is at an infinite diſtance: *the tendency to oſcillate will then ceaſe, and if any extraneous force produce motion, it will be completely rotatory.*

Cor. When the vibrations are to be the quickeſt poſſible $x=\frac{2v^2-6vy+6y^2}{3v-6y}$, muſt be a minimum; or its fluxion making

y variable $=0$. Hence $(12\,y\,\dot{y} - 6\,v\,\dot{y}) \cdot (3\,v - 6\,\dot{y}) - 6\,\dot{y}\,(2\,v^2 - 6\,v\,y + 6\,y^2) = 0$, and $y = \frac{1}{2}\,v \pm \frac{1}{2}\,v\,\sqrt{\frac{1}{3}}$. When S is taken between A and G the negative ſign manifeſtly obtains, and $y = \frac{1}{6}\,v\,(1 - \sqrt{\frac{1}{3}}) = {\cdot}211325\,v$.

III. Let ACB (fig 5, pl. XIV.) be an angular pendulum, compoſed of two thin cylindric wires AC, CB, vibrating upon the point of ſuſpenſion C, in the plane of the figure.

Biſect AC, and CB, in g, γ, and biſect $g\,\gamma$ in G, ſo ſhall g, γ, G, be the centres of gravity of AC, CB, and of AC+CB. In this inſtance we may have recourſe to art. 368. Denote AC or CB by a, and the natural ſecant of angle $\text{ACG} = \frac{1}{2}\,\text{ACB}$, by s; then o and o′ being the centres of oſcillation of the ſeparate maſſes AC, CB, we have (ex. I.) $\text{C}\,o = \text{C}\,o' = \frac{2}{3}\,a$: and by trigonometry, $s : 1 :: \frac{1}{2}\,\text{AC} : \text{CG} = \frac{a}{2\,s}$. Hence we have

$$\frac{\text{AC} \cdot \frac{1}{2}\,\text{AC} \cdot \frac{2}{3}\,\text{AC} + \text{CB} \cdot \frac{1}{2}\,\text{CB} \cdot \frac{2}{3}\,\text{CB}}{(\text{AC}+\text{CB})\,\frac{a}{2\,s}} = \frac{(a \cdot \frac{1}{2}\,a \cdot \frac{2}{3}\,a)^2}{2\,a\,\frac{a}{2\,s}} = \frac{2}{3}\,a\,s = \text{CO}.$$

COR. 1. Since CO varies with the ſecant of $\frac{1}{2}$ ACB, it follows that the time of vibration of the angular pendulum may be increaſed *ſine limite*, while its length remains conſtant; the time of vibration depending upon the magnitude of the angle ACO.

COR. 2. When the angle ACO vaniſhes, or AC and CB coincide throughout, the time of vibration will be equal to that of a ſimple pendulum whoſe length is $\frac{2}{3}$ AC: when ACO is a right angle, the time of vibration is infinite, or the pendulum will reſt upon its centre of gravity in all poſitions.

COR. 3. Hence if AC=CB be given, and the time of vibration, the angle ACB may readily be found: for the above expreſſion for CO gives $s = \frac{3\,\text{CO}}{2\,a}$.

Thus, if AC=15 inches, and the time were 1 ſecond, CO would be $39\frac{1}{8}$ inches, and $s = \frac{117\frac{3}{8}}{30} =$ sec. of 75° $11\frac{1}{2}'$ nearly: hence ACB=150° 23′.

If, while AC=15 inches, the time of vibration ſhould be 5 ſeconds, then $\text{CO} = 25 \times 39\frac{1}{8}$, and $s = \frac{75 \times 39\frac{1}{8}}{30} = 97.8125 =$ ſec. of 89° $24\frac{3}{4}'$; conſequently ACB=178° $49\frac{1}{2}'$ nearly.

IV. Let the ſolid formed by the rotation of the curve V m M about its abſciſs VA vibrate about an axis paſſing through C; to find the centre of oſcillation.

Put CV$=d$, the variable abſciſs V $a=x$, its correſponding ordinate $a\,m=y$, VA$=a$, AM$=r$, $3\cdot1416=\pi$; then, as may eaſily be ſhewn (though that part of the inveſtigation is omitted, to ſave room,) the ſum of the products of each particle in the circle $m\,n$ into the ſquare of its diſtance from the axis $=$ $(\mathrm{C}a^2+\frac{1}{4}\,a\,m^2)\times\pi\,y^2=\pi\left((d+x)^2\times y^2+\frac{1}{4}\,y^4\right)$. Hence $\pi\,\dot{x}\times\left((d+x)^2\times y^2+\frac{1}{4}\,y^4\right)$ is the fluxion of the ſum of all ſuch products for the whole body. Conſequently we ſhall have

$$\mathrm{CO}=\frac{\int\pi\,\dot{x}\times\left((d+x)^2\times y^2+\frac{1}{4}y^4\right)}{\mathrm{B\,.\,CG}}.$$

V. Let the general theorem in the preceding example be applied to the caſe of the cone whoſe vertex is V, and radius of its baſe AM.

In this caſe V m M will become a right line, ſo that $a:r::x:y=\frac{r}{a}\,x=n\,x$, putting $n=\frac{r}{a}$. Hence $\int\pi\,\dot{x}\times(\overline{d+x}^2\times y^2+\frac{1}{4}y^4)=\int\pi\,\dot{x}\times(\overline{d+x}^2\times n^2\,x^2+\frac{1}{4}\,n^4\,x^4)=\frac{1}{3}\,\pi\,d^2\,n^2\,x^2+\frac{1}{2}\,\pi\,d\,n^2\,x^4+\frac{1}{5}\,\pi\,n^2\,x^5+\frac{1}{20}\,\pi\,n^4\,x^5$. But, by the rules of menſuration, the body B$=\frac{1}{3}\,\pi\,n^2\,x^3$; and (art. 124.) VG$=\frac{3}{4}\,x$, whence CG$=d+\frac{3}{4}\,x$. Therefore B . CG$=\frac{1}{3}\,\pi\,d\,n^2\,x^3+\frac{1}{4}\,\pi\,n^2\,x^4$, by which dividing the preceding fluent, there reſults CO$=\frac{20\,d^2+30\,d\,x+12\,x^2+3\,n^2\,x^2}{20\,d+15\,x}$; which for the whole cone when $x=a$, and $n\,x=y=r$, is transformed to this:

$$\mathrm{CO}=\frac{20\,d^2+30\,a\,d+12\,a^2+3\,r^2}{20\,d+15\,a}.$$

Cor. 1. When C coincides with V, or the cone is ſuſpended by its vertex, d vaniſhes, and the expreſſion becomes

$$\mathrm{CO}=\mathrm{VO}=\frac{4\,a^2+r^2}{5\,a}=\frac{4}{5}\,a+\frac{r^2}{5\,a}.$$

Cor. 2. If, when d vaniſhes, r be$=a$, then will CO$=a$. Conſequently, *if a right-angled cone be ſuſpended at its vertex, its centre of oſcillation will coincide with the centre of its baſe, and the cone will vibrate in the ſame time as a ſimple pendulum whoſe length is equal to the altitude of the cone.*

Remark. The concluſion in the laſt corollary flows alſo very naturally from the geometrical method of finding the centre of oſcillation of a cone ſuſpended at its vertex; which is this: *Biſect* AN, *the radius of the baſe* (fig. 7. pl. XIV.), *in* F; *join* VF, *and make* IF$=\frac{1}{5}$ VF; *then* IO *drawn perpendicular to* VF *will meet* VA *or* VA *produced, in* O, *the centre of oſcillation.*

For VF $= \sqrt{\text{VA}^2 + \text{AF}^2} = \sqrt{a^2 + \frac{1}{4}r^2}$, and VI $= \frac{4}{5}\sqrt{a^2 + \frac{1}{4}r^2}$. But the triangles VAF, VIO, are ſimilar, therefore VA : VF : : VI : VO, that is, $a : \sqrt{a^2 + \frac{1}{4}r^2} :: \frac{4}{5}\sqrt{a^2 + \frac{1}{4}r^2} : \frac{\frac{4}{5}a^2 + \frac{1}{5}r^2}{a} = \frac{4}{5}a + \frac{r^2}{5a} =$ VO, which is the ſame expreſſion as in cor. 1.

When O coincides with A, we have (fig. 8.) VI : IO : : IO : IF : : VA : AF. So that IO $= \sqrt{\text{VI} . \text{IF}} = \sqrt{4\,\text{IF} . \text{IF}} = 2\,$IF; therefore, VA $=$ 2 AF, and conſequently VA$=$AN, by conſtruction: which agrees with cor. 2.

COR. 3. *When* VA *is leſs than* AN, O *falls below* A; *and when* VA *is greater than* AN, O *falls above* A.

COR. 4. From the above it likewiſe appears, that if cones be made to vibrate as pendulums, about their vertices as points of ſuſpenſion, the time of vibration may be increaſed *ſine limite*, while VN, the ſlant ſide of the cone, remains conſtant. In practice, however, this cannot be carried ſo far in conical pendulums as the ſimilar one in angular pendulums, noticed in cor. 1. exam. III.

COR. 5. Whether a right-angled cone be ſuſpended at its baſe or its vertex, it will vibrate in equal times (art. 306. cor. 5.)

Theſe examples muſt ſuffice as general ſpecimens of the method of finding the centre of oſcillation: we ſhall here add the reſults of other inveſtigations in ſome of the moſt uſeful caſes.

In ſurfaces when the vibration is flatwiſe, or perpendicular to the plane of the figure, the diſtance CO of the centre of oſcillation from the point of ſuſpenſion will be as below:

In an isosceles triangle suspended at its vertex, ...	$\frac{3}{4}$ the altitude.
In a circle suspended at its circumference, ...	$\frac{5}{4}$ the radius.
In the common parabola suspended at its vertex, ..	$\frac{5}{7}$ the axis.
Any parabola, ditto..	$\frac{2m+1}{3m+1} \times$ axis.

When the ſurfaces are moved edgewiſe, or in the plane of the figure; then

In a circle.		$\frac{3}{4}$ of the diameter.
In a rectangle suspended by one angle, ...		$\frac{2}{3}$ of the diagonal.
In a parabola.............	its vertex, ...	$\frac{5}{7}$ axis $+ \frac{1}{3}$ param.
Ditto	middle of base	$\frac{4}{7}$ axis $+ \frac{1}{2}$ param.

In a ſquare pyramid whoſe altitude is a, and ſide of its baſe $2b$, CO$= \frac{4}{5}a + \frac{4b^2}{15a}$, the vibrations being performed in vertical planes parallel to that paſſing through a ſide of the baſe.

If a ſphere whoſe radius is r be ſuſpended at the diſtance d from its centre, then is δ the diſtance from the point of ſuſpenſion to the centre of oſcillation $=d+\frac{2r^2}{5d}$. If the point of ſuſpenſion be at the extremity of a radius, then $\delta=\frac{7}{5}r$. When δ and r are given, we have $d=\frac{1}{2}\delta\pm\sqrt{\frac{1}{4}\delta^2-\frac{2}{5}r^2}$; whence we learn that a ſphere may be ſuſpended at two different diſtances from the axis of motion, and yet *vibrate in the ſame time.* When $\frac{1}{4}\delta^2=\frac{2}{5}r^2$ the radical expreſſion vaniſhes, and $d=\frac{1}{2}\delta=r\sqrt{\frac{2}{5}}=\cdot632495\,r$ is the only value of d; the point of ſuſpenſion being then the centre of gyration, and δ, which varies with the time of vibration, is a *minimum.*

If the weight of the thread is to be taken into the account, we have the following diſtance between the centre of the ball and that of oſcillation, where B is the weight of the ball, d the diſtance between the point of ſuſpenſion and its centre, r the radius of the ball, and w the weight of the thread or wire, $\text{GO}=\frac{(\frac{1}{3}w+\frac{2}{5}\text{B})\,4r^2-\frac{1}{6}w\,(2dr+d^2)}{(\frac{1}{2}w+\text{B})\,d-rw}$; or, if B be expreſſed in terms of w conſidered as a unit, then $\text{GO}=\frac{\frac{1}{6}d}{\text{B}+\frac{1}{2}}$.

If two weights W, W′, be fixed at the extremities of a rod of given length WW′, C being the centre of motion between W and W′; then if $d=\text{CW}$, $\text{D}=\text{CW}'$, and m the weight of an unit in length of the rod, we ſhall have $\text{CO}=\frac{m\text{D}^3+3\text{W}'\text{D}^2+md^3+3\text{W}d^2}{m\text{D}^2+2\text{W}'\text{D}-md^2-2\text{W}d}$; the radii of the balls being ſuppoſed very ſmall in compariſon with the length of the rod.

In the bob of a clock pendulum, ſuppoſing it two equal ſpheric ſegments joined at their baſes, if the radii of thoſe baſes be each $=\varrho$, the height of each ſegment v, and d the diſtance from the point of ſuſpenſion to G the centre of the bob, then is $\text{GO}=\frac{1}{3}d\,.\,\frac{\varrho^4+\frac{1}{2}\varrho^2v^2+\frac{1}{10}v^4}{\varrho^2+\frac{1}{3}v^2}$; which ſhews the diſtance of the centre of oſcillation below the centre of the bob.

If r the radius of the ſphere be known, the latter expreſſion becomes $\text{GO}=\frac{\frac{2}{3}r^2v-\frac{1}{2}rv^2+\frac{1}{10}v^3}{d(r\quad\frac{1}{3}v)}$.

The Centre of Gyration.

312. Ex. I. Let a ſlender rod whoſe length is $\text{CP}=l$ (fig. 2.) revolve on the point C; to find its centre of gyration.

Let any variable diſtance $\text{CA}=x$; then will the general expreſſion for (art. 310.) $\sqrt{\frac{\int pr^2}{\text{B}}}$, become $\sqrt{\frac{\int x^2\dot{x}}{l}}=\sqrt{\frac{x^3}{3l}}$; and this when $x=l$, gives $\text{CR}=\sqrt{\frac{1}{3}l^2}=l\sqrt{\frac{1}{3}}=\cdot57735\,l$.

The ſame concluſion may be deduced from cor. 1. art. 310. and ex. 1. art. 311. For $CG = \frac{1}{2}l$, and $CO = \frac{2}{3}l$: conſequently $CR = \sqrt{\frac{1}{2}l \cdot \frac{2}{3}l} = l\sqrt{\frac{1}{3}}$.

Ex. II. To determine the centre of gyration in a circle, or in a circular wheel of uniform thickneſs, ſo that its weight may be as its area.

Let $\pi = 3{\cdot}1416$, and r the radius of the circle; then is $\pi r^2 =$ the area of the circle; its fluxion is $2\pi r\dot{r}$; and therefore $\int p r^2 = \int 2\pi r^3 \dot{r}$. Conſequently $\frac{\int p r^2}{B} = \int \frac{2\pi r^3 \dot{r}}{2\pi r \dot{r}} = r\sqrt{\frac{1}{2}} = \frac{1}{2}r\sqrt{2} = {\cdot}707107\, r = CR$; the diſtance from C the centre of motion, to R the centre of gyration.

Ex. III. Let ABDE, and *a b d e* (fig. 9. pl. XIV.), be two concentric circles whoſe reſpective radii are R, r;—if the plane or ſolid wheel *a b e d* be taken away, and the ring ABDE *a b d e*, revolve about an axis paſſing through C perpendicular to the plane of the figure, it is then propoſed to determine the diſtance CR.

Here $\pi R^2 - \pi r^2 =$ the area of the annulus, the fluxion of which is $2\pi R\dot{R}$, R being ſuppoſed variable: conſequently $2\pi R^3\dot{R}$ is the fluxion of all the $p r^2$. And by the general rule we have $CR = \sqrt{\frac{\int . 2\pi R^3 \dot{R}}{\pi R^2 - \pi r^2}} = \sqrt{\frac{R^4}{2R^2 - 2r^2}}$.

Cor. 1. When r vaniſhes, or the ring becomes a circle, we have $CR = R\sqrt{\frac{1}{2}}$, as in the preceding example.

Cor. 2. The ſectors C *a b*, CAB, being to each other as the areas of their reſpective circles, if $\frac{1}{n}\pi$ be put for the correſponding part of the circumference, the ſame reſult will be obtained with reſpect to the ſectors as to the whole circles; ſo that if the part AB *b a* revolved about the centre C, its centre of oſcillation R′ would be determined by means of the equation $CR' = \sqrt{\frac{R^4}{2R^2 - 2r^2}}$.

Ex. IV. To find the centre of gyration of a cone which revolves about its vertex.

Here we may again have recourſe to cor. 1. art. 310. For we know that VG, the diſtance from the vertex to the centre of gravity, is $= \frac{3}{4}a$ (art. 124.); and $VO = \frac{4}{5}a + \frac{r^2}{5a}$ (ex. v. art. 311.). Hence $VR = \sqrt{VG . VO} = \sqrt{\frac{12a^2 + 3r^2}{20}} = \frac{1}{2}\sqrt{\frac{12}{5}a^2 + \frac{3}{5}r^2}$.

Cor. When $a=r$, as in the right-angled cone, we ſhall have $\mathrm{VR}=\sqrt{\frac{12}{5}a^2+\frac{3}{5}a^2}=a\sqrt{3}$.

The reſults in a few other uſeful caſes are as follow:

In the periphery of a circle revolving about the diam.. $\mathrm{CR}=\mathrm{rad.}\sqrt{\frac{1}{2}}$.
In the plane of a circle .... ditto.. $\mathrm{CR}=\frac{1}{2}$ rad.
In the ſurface of a ſphere .... ditto.. $\mathrm{CR}=\mathrm{rad.}\sqrt{\frac{2}{3}}$.
In a ſolid ſphere .... ditto.. $\mathrm{CR}=\mathrm{rad.}\sqrt{\frac{2}{5}}=\frac{7}{11}r$ nearly.
In a cone.................................. the axis.. $\mathrm{CR}=r\sqrt{\frac{3}{10}}=\cdot 54772\,r$.

In a ſtraight lever whoſe arms are B and b, the diſtance $\mathrm{CR}=\sqrt{\frac{\mathrm{B}^3+b^3}{3\,\mathrm{B}+3b}}$.

If the matter of any gyrating body were actually to be placed as if in the centre of gyration, it ought either to be diſpoſed in the circumference of a circle whoſe radius is CR, or into two points R, R′, diametrically oppoſite, and at diſtances from the centre each $=$ CR.

For a practical method of finding the centre of gyration, ſee art. 314. cor. 3.

313. **Prop.** *When a pendulum is at reſt, if a body impinge on it in a horizontal direction, the ſame velocity will be communicated to the point of impact, as if the maſs of the pendulum were removed, and inſtead of it an equivalent maſs were concentrated in the point of impact, the quantity of the equivalent maſs being to that of the pendulum in a duplicate ratio of the diſtances of the centre of gyration and the point of impact from the axis of motion.*

Let CADE (fig. 10. pl. XIV.) repreſent a pendulum whoſe axis of motion paſſes through C, and let any impact be impreſſed upon the point P, in an horizontal direction, and perpendicular to the vertical plane ABCD, the pendulum being at reſt. Let Q be the quantity of matter which being concentrated in P, the ſame angular velocity will be communicated by the impact, as when B the whole body is concentrated into its centre of gyration R. Then, ſince the reſiſtances ariſing from the inertia of the two maſſes at their reſpective diſtances muſt be equal, in order that the angular velocities may be equal; we muſt, as in cor. 4. art. 310, make $\mathrm{B}\,.\,\mathrm{CR}^2=\mathrm{Q}\,.\,\mathrm{CP}^2$; whence $\mathrm{Q}=\mathrm{B}\,.\,\frac{\mathrm{CR}^2}{\mathrm{CP}^2}$.

Cor. Hence V, the velocity of the impinging body I, may be determined. For, let v be the velocity communicated to the point of impact: then, by the laws of the colliſion of non-elaſtic bodies (chap. v. art. 343.), $\mathrm{I}:\mathrm{I}+\mathrm{Q}::v:\mathrm{V}=v\,.\,\frac{\mathrm{I}+\mathrm{Q}}{\mathrm{I}}$. To determine v, meaſure the arc deſcribed by O, the centre of oſcillation in its aſcent after impact; and call its verſed ſine s, then (arts. 243. 263.) the velocity of the centre of oſcillation at its loweſt point $=\sqrt{2gs}$: whence, the velocity of the point of impact $=\frac{CP}{CO}\cdot\sqrt{2gs}$.

SCHOLIUM.

On the principles developed in this propoſition and corollary, the celebrated mathematician Mr. Benjamin Robins founded the uſe of his ingenious contrivance, the *Balliſtic Pendulum*, for aſcertaining the velocities of military projectiles. (New Principles of Gunnery, prop. 8.) This machine conſiſts of a large block of wood annexed to the end of a ſtrong iron ſtem having a croſs ſteel axis at the other end, placed horizontally, about which the whole vibrates together. When the machine is at reſt, a ball diſcharged horizontally from a piece of ordnance ſtrikes and enters the block, and cauſes the pendulum to vibrate more or leſs, according to the velocity of the projectile; and by the extent of the vibration the velocity of the ball at the time of the impact becomes known. A more minute and particular deſcription of this inſtrument may be ſeen in Dr. Hutton's Tracts, vol. I. with the ſeveral computations, taking into the account the change in the centre of oſcillation occaſioned by the ball's lodging in the point P, and ſome other ſmall quantities which muſt not be neglected in practice.

314. PROP. *If motion be communicated by a hanging weight to a ſyſtem revolving on a fixed axis paſſing through the centre of gravity, and the moving force act always at a given diſtance from the axis of motion, it will generate in the revolving ſyſtem the ſame motion as the weight would acquire in the ſame time by falling freely by its gravity from a ſtate of quieſcence.*

Let DGB (fig. 11. pl. XIV.) repreſent a body or ſyſtem moveable upon a horizontal axis paſſing through its centre of gravity G; let R be the centre of gyration, P the weight of the body which gives motion to the ſyſtem whoſe weight is W, by means of a cord P d C wound round the circumference dCE. If GR be denoted by r, and G d by d, the inertia of the whole ſyſtem will be equivalent to the weight $W\frac{r^2}{d^2}$ uniformly diffuſed through the periphery dCE, every point of which manifeſtly moves with the ſame velocity as P. The moving body P being ſuppoſed either deſtitute of inertia, or incomparably leſs than the weight of the ſyſtem, the accelerating force ϕ will be $\frac{Pd^2}{Wr^2}$; conſequently in the time t, the velocity generated will be $=\phi g t=\frac{gtPd^2}{Wr^2}$; which multiplied into the quantity of matter $W\frac{r^2}{d^2}$, gives $g\,t\,P$ for the whole quantity of motion generated. But if P were to deſcend by the force of gravity, it would acquire, in the time t, the velocity $g\,t$ (art. 243.); which multiplied into the quantity of matter P, produces $g\,t\,P$, the ſame quantity of motion as before.

Cor. 1. Hence it follows, that the permanence of motion, eſtimated by the product of the quantity of matter and velocity, obtains in bodies which revolve on fixed axes.

Cor. 2. If P be ſuppoſed to poſſeſs inertia, the quantity $P\,d^2$ muſt be added to the denominator of the fraction expreſſing the accelerating force, which will then become $\frac{P\,d^2}{P\,d^2+W\,r^2}$. This leads to a uſeful practical purpoſe, explained in the next corollary.

Cor. 3. Hence, *to find the centre of gyration in any ſyſtem experimentally.* Suppoſe s to be the ſpace deſcribed by the weight P deſcending from reſt (as in the prop.) in t ſeconds: then the accelerating force being as in cor. 2. we ſhall have $s=\frac{1}{2}\varphi\,t^2=\frac{g\,P\,t^2\,d^2}{2\,P\,d^2+2\,W\,r^2}$; whence we obtain $r=\sqrt{\frac{gP\,t^2\,d^2-2\,s\,P\,d^2}{2\,s\,W}}$; in which equation, if the numerical values of s, d, P, W, as reſulting from the experiment, be uſed, r becomes known.

315. Prop. *Let the body* DB (fig. 11. pl. XIV.), *whoſe centre of gravity* G *is the centre of the circle* CE d, *be attached to a fixed point* A *by means of a cord* ACE d, *wound about* CE d *as an axle; and let* O *be the centre of oſcillation when* C *is the point of ſuſpenſion or momentary centre of motion of the body: then, if while the body deſcends by the force of gravity, it is made to turn round by the unwinding of the cord* ACE d, *the ſpace actually deſcended by the body* DB, *is to the ſpace deſcribed in the ſame time by a body falling freely, as* CG *to* CO.

Through the point of contact C, and the centre of gravity G, draw the horizontal line CGO, and ſuppoſe the motion of the body to commence when theſe centres are ſo ſituated; then (art. 305.) the angular velocity of the body at ſuch commencement of its motion will be the ſame as if the whole were placed at O: and if a body were placed in O, its initial velocity would be the ſame as that of a body falling freely. Drawing, therefore, C*go* indefinitely near CGO, and the ſmall arcs O*o*, G*g*, having the common centre C, we ſhall have, veloc. of O : veloc. of G :: O*o* : G*g* :: OC : GC; but the velocity of the deſcending body is that of its centre of gravity; therefore, velocity in free deſcent : velocity of body :: OC : GC. Now, ſince the points C, G, O, are always in a horizontal line, and the radius CG is given as well as the diſtance CO; the velocity of a body falling freely, and of the body unwinding from the thread, is always in the ratio of CO to CG; and conſequently, when CO to CG is a conſtant ratio, the whole ſpaces deſcribed in a given time will be as ſtated in the propoſition.

COR. 1. *The weight of the body* DB, *is to the tenſion of the cord* AC, *as* CO *to* GO.

For, if the body were ſupported at O as well as C, G being the centre of gravity, or eſtimate place of the body, we ſhould have (art. 133. cor. 6.) weight of DB : preſſure at C : : CO : GO. But, if the point O be ſet free, the force acting there will generate a motion about C whilſt the preſſure there, and conſequently the tenſion of the cord, will be neither increaſed nor diminiſhed.

COR. 2. *This motion of the body* DE *by unwinding the thread is uniformly accelerated, and the tenſion of the cord is invariable through the whole deſcent.*

COR. 3. *If any body whoſe tranſverſe ſections are circular run down an inclined plane, while the cord or ribband* CE d *unfolds; or if ſuch round body roll down an inclined plane, being hindered by friction from ſliding: the ſpace it deſcribes in any time, is to the ſpace deſcribed by a body ſliding down freely without friction, as* CG *to* CO.

For the forces which generate the motions are both decreaſed in the ſame ratio; that is, as the abſolute gravity to the relative gravity upon the plane; therefore, the ſpaces deſcribed will, in either caſe, remain in the ratio of CG to CO.

SCHOLIUM.

The force by which ſpheres, cylinders, &c. are cauſed to revolve as they move down an inclined plane (inſtead of ſliding) is the adheſion of their ſurfaces occaſioned by the preſſure againſt the plane: this preſſure is part of the body's weight; for the weight being reſolved into its components, one in the direction of the plane, the other perpendicular to it, the latter is the force of the preſſure; and, while the ſame body rolls down the plane, will be expreſſed by the coſine of the plane's elevation. Hence, ſince the coſine decreaſes while the arc or angle increaſes, after the angle of elevation arrives at a certain magnitude, the adheſion may become leſs than what is neceſſary to make the circumference of the body revolve faſt enough; in this caſe the body deſcends partly by ſliding and partly by rolling. And the ſame may happen in ſmaller elevations, if the body and plane are very ſmooth. But at all elevations the body may be made to roll by the uncoiling of a thread or ribband wound about it.

316. If W denote the weight of a body, s the ſpace de-

ſcribed by a body falling freely, or ſliding freely down an inclined plane, according as the ſuppoſition is that in the prop. or in cor. 3. then the ſpaces deſcribed by rotation in the ſame time by the following bodies, will be in theſe proportions.

1. A hollow cylinder, or cylindrical ſurface, $S=\frac{1}{2}s$; tenſion of the cord in the firſt caſe $=\frac{1}{2}W$.

2. In a ſolid cylinder, $S=\frac{2}{3}s$, tenſ. $=\frac{1}{3}W$.

3. In a ſpheric ſurface, or thin ſpherical ſhell, $S=\frac{3}{5}s$, tenſ. $=\frac{2}{5}W$.

4. In a ſolid ſphere, $S=\frac{5}{7}s$, tenſ. $=\frac{2}{7}W$.

Something of the ſame kind as we have ſtated in the propoſition obtains in common pendulous bodies. *A ball hung by a flexible thread not only oſcillates, but alſo makes part of a rotation*; whence its oſcillations differ from thoſe of a heavy point hanging by the ſame thread. The curious reader who wiſhes to ſee how oſcillations of this nature are to be determined, ſuppoſing the arcs deſcribed to be very ſmall, is referred to *Bezout's Mechanics*, art. 732. The inveſtigation, being long, is omitted here, in order to make room for more uſeful matter.

317. Def. The *centre of percuſſion* is that point in a body revolving about an axis, at which, if it ſtruck an immoveable obſtacle, all its motion would be deſtroyed, or it would not incline either way: ſo that if at the inſtant of the impact the ſupports of the axis were annihilated, the body would remain in abſolute reſt.

During the vibration of a ſyſtem of bodies round a fixed axis paſſing through C (fig. 12. pl. XIV.) if ſuch an obſtacle be oppoſed to any point O, as entirely to deſtroy the motion of that point, every other particle of the ſyſtem will endeavour, by its inertia, to proceed in the direction of its motion, that is, of the tangent of the circular arc it was deſcribing the inſtant O was ſtopped. Theſe forces, therefore, will act on the ſyſtem to turn it round O; and if the ſum of the forces on each ſide of O ſhould be unequal, the motion of the ſyſtem will not be deſtroyed when O is ſtopped: but ſince the forces which act upon the pendulous body between O and C have an effect to continue the motion of the ſyſtem, contrary to thoſe which are impreſſed on the other ſide of O, if this point O be ſo ſituated that the ſum of the forces to turn the ſyſtem round O, on each ſide of that point, may be exactly equal, then the inſtant in which O is ſtopped, the whole motion of the ſyſtem will be deſtroyed.

Cor. 1. When a pendulum, vibrating with a given angular velocity, ſtrikes an obſtacle, the effect of the impact will be the greateſt if it be made at the centre of percuſſion.

For, in this caſe the obſtacle receives the whole revolving motion of the pendulum; whereas, if the blow be ſtruck in any other point, a part of the motion of the pendulum will be employed in endeavouring to continue the rotation.

Cor. 2. If a body revolving on an axis ſtrike an immoveable obſtacle at the centre of percuſſion, the point of ſuſpenſion will not be affected by the ſtroke.

We can aſcertain this property of the point O when we give a ſmart blow with a ſtick. If we give it a motion round the joint of the wriſt only, and, holding it at one extremity, ſtrike ſmartly with a point conſiderably nearer or more remote than $\frac{2}{3}$ of its length, we feel a painful wrench in the hand: but if we ſtrike with that point which is preciſely at $\frac{2}{3}$ of the length, no ſuch diſagreeable ſtrain will be felt. If we ſtrike the blow with one end of the ſtick, we muſt make its centre of motion at $\frac{1}{3}$ of its length from the other end; and then the wrench will be avoided.

318. Prop. *The diſtance of the centre of percuſſion from the axis of motion is equal to the diſtance of the centre of oſcillation from the ſame: ſuppoſing that the centre of percuſſion is required in a plane paſſing through the axis of motion and centre of gravity.*

Let CBE (fig. 12.) be a plane paſſing through the centre of gravity G of the body, and perpendicular to the axis of ſuſpenſion which paſſes through C; and conceive the whole body to be projected upon this plane in lines perpendicular to it, or parallel to the axis of motion; for then, as each particle will fall at the ſame diſtance from the axis as in the body itſelf, the effect from the rotatory motion will not be changed, neither will the place of the centre of gravity. Through C and G draw the line CGON, and let P be the place of one of the particles p compoſing the ſyſtem. Now, ſince the angular motion of all the particles is the ſame, the abſolute velocity will be proportional to the diſtance from the axis of motion; and if at the diſtance 1 the velocity be expreſſed by unity, the velocity of p will manifeſtly be denoted by PS, and its quantity of motion will be $p \cdot PC$, which will act in the direction PR perpendicular to PC: produce PR till it meets OD, drawn parallel to PC, in D; then the force $p \cdot PC$ acting in the direction PR will act upon O as though it had the advantage of the lever OD, and conſequently, $p \cdot PC \cdot OD$ will repreſent the force of the particle p to move the ſyſtem round O. But by reaſon of the ſimilar triangles CPR, ODR, we have RC : CP : : RO : $OD = \frac{CP \cdot RO}{CR} = CP \cdot \frac{CO - CR}{CR}$; and, if PA be perpendicular to

CO, we ſhall have $RC : CP :: CP : CA = \frac{CP^2}{CR}$. Hence the entire force $p . PC . OD$ becomes $= p . PC^2 . \frac{CO - CR}{CR} = P . CA . CO - p . PC^2$. But, when O is the centre of percuſſion the ſum of all the $p . CA . CO - p . PC^2$, muſt be equal to zero, or $\int p . CA . CO = \int p . PC^2 = \int p\, r^2$. Whence it follows that $CO = \frac{\int p\, r^2}{\int p . CA} = \frac{\int p\, r^2}{B . CG}$; the denominators of the two fractions being equal, by arts. 108. 109. and this value of CO obviouſly correſponds with that given for the centre of oſcillation in art. 306.

Cor. 1. If the body be ſymmetrical with regard to the plane BGE, or if it be a ſolid of rotation, the centre of percuſſion found in the axis of the body will coincide with the centre of oſcillation.

Cor. 2. If the centre of percuſſion be required in a plane which does not paſs through G, as C*o* for inſtance, we muſt proceed thus: from G, the centre of gravity, let fall on C*o* the perpendicular G*g*; and, by the ſame argument as above, $C\,o = \frac{\int p\, r^2}{\int p . C\,a} = \frac{\int p\, r^2}{B . C\,g}$. Now, $C\,o : CO :: CG : C\,g$; hence the angles *o*O C and G*g*C are equal, and, conſequently, the former is a right angle.

Cor. 3. Hence it follows that *a body has ſeveral centres of percuſſion* according to the plane paſſing through the axis of motion in which the impact is made, *and the right line* O o, *at right angles to* CO, *is their locus.*

319. Prop. *If* CB (fig. 13. pl. XIV.) *be the axis of a body's motion,* CAO *a plane perpendicular to* CB *and paſſing through the centre of gravity* G, p A *a perpendicular let fall from any particle* p *of the body on the plane* CA, *and* P *the centre of percuſſion, then will* $PO = CB =$ *the ſum of all the* $\frac{p . A\,p . AG}{\text{Body} \times CG}$.

For the ſum of all the forces with which the body is liable to be turned in one direction round PB as an axis = ſum of all the $p . CA . (CB - A\,p) = \int p . CA . (CB - A\,p)$, and the ſum of all the forces which tend to turn it in the contrary direction = ſum of all the $p . CA\ (A\,p - CB) = \int . p . CA\ (A\,p - CB)$. Therefore, in the caſe of no motion either way, we have $\int p . AC . CB = \int p . A\,p . AC$. But, $\int p . CB . AC =$

$\int p\,.\,CB\,.\,CG + \int p\,.\,CB\,.\,GA$; and $\int p\,.\,CB\,.\,GA = o$, from the nature of the centre of gravity. In like manner, $\int p\,.\,pA\,.\,AC = \int p\,.\,pA\,.\,AG + \int p\,.\,pA\,.\,CG$, and $\int p\,.\,pA\,.\,CG = o$: conſequently, $\int p\,.\,CB\,.\,CG = \int p\,.\,pA\,.\,AG$; and $CB =$ $PO = \int \frac{p\,.\,pA\,.\,AG}{pCG} = \frac{\int p\,.\,pA\,.\,AG}{B\,.\,CG}$.

COR. Hence the centre of percuſſion of a body turning round the axis CB, is determined by theſe conditions. 1ſt. It is in a line PO paſſing through the centre of oſcillation and parallel to CB. 2dly. Its diſtance OP from the centre of oſcillation is $\frac{\int p\,.\,pA\,.\,AG}{B\,.\,CG}$.

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320. As ſeveral writers on mechanics have aſſerted that the centres of percuſſion and oſcillation reſide univerſally in the ſame point, we ſhall, in addition to the preceding inveſtigations, give a ſimple inſtance here, in which the contrary will be obvious, if we adhere to the firſt definition in art. 305. To this end let two equal balls B, B′ (fig. 14. pl. XIV.) (of evaneſcent magnitude with reſpect to their diſtance from each other), be connected with an axis of motion CC′, by inflexible lines CB (=3) and C′B′ (=7), void of gravity, perpendicular to CC′ (=6), and parallel to each other; to find the reſpective diſtances of the centres of gravity, oſcillation, and percuſſion, of this ſyſtem from each other. Join BB′ and biſect it in G, the centre of gravity, ſo ſhall $Gg = 5$. Then (art. 306.) $\frac{B\,.\,BC^2 + B'\,.\,B'C'^2}{(B+B')\,Gg} = \frac{9+49}{2\times5} = 5{\cdot}8 = gO$, the diſtance of the centre of oſcillation from the axis; whence $GO = {\cdot}8$. Through O draw *bb′* parallel to CC′, then (art. 318.) will the centre of percuſſion P be ſomewhere in this line; and, by the laſt article we find $PO = \frac{B\,.\,BA\,.\,AG + B'\,.\,B'A'\,.\,A'G}{(B+B)\,gG} = \frac{6+6}{10} = 1.2$. Here the products 6 and 6 are added together, becauſe, while B and B′ are on oppoſite ſides of Gg, they are alſo on contrary ſides of a line paſſing through G parallel to CC′; ſo that, if BA be +, B′A will be −, and AG being +, A′G will be −, and conſequently BA . AG, and B′A′ . A′G, are both affirmative products. — . . . Or, the centre of percuſſion P muſt be ſo ſituated that $B\,.\,BC\,.\,bP$ muſt be $= B'\,.\,B'C'\,.\,b'P$; whence, ſince $B = B'$, we have $B'C' : BC :: bP : Pb' = \frac{3}{7}bP = \frac{3}{10}bb' = 1{\cdot}8$. Conſequently, $OP = 3 - 1{\cdot}8 = 1{\cdot}2$; and $GP = \sqrt{{\cdot}8^2 + 1{\cdot}2^2} = 1{\cdot}4422$; the triangle GOP being right-angled at O. See alſo arts. 392, 393.

321. Prop. *In the wheel and axle, when a weight* P *acting at the diſtance* R, *elevates a weight* W *acting at the diſtance* r, *from the mathematical axis, the preſſure ſuſtained by that axis will be* = $\mathrm{PW} \cdot \frac{(\mathrm{R}+r)^2}{\mathrm{PR}^2+\mathrm{W}r^2}$; *the weight of the wheel and axle and the friction of the cord, &c. not being conſidered.*

Suppoſe P and W to be at their reſpective extremities of the horizontal line paſſing through the centre of motion, and in that ſituation let o and g be the reſpective diſtances of their centres of oſcillation and of gravity from the centre of motion. Then, ſince the whole ſyſtem revolves with the ſame angular velocity as if P + W were placed at the diſtance o from the centre of motion (art. 305.); and, ſince the accelerative forces are as the velocities generated in a given time (art. 225.) or as the diſtances from the centre of motion; we have $o : g :: \mathrm{P}+\mathrm{W} : \frac{g}{o} \times (\mathrm{P}+\mathrm{W})$, the force with which the centre of gravity deſcends. But $o = \frac{\mathrm{PR}^2+\mathrm{W}r^2}{\mathrm{PR}-\mathrm{W}r}$ (art. 306.) and $g = \frac{\mathrm{PR}-\mathrm{W}r}{\mathrm{P}+\mathrm{W}}$ by the nature of the centre of gravity. Therefore $\frac{g}{o} = \frac{(\mathrm{PR}-\mathrm{W}r)^2}{\mathrm{PR}^2+\mathrm{W}r^2} \times \frac{1}{\mathrm{P}+\mathrm{W}}$, and $\frac{g}{o} \times (\mathrm{P}+\mathrm{W}) = \frac{(\mathrm{PR}-\mathrm{W}r)^2}{\mathrm{PR}^2+\mathrm{W}r^2}$. Now the force with which the centre of gravity tends to deſcend is manifeſtly that by which the preſſure upon the axis P + W ariſing from the weight is diminiſhed by the motion: conſequently the preſſure upon the axis is $= \mathrm{P}+\mathrm{W} - \frac{(\mathrm{PR}-\mathrm{W}r)^2}{\mathrm{PR}^2+\mathrm{W}r^2} = \mathrm{PW}\,\frac{(\mathrm{R}+r)^2}{\mathrm{PR}^2+\mathrm{W}r^2}$.

Cor. 1. If $\mathrm{R}=r$, as in the caſe of the ſingle fixed pulley, then the preſſure $= \frac{4\,\mathrm{PW}}{\mathrm{P}+\mathrm{W}}$.

Cor. 2. The ſame reaſoning will evidently apply to the preſſure upon the fulcrum of a ſtraight lever.

322. Def. The centre of *ſpontaneous rotation*, or ſpontaneous gyration, is that point which remains at reſt the inſtant any body is ſtruck, or the point about which the body begins to revolve.

When a body B (fig. 15. pl. XIV.) of any ſhape whatever receives an impulſe, the direction of which does not paſs through the centre of gravity, and takes, in conſequence, the two motions of which we have before ſpoken (art. 300.), it is evident that, for an inſtant of time, we may conſider it as having only one motion, namely, a motion of rotation about a point or fixed axis C, which may be either within the body or out of it, according to the ſhape of the body, and the diſtance GS between the centre of gravity and the direction of the impact. If while the line GS is tranſported parallel to itſelf from GS to G'S', we imagine that it turns about the moveable point G, as the particles of the body have greater or leſs velocities as they are more or leſs diſtant from G, it is manifeſt that there is upon SG a certain point C, which will be found to deſcribe from C' towards C an arc equal to GG', which during an evaneſcent inſtant may be regarded as a right line; in that caſe the point C will have been carried as far backward by its motion of rotation as it will have been advanced parallel to GG' by the velocity common to all the parts of the body; the point C has, therefore, during this inſtant, been actually at reſt in C'; and may, conſequently, be conſidered as a fixed point about which the body during ſuch inſtant has a rotatory motion.

323. PROP. *The centre of ſpontaneous rotation is the ſame with the centre of ſuſpenſion correſponding to the centre of percuſſion, the centre of percuſſion being that point where the body is ſtruck.*

The arcs CC', S'I, which the points C and S deſcribe during an inſtant, being regarded as right lines perpendicular to GS or parallel to GG', the ſimilar triangles CC'G, G'S'I, give GS : GC : : SI : GG': now the velocity GG' will be expreſſed by $\frac{\varphi}{B}$, and the velocity S'I by $\frac{\varphi \,.\, GS^2}{\int p r^2}$ (art. 301.); therefore GS : GC : : $\frac{\varphi \,.\, GS^2}{\int p r^2} : \frac{\varphi}{B}$; and conſequently $GC = \frac{\int p r^2}{B \,.\, GS}$, the r, r', &c. being reckoned from G. Hence $CS = GS + \frac{\int p r^2}{B \,.\, GS} = \frac{B \,.\, GS^2 + \int p r^2}{B \,.\, GS}$, which value of CS correſponds with the value of CO in art. 318; becauſe when we have the momentum of inertia with regard to an axis paſſing through the centre of gravity G, we obtain that momentum with regard to any other axis SI parallel to the former, by adding to it the product of the maſs into the ſquare of the diſtance GS between the two axes.

The ſame concluſion may be otherwiſe deduced, thus:—The action of the body againſt an immoveable obſtacle in the centre of percuſſion muſt have the ſame effect upon the body as if the body had been at reſt and the obſtacle had ſtruck it; in which latter caſe the centre of ſuſpenſion would not be affected, and therefore it becomes the centre of ſpontaneous rotation.

COR. 1. Since φ the force does not enter the above value of CS, we ſee that *the poſition of the centre of ſpontaneous rotation is independent of the magnitude of the impact.*

COR. 2. *The diſtance CG is always greater as the force φ, or the reſult of all the forces, acts nearer the centre of gravity; and vice verſa.* So that when φS coincides with GG, GC will be infinite; or there will be no motion of rotation.

COR. 3. *If an impact be made on any point of the axis of a ſymmetrical body, or a ſolid of rotation, and that point be conſidered as the point of ſuſpenſion, the correſponding centre of oſcillation will be the centre of ſpontaneous rotation.*

This will be evident on comparing this propoſition with art. 306. cor. 5. and 318. cor. 1.

COR. 4. *To whatever point of a right line drawn through the centre of gravity of a body the impact be applied, the velocity of the centre of gravity will be the ſame.*

For the expreſſion $\frac{\varphi}{B}$ denoting the velocity, contains no diſtance. The ſame thing will readily appear otherwiſe, thus: To whatever point of QP (fig. 15.) the force φ is applied, the incipient motion $Q \cdot Qq + P \cdot Pp$ will be the ſame by the ſecond law of motion, the particles Q, P, being connected by the inflexible line QP; and, conſequently, the velocity of the centre of gravity, or GG′, is always the ſame as if both particles were placed at G, and impelled by the ſame force φ; for $GG' = \frac{P \cdot Pp + Q \cdot Qq}{P+Q}$, by the nature of the centre of gravity. And the ſame thing may be ſhewn of as many other points as we pleaſe.

COR. 5. *Hence, in rotations about a centre of ſpontaneous gyration, the permanency of the quantity of motion obtains.*

324. PROP. *When φ S, the direction of the impact, paſſes through the centre of the impelling body, the centre of gravity of the body ſtruck will move with a velocity equal to the product of the quantity of motion of the impelling body into the diſtance between the centres of gravity and ſpontaneous rotation, divided by the ſums of the products of the impelling body into the diſtance of the point of impact from the centre of ſpontaneous rotation, and of the impelled body into the diſtance between the centres of ſpontaneous rotation and of gravity.*

Let the quantity of matter of the inpinging body be b, its velocity v, or $bv=\varphi$; and when the body B is ſtruck in the direction ϕ S (in which the centre of the body b is always found,) let the velocity of its centre of gravity be V, the centre of ſpontaneous converſion being at C. Then CG : CS : : V : the velocity of the point S, which is therefore $= V \,.\, \frac{CS}{CG}$; conſequently, $v - V \,.\, \frac{CS}{CG}$ = the velocity loſt by b in the direction ϕ S: whence, by the third law of motion, $b \times \frac{v\,.\,CG - V\,.\,CS}{CG} = B\,.\,V$, and, by reduction, $V = \frac{b\,.\,v\,.\,CG}{B\,.\,CG + b\,.\,CS}$.

COR. If the inertia of the ſtriking body be evaneſcent, the velocity V will become $\frac{bv}{B}$ or $\frac{\phi}{B}$; being *the ſame as would be generated in the centre of gravity if the body* b *impinged directly on it with the velocity* v.

325. PROP. *The conditions in the laſt corollary being retained, the angular velocity of the centre of the ſyſtem about the centre of gravity, is equal to the momentum of the impelling body, divided by twice the product of the maſs of the impelled body and the diſtance* GC, *into the periphery of a circle whoſe diameter is unity.*

If a fixed axis paſſed through C, the centre of gravity would deſcribe a circle whoſe radius is CG with the velocity $\frac{bv}{B}$. But in the preſent caſe the motion of the ſyſtem will (art. 322.) be compounded of the uniform rectilinear motion of the centre of gravity in the direction GG′, perpendicular to CS, and the angular motion $x\,G'\,p = GCG$, generated round the centre of gravity. And, ſince the periphery of a circle whoſe radius is CG is $2\pi \,.\, CG$, we have this analogy, $\frac{bv}{B} : 2\pi \,.\, CG : : 1^s : \frac{2\pi\,.\,B\,.\,CG}{bv}$, the time of one revolution in ſeconds: whence it follows, that the number of revolutions, or parts of a revolution, in a ſecond, or the angular velocity U, will be $1 \div \frac{2\pi\,.\,B\,.\,CG}{bv} = \frac{bv}{2\pi\,.\,B\,.\,CG}$.

COR. 1. Since C is the centre of percuſſion to S as a centre of motion, if Q be the centre of gyration with reſpect to G as a centre of motion, (that is, if Q be the principal centre of gyration,) we have (art. 310. cor. 2.) $GC \,.\, GS = GQ^2$, or $CG = \frac{GQ^2}{GS}$; this value of CG being ſubſtituted for it in the preceding expreſſion for the angular velocity, it becomes $U = \frac{b\,.\,v\,.\,GS}{2\pi\,.\,B\,.\,GQ^2}$.

Cor. 2. *The centre of ſpontaneous rotation, during the motion of the ſyſtem, deſcribes the common cycloid.*

For the motion of any point in the ſyſtem is compounded of the uniform rectilineal motion of the centre of gravity, and of the angular motion generated round that centre: but (art. 322.) the velocity with which the centre of ſpontaneous rotation would move round the centre of gravity, if there only exiſted a rotatory motion in the ſyſtem, would be equal to that with which the centre of gravity would move round it, if the centre C were fixed: conſequently, ſince the centre C has both a rotatory and progreſſive motion, each of which is equal to that of the centre of gravity, it will deſcribe a cycloid.

326. Prop. *In the body or ſyſtem* B (fig. 15.) *to which, when quieſcent, motion has been communicated by the impulſe of a force* φ *without inertia, that is, rectilinear motion to the centre of gravity meaſured by the ſpace* V *which that centre would deſcribe uniformly in any given time, and angular motion meaſured by the revolutions* U, *or parts of a revolution, which it would deſcribe uniformly round* G *in the ſame time; then, if the notation in the preceding propoſitions be retained, and* Q *be the principal centre of gyration when the ſyſtem revolves about its centre of gravity, the perpendicular diſtance from the centre of gravity at which the impelling force muſt act ſo as to have generated theſe progreſſive and rotatory motions will be* $GS = \frac{2\pi . U . GQ^2}{V}$.

Let φ S be the direction of the impulſe, and let φ be equal to the momentum of an evaneſcent body b moving with the velocity v, B being the weight of the ſyſtem; then (art. 324. cor.) the velocity communicated to the centre of gravity will be $= \frac{b\,v}{B}$; and the angular velocity generated in the ſyſtem (art. 325. cor. 1.) $= \frac{b . v . GS}{2\pi . B . GQ^2}$. But, by the prop. the velocity communicated to the centre of gravity is V, and the angular motion, that is the number of revolutions, or parts of a revolution, deſcribed while the centre of gravity paſſes over the ſpace V, is U: ſo that, from the conditions there ariſes this equation, $U = \frac{b . v . GS}{2\pi . B . GQ^2} = \frac{V . GS}{2\pi . GQ^2}$, by putting V for its equal $\frac{bv}{B}$. Hence $GS = \frac{2\pi U . GQ^2}{V}$.

Cor. If the body B be a ſphere whoſe radius is r, then, art. 312. $GQ^2 = \frac{2}{5} r^2$, and if u be the abſolute velocity of rotation of an equator of the ſphere, we have $U = \frac{u}{2\pi r}$; whence the preceding value of GS is transformed to this, $GS = \frac{2}{5} r . \frac{u}{V}$.

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327. This propoſition may be applied to the double motion of the planets. The earth revolves about an axis paſſing through its centre of gravity, while, by a motion of tranſlation, that centre is carried on in free ſpace in an orbit nearly circular. And a ſimilar kind of double motion has been diſcovered in ſeveral of the planets; and analogy leads us to believe it obtains in the others. Now, ſuppoſing the bodies of the planets to be ſpherical, as they are nearly, the uſe of this propoſition at once appears. Having given, for inſtance, the magnitude of an impulſe with reſpect to the maſs of the earth, and the direction φ S in which it was applied, at any given diſtance SG from the centre of gravity, the angular motion round G would be inferred: and, converſely, if the actual rotatory velocity of the earth's equator and the velocity of its orbit be aſcertained, the diſtance GS from the centre at which it may have received a ſingle impulſe φ S, adequate to produce the double motion, may readily be found.

Thus, any point in the earth's equator paſſes over 25020 miles by rotation in a ſidereal day; and, if the mean diſtance of the earth from the ſun be 95 millions of miles, the earth will paſs over nearly 596904000 miles by its orbital motion, in a year, or about 366 ſidereal days: hence $596904000 \div 366 = 1630852$, will be $= \mathrm{V}$ in our theorem, while $25020 = u$. Conſequently $\mathrm{GS} = \frac{2}{5} r \cdot \frac{u}{\mathrm{V}} = \frac{r}{163 \cdot 2}$. So that if an impulſe be impreſſed on a quieſcent ſphere, and the direction of the force ſhould be at a diſtance GS from its centre of gravity of about $\frac{1}{163}$ part of the radius, the angular motion of that ſphere, and the abſolute motion of its centre, will have the ſame relation to each other, as thoſe which actually obtain in the earth.

The time of the rotations of Mercury, Venus, Herſchel, and the other planets, is unknown; but for the following planets it is aſcertained; ſo that by the ſame theorem we obtain theſe values of GS. Mars $\frac{r}{195}$; Jupiter $\frac{r}{2 \cdot 8125}$; Saturn $\frac{r}{2588}$; the moon $\frac{r}{555}$. We have not ſufficient data to determine this for the ſun. "But (as Dr. Robiſon remarks) the very circumſtance of his having a rotation in 27 d. 7 h. 47 m. makes it very probable, that he, with all his attending planets, is alſo moving forward in the celeſtial ſpaces, perhaps round ſome centre of ſtill more general and extenſive gravitation: for, the perfect oppoſition and equality of two forces, neceſſary for giving a rotation without a progreſſive motion, has the odds againſt it of infinity to unity. This corroborates the conjectures of philoſophers, and the obſervations of Herſchel and other aſtronomers, who think that the ſolar ſyſtem is approaching to that quarter of the heavens in which the conſtellation Aquila is ſituated."

328. Prop. *If a body revolves about an axis paſſing through its centre of gravity with the angular velocity* U, *while this axis is carried round another axis, alſo paſſing through its centre of gravity, with the angular velocity* U, *theſe two motions compoſe a motion of every particle of the body round a third axis lying in the plane of the other two, and inclined to each of the former axes in angles whoſe ſines are inverſely as the angular velocities round them; and the angular velocity* v *about this new axis is to that about one of the primitive axes, as the ſine of the inclination of the latter axis, to the ſine of the inclination of the new axis to the other primitive axis.*

Thus, if a body turns round an axis AG *a* (fig. 1. pl. XV.) paſſing through its centre of gravity G with the angular velocity U, while this axis is carried round another axis BG *b* with the angular velocity *u*, and if GD be taken to GE as U to *u*, (the points B and E being taken on that ſide of the centre where they are moving towards the ſame ſide of the plane of the figure,) and the line DE be drawn, the whole and every particle of the body will be in a ſtate of rotation about a third axis CG *c*, parallel to DE, lying in the plane of the other two; and the angular velocity *v* about the axis CG *c*, will be to U, and to *u*, as DE is to GD and to GE.—For, let P be any particle of the body, and ſuppoſe a ſpherical ſurface whoſe centre is G to paſs through P. Draw PR perpendicular to the plane of the figure: then is PR the common ſection of the circle of rotation IP *i* round the axis A *a*, and the circle KP *k* of rotation round the axis B *b*. Let F and O be the centres of theſe circles of rotation, and I *i*, K *k*, their diameters. Draw the radii PF, PO, and the tangents PM, PN: theſe tangents are in a plane MPN which touches the ſphere in P, and cuts the plane of the axis in a line MN, to which a line drawn from G through R would be perpendicular. Suppoſe PN to repreſent the velocity of rotation of the point P about the axis B *b*, while P *f* repreſents its velocity of rotation about A *a*; and complete the parallelogram PN *t f*: then is P *t* the direction and velocity of the reſultant of the compoſition of PN and P *f*, and it is manifeſtly in the ſame plane as the conſtituent lines. Let perpendiculars *f* F, *t* T, be drawn to the plane of the axis, and the parallelogram PN *t f* will be orthographically projected on that plane, its projection being alſo a parallelogram RNTF, (art. 68.) *. Draw the diagonal RT. Then, ſince PR is perpendicular to the plane of the primitive axis, PR t T is ſo likewiſe: and, conſequently, the compound motion P t is in the plane of a circle of revolution about ſome axis ſituated in the plane of the other two. Produce TR, and draw GC interſecting it perpendicularly in H; and let LP l be the circle of rotation, its diameter being L l = 2 LH: then is P t a tangent, and perpendicular to PH; and it will meet RT in ſome point Q of the line MN. The particle P is in a ſtate of rotation about the axis CG c, and its velocity is to the velocities round A a, or B b, as P t to PF or to PN. Now PN the tangent is perp. to OP, and PR is perp. to ON: therefore, OP : PN : : PR : RN, and RN $= \frac{\text{PR . PN}}{\text{OP}}$. But the velocity of P about the axis B b is u . OP: whence RN $= \frac{\text{U . OP . PR}}{\text{OP}} = u$. PR. In like manner RF = U . PR. Conſequently RF : RN : : U : u : : GD : GE. But NT : RN : : ſin. NRT : ſin. NTR, and GD : GE : : ſin. GED : ſin. GDE; hence, ſin. NRT : ſin. NTR : : ſin. GED : ſin. GDE. Now ſince NR is perpendicular to EG, and NT (parallel to IF) perpendicular to DG, we have RNT=EGD. Hence TR is perpendicular, and C c parallel to ED, and the rotation of the particle P is about an axis parallel to ED. Alſo, ſince RN, RF, RT, are as the velocities u, U, v, about theſe different axes, and vary reſpectively, as EG, DG, DE, we have v : U : u : : ED : GD : GE, which was to be demonſtrated.

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* The coincidence of F with the centre of the circle IP *i* is purely accidental.

COR. 1. Hence, if every particle of a body, whether ſolid or fluid, receives at the ſame inſtant two ſeparate impulſes, the one competent to the production of a motion of the particle round an axis with a certain angular velocity, and the other competent to the production of a rotatory motion about another axis with a certain velocity, the combined effect of all theſe impulſions will be a motion of the whole ſyſtem about a third axis given in poſition, with an angular velocity which is alſo given; and this motion will obtain *without any ſeparation or diſunion of parts*, except ſuch as may be occaſioned by the action of the centrifugal forces reſulting from the rotation.

COR. 2. Hence, alſo, if a body be turning round any axis, and every particle in one inſtant get preciſely ſuch an impulſe as is competent to produce a given angular velocity round another axis, the body will turn round a third axis given in poſition with a given angular velocity.

COR. 3. Laſtly, when a rigid body acquires a rotation about an axis by an impulſe on one part of it, and either at the ſame time or afterwards receives an impulſe on any part, which, alone, would have produced a certain rotation about another axis, the joint effect of theſe impulſes will be a rotation about a third axis, in conformity with this propoſition. For when a rigid body acquires a motion about an axis, not by the ſimultaneous impulſe of the preciſely competent force on each particle, but by an impulſe on one part, there has been propagated to every particle (by means of the coheſive forces) an impulſe preciſely competent to the production of that motion which the particle actually acquires; and when a rigid body already turning round an axis A *a* receives an impulſe which makes it actually turn about another axis C *c*, there has been propagated to each particle a force preciſely adequate to the production, not of the motion, but of the *change* of motion which takes place in that particle, that is, a force which, when compounded with the inherent force of its primitive motion, produces the new motion; that is, (by the prop.) a force which alone would have cauſed it to turn about a third axis B *b*, with a rotation making the other component of the actual rotation about C *c*.

This elegant theorem, the enunciation of which is due to P. Friſi, is very important, and gives a great extenſion to the doctrine of the compoſition of motion. It is of great uſe in many curious problems, and particularly that of the preceſſion of the equinoxes. Such as wiſh for further information on various points connected with the ſubjects of this chapter, may conſult *Simpſon's Tracts*, *Friſi's Coſmographia*, *Mr. Vince's Paper in the Phil. Tranſ.* 1780, *Landen's Memoirs*, *Atwood on the Rotatory Motion of Bodies*, and the article ROTATION in the *Encyclopædia Britannica*.

CHAPTER V.

Phyſico-Mathematical Theory of Percuſſion.

329. DEFS. IN the ordinary theory of percuſſion, or colliſion, bodies are regarded as either *hard*, *ſoft*, or *elaſtic*. A *hard* body is that whoſe parts do not yield to any ſtroke or percuſſion, but retains its figure unaltered. A *ſoft* body is that whoſe parts yield to any ſtroke or impreſſion, without reſtoring themſelves again; the ſhape of the body remaining altered. An *elaſtic* body is that whoſe parts yield to any ſtroke, but preſently reſtore themſelves again, ſo that the body regains the ſame figure as before the ſtroke. When bodies which have been ſubjected to a ſtroke or a preſſure return only in part to their original form, the *elaſticity* is then *imperfect*: but if they reſtore themſelves entirely to their primitive ſhape, and employ juſt as much time in the reſtoration as was occupied in the compreſſion, then is the elaſticity *perfect*.

It has been cuſtomary to treat only of the colliſion of bodies perfectly hard or perfectly elaſtic: but as there do not exiſt in nature any bodies (which we know) of either the one or the other of theſe kinds, the uſual theories are of no ſervice in practical mechanics. In fact, if the effects which are experienced in the mutual impact of ſuch bodies as nature furniſhes were the ſame as thoſe which would reſult from bodies perfectly hard, it would appear impoſſible to oppoſe to the moving bodies any preſſure capable of deſtroying their effects. For the quantity of motion, or momentum of a moving body, being valued by the product of its maſs and velocity, while the effect produced by a body which merely preſſes is eſtimated by the product of its maſs into a velocity denoted by 0; one of the factors of the latter product will be nothing, while both factors of the former will be finite, and of conſequence the magnitudes are ſuch as cannot be compared.

Yet we may ~~eaſily~~ be convinced by daily experience, that the advantage which bodies moving ſwiftly have over thoſe which oppoſe merely a reſiſtance of preſſure, though very great, is by no means infinite. Numerous circumſtances which will be ſuggeſted to every mind prove that, phyſically ſpeaking, we may balance a percuſſive force by one of mere preſſure, and even that the latter may overcome the former and produce a greater effect *. It argues nothing to aſcribe any part of this to *friction;* for friction itſelf is a ſpecies of preſſive force, or is, certainly, more analogous to preſſure than to percuſſion.

The cauſes to which we ſhould attribute the difference which ſubſiſts between the effects of impact and of ſimple preſſure may be readily traced. When a body ſuppoſed perfectly hard in the courſe of its motion ſtrikes another body, perfectly hard alſo, the variation of motion ought to be produced in an indiviſible inſtant in ſuch ſort, that between the initial velocity and the velocity after the ſhock there ſhall not be any intermediate velocity. But if the motion of the body were modified by a preſſure, or by a conſtant force, as gravity, then it would change by ſenſible degrees, and have undergone a determinate variation at the end of a certain time.

It is therefore the law of continuity which diſtinguiſhes the effects of preſſion and thoſe of impact, when the *hardneſs* is infinite: but as ſuch hardneſs no-where exiſts, ſince matter always poſſeſſes a certain degree of elaſticity, and a limited coheſion of particles which may be ſurmounted, we may enquire if percuſſion, conſidered phyſically, conforms to the law of continuity. Now when a body ſtrikes another, two effects have place in each. 1ſt. The parts in contact yield to the action of the ſtroke, and become compreſſed, ſo that the figure of the bodies is altered by a flattening or impreſſion, which obtains in the parts in contact, and in the neighbourhood of thoſe parts. 2d. When the flattening or impreſſion has arrived at the greateſt degree of which the bodies are ſuſceptible, then their inherent elaſticity tends to deſtroy the impreſſion, and effaces it either wholly or in part; this produces a mutual action and reaction of the bodies, which is continued till they are no longer in contact *. Hence it appears, that the law of continuity is followed actually in the impact of bodies; and that no alteration in their motion takes place, without the previous taking place of all the intermediate alterations. It is true this alteration is always produced during an extremely ſhort interval of time, and this occaſions the great diſproportion which is obſerved between the effects of impact and thoſe of preſſure: but it is, notwithſtanding, inconceivable, as Mr. Atwood remarks, "that any really exiſting body ſhould paſs from quieſcence into finite motion, or from one degree of finite motion to another, without having poſſeſſed all the intermediate degrees of velocity:" and hence it follows that the phenomena of colliſion may be conſidered of a

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* Thus, in driving the piles for the foundation of the bridge of ſome mills, where the ſoil upon which the bridge was to be erected was a pure ſand of *uniform* denſity to the depth of 47 feet, no piles could be driven lower than 15 feet, although the rammers in the engines employed were enormouſly great. So that percuſſion is brought into a ſtate of equilibrium with mere preſſure, and cannot therefore be infinitely greater. Indeed it is not a little inconſiſtent that many authors, while they admit that impact and preſſure are comparable in the theory of Hydrodynamics, and Pneumatics (as when, for example, they aſſume *the weight of a column of water equal to the force of impact* of that fluid on a wheel), ſhould deny the ſame thing when writing on Dynamics.

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* A very diſtinct view of this proceſs has been given by Dr. Robiſon in the article IMPULSION, *Supp. Ency. Britan.* from which the following is extracted:

"As ſoon as the bodies come into ſenſible contact, compreſſion *muſt* begin; for we may ſuppoſe the bodies to be two balls, which will therefore touch only in one point. The mutual preſſure, which is neceſſary in order to produce the retardation of A, and the acceleration of B, is exerted only on the foremoſt particle of A, and the hindmoſt particle of B; but no atom of matter can be put in motion, or have its motion any way changed, unleſs it be acted on by an adequate force. The force urging any individual particle muſt be preciſely competent to the production of the very change of motion which obtains in that particle. Except the two particles which come into contact in the colliſion, all the other particles are immediately actuated by the forces which connect them with each other; and the force acting on any one is generally compounded of many forces which connect that particle with thoſe adjoining. Therefore, when A overtakes B, the foremoſt particle of A is immediately retarded: the particles behind it would move forward, if their mutual connection were diſſolved in that inſtant; but, this remaining, they only approach nearer to the foremoſt ſtriking particles, and thus make a compreſſion, which gives occaſion for the inherent elaſticity to exert itſelf, and, by its reaction, retard the following particles. Thus each ſtratum (ſo to conceive it), continuing in motion, makes a compreſſion, which occaſions the elaſticity to react, and, by reacting, to retard the ſtratum immediately behind it. This happens in ſucceſſion: the compreſſion and elaſtic reaction begin in the anterior ſtratum, and take place in ſucceſſion backward, and the whole body gets into a ſtate of compreſſion. Things happen in the ſame manner in B, but in the contrary direction, the foremoſt ſtrata being the laſt which are compreſſed. All this is done in an inſtant (as we commonly, but inaccurately, ſpeak), that is, in a very ſmall and inſenſible moment of time; but in this moment there is the ſame gradual compreſſion, increaſe of mutual action, greateſt compreſſion, common velocity, ſubſequent reſtitution, and final ſeparation, as in the caſe of bodies with a ſlender ſpring interpoſed, or even in the caſe of mutually repelling magnets. In all the caſes, the changes of motion are produced by the elaſticity or the repulſion, and not by the transfuſion of the force of motion. The changing force is indeed inherent in the bodies, but not becauſe they are in motion; the uſe of the motion is to give occaſion, by continued compreſſion, for the continued operation of the inherent elaſticity."

kindred claſs to thoſe which are occaſioned by accelerating or retarding forces, and act by inſenſible degrees in order to produce a finite effect.

After theſe preliminary obſervations, we ſhall now preſent the reader with a phyſical theory of percuſſion, which will comprehend all the circumſtances of the motion, both during and after the ſtroke. From the ſame ſource the laws of the colliſion of hard, ſoft, and perfectly or imperfectly elaſtic bodies, will be deduced. All that we ſhall give which is different from the uſual theory we wiſh to be conſidered as originally due to *Don George Juan*, a Spaniſh author, and publiſhed in 1771, a work entitled *Examen Maritimo.* The ſame theory has been lately adopted by *M. Prony*, and ſome other French authors. As the mode of inveſtigation is analytical throughout, it ſhall be preſented nearly in its original ſtate, without announcing the various particulars as they ariſe, in ſeparate propoſitions.

330. Def. By the *Depth of the impreſſion* we mean its greateſt lineal meaſure in the direction of the motion; as, if the whole impreſſion were a ſpheric ſegment the *depth* would be its verſed ſine. The *Amplitude of the impreſſion* is the greateſt ſection which can be taken perpendicularly to the direction of the motion: in the caſe of a ſpherical body it would be the circular baſe of a ſegment.

Through the whole of this inveſtigation the various quantities will be thus repreſented:

m and μ, the two bodies which ſtrike each other;
b and β, the amplitudes of impreſſion;
r and ρ, the comparative hardneſs of the bodies;
x and z, the depths of the impreſſions;
f and φ, the quantities of motion which the bodies acquire in a unit of time, in virtue of the action of the conſtant powers;
W and w, the velocities with which the ſhock commences;
u and v, the velocities in any inſtant of the ſhock;
s and σ, the ſpaces deſcribed in the ſame time;
t, that time.

Any other characters that may be adopted will be explained as they firſt occur.

331. Let it be ſuppoſed that the body m follows and ſtrikes the body μ, and, of conſequence, that the velocity W is greater than w; without this the impact can never take place, unleſs the velocity w be negative, or contrary to W: but, for the greater facility, we ſhall ſuppoſe all along (except it be otherwiſe expreſſed) that the powers f and φ, as well as the velocities W and w, are poſitive; for it will be eaſy to render either of the quantities negative in the reſult of the computation.

Let it be ſuppoſed, moreover, that the two bodies are moving in the ſame direction, and that the impact is ſo effected that it does not produce any gyratory motion; otherwiſe we ſhould render the inveſtigation very intricate and perplexing.

Finally, let the bodies m and μ be conſidered as ſo large that the impreſſions made upon them by the ſhock ſhall not produce any ſenſible change in the place of the centre of gravity, or that the motions of theſe centres ſhall not be ſenſibly affected by the relative change of ſituation in the various moleculæ of the bodies.

322. All the quantities mentioned in art. 330. may be readily introduced into the calculus, ſince each of them has a certain relation to determinate units; the quantities r and ρ only, which repreſent the degrees of hardneſs, appear to preſent ſome little difficulty. Now to form an idea of hardneſs in that point of view under which we muſt conſider it, that is to ſay, as a *meaſurable* object, we may obſerve, that ſince this property is one of the moſt intimate and hidden properties of bodies, and ſince to meaſure is to compare, we cannot (properly ſpeaking) meaſure hardneſs, becauſe in an unknown object we can make no compariſon. But as hardneſs is only intereſting to us by its effects, it is the meaſure of theſe effects which it is important for us to know; and this meaſure for the reſpective bodies we denote by r and ρ.

If a perfectly hard body ſtrikes another which is not perfectly hard, and if the ſurface of contact to which the direction of the motion is ſuppoſed perpendicular is plane and conſtantly equal to a given ſurface, as a foot ſquare (the foot being the unit), the reſiſtance of the body ſtruck will take from the impinging body, during an evaneſcent inſtant of time $\dot{t}$, a quantity of motion infinitely ſmall, which let be called ω. If ω be multiplied by the fraction expreſſing the ratio of the unit of time to the element of time, the product $\omega \frac{1}{\dot{t}}$ will be preciſely equal to the quantity ρ, which denotes the hardneſs of the body ſtruck during the inſtant which correſponds to the value of ω, or rather, which denotes the effect that this hardneſs would produce by continuing its action for a unit of time. We have, therefore, $\frac{\omega}{\dot{t}} = \rho$, or $\omega = \rho \dot{t}$.

Thus it appears that ω is independent of the maſſes and of the velocities, whether abſolute or relative (12. 15.); it repreſents a quantity of motion which may reſult from an infinite number of combinations of maſſes and velocities; ω, therefore, depends ſolely on the nature of the body ſtruck: and although the maſſes and the velocities of the two bodies enter into the value of ϱ, they do ſo merely as cauſes producing the impreſſion, and we determine ρ from experiments by the relation which they bear to this impreſſion.

A ſimilar reaſoning will apply to the quantity r, by ſuppoſing that the body ſtruck is perfectly hard, while the impinging body is not; and hence will ariſe the analogous equation $\omega'=r\dot{t}$.

If the ſurface of contact be not plane, we muſt ſubſtitute for it the amplitude of the impreſſion: but this ſubſtitution will change nothing; for all the points ſtruck in the direction of the motion being contained in the amplitude of impreſſion, it may repreſent the ſum of all thoſe points.

333. *To find the quantities of motion loſt*, viz. the values of ω and ω, which anſwer to any amplitudes of impreſſion whatever b and β, we muſt make theſe proportions; $1 : \beta :: \rho\dot{t} : \beta\varrho\dot{t} = \omega$, and $1 : b :: r\dot{t} : br\dot{t} = \omega'$; then will the equations $\omega=\beta\rho\dot{t}$, and $\omega'=br\dot{t}$ obtain, whatever are the amplitudes of impreſſion.

In conſequence of the equality of action and reaction, the values $\omega=\beta\rho\dot{t}$, and $\omega'=brt$, of motion loſt by the impinging bodies, are likewiſe thoſe of the *motion gained by the bodies ſtruck.*

334. *To find the value of the motion loſt by the body, or of the percuſſive force*, when neither of the two bodies is perfectly hard. In this caſe both are compreſſible, and if F be the fluxion of the quantity of motion, we ſhall have to determine $\frac{1}{\dot{t}}$ F or $\frac{F}{\dot{t}}$. Now it is manifeſt that F will be always the greater, as $\beta\rho$ and br are greater; and thence (ſuppoſing $\dot{t}$ conſtant) $\frac{F}{\dot{t}}$ will vary as $\beta\varrho br$. So that when $\varrho=\infty$, or is infinite, we muſt have $F=rb\dot{t}$ (333), and when $r=\infty$, we have $F=\varrho\beta\dot{t}$: let us, then, multiply $\beta\rho br$ by ſuch a quantity $\frac{1}{\beta\varrho+br}$ as will ſatisfy theſe two conditions, and there will reſult $\frac{F}{\dot{t}}=\frac{\beta\varrho br}{\beta\varrho+br}$. And if $\frac{F}{\dot{t}}$, or the force of percuſſion, be denoted by P, we have $P=\frac{\beta\varrho br}{\beta\varrho+br}$; an expreſſion for the percuſſive force in any inſtant whatever.

335. *The fluxions of the impreſſions are reciprocally as the degrees of hardneſs.* The fluxions $b\dot{x}$, $\beta\dot{z}$, of the impreſſions are produced at every inſtant by the ſame power $P\dot{t}$ acting

equally on the two bodies; and to produce theſe elementary impreſſions the force P $\dot{t}$ has to overcome the powers $r\dot{t}$, and $\varrho\dot{t}$; thus the effects $b\dot{x}$ and $\beta\dot{z}$ are the greater as the powers $r\dot{t}$ and $\varrho\dot{t}$ are the leſs; and as theſe effects are produced by the ſame cauſe, they will be in the inverſe ratio of the obſtacles $r\dot{t}$ and $\varrho\dot{t}$, which oppoſe them; we have, therefore, $b\dot{x} : \beta\dot{z} :: \varrho\dot{t} : r\dot{t} :: \varrho : r$; whence $br\dot{x} = \beta\varrho\dot{z}$.

336. Since the body m follows the body μ, and by the impact cauſes impreſſions in both whoſe depths are x and z, the ſpace s run over by the body m muſt be equal to the ſpace σ deſcribed by the body μ *plus* the ſum $x+z$ of the depths of impreſſion: hence $s=\sigma+x+z$.

Elaſtic bodies aſſume again at the end of the ſhock the figures which they had at its commencement; ſo that in this caſe the impreſſions vaniſh, and *at the end of the ſhock of elaſtic bodies* $s=\sigma$.

337. From the equation $s=\sigma+x+z$, or $s-\sigma=x+z$, we deduce $\dot{s}-\dot{\sigma}=\dot{x}+\dot{z}$: but the property of variable motion (232. I.) gives $u\dot{t}=\dot{s}$, and $v\dot{t}=\dot{\sigma}$; we have, therefore, $(u-v)\dot{t}=\dot{s}-\dot{\sigma}=\dot{x}+\dot{z}$, and $\dot{t}=\frac{\dot{x}+\dot{z}}{u-v}$, *the value of the element of the time.*

When the impreſſions arrive at their greateſt value we have $x+z=a$ max. or $\dot{x}+\dot{z}=0$, and the equation $u-v=\frac{\dot{x}+\dot{z}}{\dot{t}}$ becomes $u-v=0$: conſequently, *in the inſtant when the greateſt impreſſions are produced the velocities of the two bodies are equal.*

338. The accelerating forces which act upon the body m are f and P; and as P tends to retard its motion, the accelerating force reſulting from the joint operation of the two will be $f-$P: for the ſame reaſon, contrarily applied, the accelerating force of μ will be $\phi+$P. Recollecting, therefore, that f, ϕ, and P, may repreſent the products of maſſes and velocities, we have, by the nature of variable motion (art. 232.) $m\dot{u}=(f-P)\dot{t}$; and $\mu\dot{v}=(\phi+P)\dot{t}$; the ſum of theſe equations is $(f+\varphi)\dot{t}=m\dot{u}+\mu\dot{v}$, the fluent of which is $(f+\varphi)t=mu+\mu v+C$. To determine the conſtant quantity C, it may be obſerved that when $t=0$, $u=W$, and $v=w$; and then the equation reduces to $mW+\mu w+C=0$; ſo that $C=-mW-\mu w$, and the corrected fluent is $(f+\varphi)t=m(u-W)+\mu(v-w)$; whence it is

eaſy to obtain $v = \frac{(f+\varphi)\,t + m\,W + \mu\,w - m\,u)}{\mu}$, *an equation which expreſſes the relation between the velocities, in any inſtant whatever of the ſhock of bodies, whether elaſtic or non-elaſtic.*

339. If the bodies are not elaſtic the ſhock ceaſes at the inſtant when the impreſſion is greateſt; but in this inſtant the velocities are equal (art. 337.). Therefore, in the preceding equation, we muſt make $u = v$, and there will ariſe $u = v = \frac{(f+\varphi)\,t + m\,W + \mu\,w}{m+\mu}$, an equation expreſſing *the common velocity after the ſhock of non-elaſtic bodies.*

340. In like manner we might deduce from the final equation in art. 338. the velocities of elaſtic bodies after the ſhock. But as we would ſtate the queſtion generally, we muſt determine the velocities of both bodies, whether they are elaſtic or not, in any inſtant of the ſhock.

The equations $(f-P)\,\dot{t} = m\,\dot{u}$ and $(\varphi+P)\,\dot{t} = \mu\,\dot{v}$ (art. 338.) give $\dot{u} = \frac{(f-P)\,\dot{t}}{m}$, and $v = \frac{(\varphi+P)\,\dot{t}}{\mu}$. Taking the latter of theſe from the former, we have $\dot{u} - \dot{v} = \left(\frac{f-P}{m} - \frac{\varphi+P}{\mu}\right)\dot{t}$; and this, multiplied by $m\,\mu$, gives $m\,\mu\,(\dot{u} - \dot{v}) = (\mu\,f - m\,\varphi - \mu\,P - m\,P)\,\dot{t}$. Subſtituting, in this equation, for $\dot{t}$ its value in art. 337. and for P and $\dot{x}$ their values in art. 334. 335. it will be transformed to this, $m\,\mu\,(u-v)\,.\,(\dot{u} - \dot{v}) = (\mu\,f - m\,\phi)\frac{\beta\,\varrho + b\,r}{b\,r}\,\dot{z} - (m+\mu)\,\beta\,\varrho\,\dot{z}$. Taking the fluents, we have $\frac{1}{2}\,m\,\mu\,(\overline{u-v}^2) - \overline{W-w}^2 = (\mu\,f - m\,\phi)\int\frac{\beta\,\varrho + b\,r}{b\,r}\,\dot{z} - (m+\mu)\int\beta\,\varrho\,\dot{z}$. Now, $\int\frac{\beta\,\varrho + b\,r}{b\,r}\,\dot{z} = x + z$; for $\frac{\beta\,\varrho + b\,r}{b\,r}\,\dot{z} = \frac{\beta\,\varrho\,\dot{z}}{b\,r} + \dot{z} = \dot{x} + \dot{z}$, becauſe $\dot{x} = \frac{\beta\,\varrho\,\dot{z}}{b\,r}$, from art. 335. Conſeq. we have, by ſubſtitution, $\frac{1}{2}\,m\,\mu\,(\overline{u-v}^2 - \overline{W-w}^2) = (\mu\,f - m\,\phi)\,.\,(x+z) - (m+\mu)\int\beta\,\varrho\,\dot{z}$. Hence $u - v = \pm\left(\overline{W-w}^2 + \frac{2\,(\mu\,f - m\,\varphi)\,.\,(x+z)}{m\,\mu} - \frac{2\,(m+\mu)\int\beta\,\varrho\,\dot{z}}{m\,\mu}\right)^{\frac{1}{2}}$. Here the poſitive ſign obtains previous to the inſtant of the greateſt impreſſion, and the negative ſign after that inſtant: for at the inſtant of the greateſt impreſſion we have $u=v$ (art. 337.); and after that inſtant the re-eſtabliſhing of the compreſſed parts accelerates the motion of μ and retards that of m, and conſequently makes u ſmaller than v.

If the value of v in art. 338. be ſubſtituted for it in the preceding equation, we ſhall, after a little reduction, find

$$u=\frac{m\,W+\mu\,w+(f+\varphi)\,t}{m+\mu}\pm\frac{\mu}{m+\mu}\left(\overline{W-w}^2+\frac{2(\mu f-m\,\varphi)\,.\,(x+z)}{m\,\mu}-\frac{2\,(m+\mu)\int\beta\,\varrho\,\dot{z}}{m\,\mu}\right)^{\frac{1}{2}}.$$

And, by a ſimilar mode of proceeding, we find

$$v=\frac{m\,W+\mu\,w+(f+\varphi)\,t}{m+\mu}\mp\frac{m}{m+\mu}\left(\overline{W-w}^2+\frac{2(\mu f-m\,\varphi)\,.\,(x+z)}{m\,\mu}-\frac{2\,(m+\mu)\int\beta\,\varrho\,\dot{z}}{m\,\mu}\right)^{\frac{1}{2}}.$$

In theſe two equations for the velocities the ſuperior ſign ſerves for any inſtant of the ſhock before that of the greateſt impreſſion, and the inferior ſign after that inſtant. Thus, at the moment of the greateſt impreſſion the quantity affected with the double ſign ± or ∓ vaniſhes, becauſe it is then paſſing from poſitive to negative, or from negative to affirmative. Theſe equations manifeſtly apply for any inſtant of the ſhock, whether the bodies be elaſtic or not.

In non-elaſtic bodies the values of u and v reduce to the common value $\frac{m\,W+\mu\,w+(f+\varphi)\,t}{m+\mu}$, which is the ſame as we found in the preceding article.

341. Bodies perfectly elaſtic, on the contrary, re-eſtabliſhing themſelves in their primitive ſtate at the end of the ſhock, we have for that inſtant x, $\dot{z}$, and $\int\beta\,\varrho\,\dot{z}$, each equal to *zero*, and thus the values of u and v for *elaſtic bodies* become

$$u=\frac{m\,W+\mu\,w+(f+\varphi)\,t}{m+\mu}-\frac{\mu}{m+\mu}(W-w)=\frac{(m-\mu)\,W+2\,\mu\,w+(f+\varphi)\,t}{m+\mu}.$$

$$v=\frac{m\,W+\mu\,w+(f+\varphi)\,t}{m+\mu}+\frac{m}{m+\mu}(W-w)=\frac{(\mu-m)\,w+2\,m\,W+(f+\varphi)\,t}{m+\mu}.$$

If we ſuppoſe $f+\varphi=0$, we may eaſily deduce from theſe values of u and v that of $m\,u^2+\mu\,v^2$; for by carrying through the calculus we have $m\,u^2+\mu\,v^2=m\,W^2+\mu\,w^2$. Hence, *in perfectly elaſtic bodies the ſums of the products of the maſſes into the ſquares of their reſpective velocities, before and after impact, are equal.*

This equality was denominated by Bernoulli *Conſervatio Vi-*

rium Vivarum, and was conſidered as a *general* law; but it is obvious from what is done here that it holds only in the caſe of perfectly elaſtic bodies. And, indeed, it is a conſequence in itſelf eaſily deducible from the third law of motion, according to the Newtonian meaſure of momentum.

342. If the bodies are not perfectly elaſtic they will not reſtore themſelves entirely to their firſt eſtate, but there will remain an impreſſion at the end of the ſhock: to find the final velocities of ſuch ſorts of bodies we muſt ſubſtitute for t, x, z, β, and ρ, their proper values. Of theſe, the laſt four may be aſcertained by experiment and the calculus; but the value of t ſeems more difficult to appreciate: nevertheleſs, this value is not incalculable, as will ſoon be ſeen. But it may be obſerved, beforehand, that whenever f and ϕ are not extremely great, or are not modifications of gravity, the value of t being very ſmall indeed, the quantity $(f+\phi)\,t$ may be neglected without ſenſible error. This quantity was made to enter the preceding formulæ in order to give them all poſſible ſtrictneſs, and that none of the phyſical circumſtances of the motion might be neglected. Now, dropping $(f+\phi)\,t$ from the expreſſion for the final velocity of the body, we have, for non-elaſtic bodies $u=\frac{m\text{W}+\mu w}{m+\mu}$; and for bodies perfectly elaſtic $u=\frac{(m-\mu)\text{W}+2\mu w}{m+\mu}$, and $v=\frac{(\mu-m)w+2m\text{W}}{m+\mu}$. Theſe are the values of the final velocities which are commonly given by writers on mechanics, and from which the uſual theorems readily flow.

343. The general theorem for non-elaſtic bodies may be thus expreſſed:

$$u=\frac{m\text{W}\pm\mu w}{m+\mu}$$

In this we conſider $w=0$ for the caſe where the body ſtruck is at reſt: the ſuperior ſign muſt be taken for the caſe in which the bodies both move the ſame way, and the inferior ſign for the caſe in which they meet each other.

Hence we deduce theſe concluſions:

I. If we ſuppoſe μ to be at reſt, the equation becomes $u=\frac{m\text{W}}{m+\mu}$, in which it is obvious u will decreaſe as μ increaſes; ſo that when $\mu=\infty$, infinity, we have $u=0$.

II. If bodies of equal maſſes are moving in oppoſite directions, our equation becomes $u=\frac{\text{W}-w}{2}$: thus *the two bodies* *move on together in the direction of that which is moving moſt ſwiftly, with a velocity equal to half the difference of the primitive velocities.*

III. If the velocities of the two bodies are equal, we have $u = W \cdot \frac{m \pm \mu}{m + \mu}$. In the caſe where both bodies move in the ſame direction, the ſuperior ſign being taken, we have $u = W$; as is evident enough, ſince there can be no ſhock.

IV. The centre of gravity of a ſyſtem of two globules, or phyſical points, which ſtrike each other, has an intereſting property of great uſe in various mechanical diſquiſitions. The diſtance X of this centre from any one of the points of the line deſcribed is $X = \frac{m d \pm \mu \delta}{m + \mu}$ (art. 108.), d and δ denoting the reſpective diſtances of the moveables from that point. Now when theſe bodies are animated by the velocities W, w, we have for theſe velocities (art. 232.) the correſponding values $\frac{\dot{d}}{\dot{t}}$ and $\frac{\dot{\delta}}{\dot{t}}$; conſequently that of the centre of gravity will be

$$\frac{\dot{X}}{\dot{t}} = \frac{m \dot{d} \div \dot{t} \pm \mu \dot{\delta} \div \dot{t}}{m + \mu} = \frac{m W \pm \mu w}{m + \mu}$$

This expreſſion being conſtant, juſtifies the concluſion that the centre of gravity of a ſyſtem of two bodies ſtriking each other has, both before and after the ſhock, a uniform motion. This anſwers to what the foreign mathematicians call *the conſervation of the centre of gravity.* And it is certainly of importance to know that, *whether bodies mutually attract each other,* (295. cor. 2.), *or whether they impinge upon each other, their common centre of gravity will, if in motion, continue to move on uniformly in a right line.*

344. The general equations for perfectly elaſtic bodies may be ſtated thus:

$$u = \frac{(m - \mu) W \pm 2 \mu w}{m + \mu} \quad . \quad . \quad . \quad . \quad v = \frac{\mp (m - \mu) w + 2 m W}{m + \mu}$$

The moſt important conſequences they furniſh are theſe.

I. *If one of the two bodies is at reſt,* then making $w = 0$ we find,

$$u = W \cdot \frac{m - \mu}{m + \mu} \quad . \quad . \quad . \quad . \quad v = \frac{2 m W}{m + \mu}$$

1. *If the bodies are equal, that which impinges remains at reſt, and the body ſtruck moves with the velocity which the other had.* For $m = \mu$ gives $u = 0$, and $v = W$.

COR. Hence it follows, that if ſeveral perfectly elaſtic bodies A, B, C, D, &c. of equal maſſes, are placed in juxta-poſition and at reſt; and if a body M of equal maſs is made to ſtrike A, all the bodies remain at reſt except the laſt, which goes off with the velocity of M: for the body A transfers all its velocity to B, B its to C, and ſo on.

2. *If the bodies decreaſe in magnitude, each will go forward after the ſtroke.* For if $m > \mu$, u and v are poſitive.

3. *If they increaſe in magnitude each will be reflected back, except the laſt, and the quantity of motion will continually increaſe.* For $m < \mu$ renders u negative, and the laſt v poſitive.

4. Here too we might ſhew that the increaſe of motion will be a maximum when the bodies increaſe in geometrical proportion; and if the number of bodies be increaſed without limit, the ultimate ratio of the velocities of A and Z (the firſt and laſt) will be that of $\sqrt{Z}$ to $\sqrt{A}$. But ſuch theorems, though undoubtedly curious, are of too little utility to tempt us to digreſs widely from the main purpoſe of this chapter.

II. *If the bodies both move the ſame way*, then we conclude from the theorems:

1. *The body ſtruck moves with a greater velocity after the ſtroke.*

2. *If the maſſes are equal the bodies change their velocities during the ſhock, and continue to move in the ſame direction.* For $m = \mu$ gives $u = w$, and $v = W$.

III. *If the bodies move in contrary directions* the inferior ſigns muſt be taken, and then

1. *If the maſſes and velocities are equal the bodies go back with the ſame velocities that each had before the ſhock.* For $m = \mu$, and $W = w$, give $u = -W$, and $v = w$.

2. If the *velocities are equal* we have $u = \frac{m - 3\mu}{m + \mu} W$, and $v = \frac{3m - \mu}{m + \mu} W$. Whence we conclude *that* μ *will ſtop a body of a triple maſs, but will recoil with a double velocity:* For if $m = 3\mu$, $u = 0$, and $v = 2W$. But if $\mu > 3m$, the maſs m does not ſtop, but recoils.

3. *If the maſſes are equal the bodies will both recoil, after having exchanged velocities.* For then $u = -w$, and $v = W$.

4. *The impinging body is ſtopped, continues its route, or recoils, according as* $W(m - \mu)$ *is equal, greater, or leſs, than* $2\mu w$. And the ſame thing holds with regard to the body ſtruck.

Many other theorems might readily be deduced from the equations at the head of this article; but it is time to return to the general inveſtigation which we have brought down to art. 342.

345. When, in the equation given in art. 340. namely,

$$\tfrac{1}{2} m \mu (\overline{u-v}^2 - \overline{W-w}^2 = (\mu f - m \varphi) . (x + z) - (m + \mu) \int \beta \varrho \dot{z}$$

we ſuppoſe the hardneſs ϱ invariable, we thence find

$$\int \beta \dot{z} = \frac{\frac{1}{2} m \mu \left(\overline{W - w}^2 - \overline{u - v}^2\right) + (\mu f - m \varphi) . (x + z)}{\varrho (m + \mu)}.$$

But $\beta \dot{z}$ being the element of the impreſſion, $\int \beta \dot{z}$ will be the total impreſſion of which this equation will furniſh the value for any inſtant whatever of the ſhock, the hardneſs being conſtant.

In this equation making $u = v$, we have

$$\int \beta \dot{z} = \frac{\frac{1}{2} m \mu (W - w)^2 + (\mu f - m \varphi) . (x + z)}{\varrho (m + \mu}.$$

And this value of $\int \beta \dot{z}$ anſwers to the inſtant of the greateſt impreſſion.

When the *body ſtruck is immoveable*, we have only to ſuppoſe $\mu = \infty$, $w = 0$, and $\varphi = 0$, and the preceding equation will become $\int \beta \dot{z} = \frac{\frac{1}{2} m W^2 + f (x + z)}{\varrho}$.

346. Since f is the quantity of motion which the accelerating force will produce in the body m in the unit of time, if we ſuppoſe that f is gravity, and the ſecond is the unit of time, the Engliſh foot being that of ſpace, we ſhall have $f = 32\frac{1}{6} m$, and the latter equation will become

$$\int \beta \dot{z} = \frac{\frac{1}{2} m W^2 + 32\frac{1}{6} m (x + z)}{\varrho} = \frac{m}{\varrho} \left(\frac{1}{2} W^2 + 32\frac{1}{6} \overline{x + z}\right)$$

If h be the height due to the velocity W, we have $W^2 = 62\frac{1}{3} h$, and

$$\int \beta \dot{z} = 32\frac{1}{6} \frac{m}{\varrho} (h + x + z).$$

When $x + z$ is very ſmall with reſpect to W^2 or to h, which commonly happens, even in bodies which are not very hard, the value of $\int \beta \dot{z}$ will be ſenſibly proportional to W^2 or to h; ſo that *the impreſſions will be nearly as the ſquares of the velocities, or as the heights due to thoſe velocities.* This will be ſtill nearer the truth when $x = 0$ and $\int \beta \dot{z} = \frac{m}{\varrho} \left(\frac{1}{2} W^2 + 32\frac{1}{6} z\right) = 32\frac{1}{6} \frac{m}{\varrho} (h + z)$: whence we conclude, that *when a hard body falls on another body which is immoveable the impreſſion made in the latter is inverſely as its hardneſs or reſiſting force, and directly as the height due to the velocity and the maſs of the impinging body conjointly.*

This concluſion correſponds with thoſe deduced from very different proceſſes, at pa. 180 of Dr. Hutton's *Select Exerciſes*, and pa. 52 of Mr. Atwood's *Treatiſe on the Motion of Bodies.*

From the equation $\int\beta\dot{z} = \frac{m}{\varrho}(\frac{1}{2}W^2 + 32\frac{1}{6}z)$ it appears that however ſmall the initial velocity W may be, provided it be a finite quantity, it will always make an impreſſion, unleſs ϱ is infinite. It will not be the ſame if $W = 0$: for from thence will reſult $\int\beta\dot{z} = \frac{32\frac{1}{6}mz}{\varrho}$ or $\frac{\int\beta\dot{z}}{z} = 32\frac{1}{6}\frac{m}{\varrho} = \frac{f}{\varrho}$; when $\int\beta\dot{z} = 0$, the depth of the impreſſion will be alſo nothing, and the equation will become $\frac{0}{0} = 32\frac{1}{6}\frac{m}{\varrho} = \frac{f}{\varrho}$: now $\frac{0}{0}$ being equal to what we pleaſe, the impreſſion may conſequently be $= 0$, while m, f, and ϱ, are any quantities whatever.

In fact, we may ſee, *à priori*, that f and ϱ being quantities of the ſame nature, the effect of each being a preſſure, may deſtroy each other at the firſt inſtant; and that it is not the ſame with regard to W, which, occaſioning a ſhock, can only be deſtroyed after it has produced an impreſſion.

The preceding formulæ accord very accurately with experiments; as will be ſufficiently obvious to thoſe who will compare them with the experiments deſcribed in s'Graveſande's *Phyſices Elementa Mathematica*, § 833, &c., and Mr. Atwood's treatiſe juſt referred to, *paſſim*.

If the two ſtriking maſſes are equal, and during the ſhock are not ſubjected to the action of any power (but their mutual reſiſtances only) we have $m = \mu$, $f = 0$, $\varphi = 0$, and $\int\beta\dot{z} = \frac{m}{4\varrho}(W - w)^2$; and when $w = 0$, $\int\beta\dot{z} = \frac{mw^2}{4\varrho}$. Which again is confirmed by many experiments.

Now ſince experiments correſpond with theſe formulæ, not only in the caſe of ſoft bodies, as clay, but even in thoſe of bodies imperfectly hard and elaſtic, when the hardneſs ϱ is ſuppoſed conſtant; it is evident that in theſe caſes the hardneſs is very nearly conſtant, and that we may ſafely ſuppoſe it ſuch, as we ſhall henceforth do.

347. The hypotheſis of invariable hardneſs transforms the equation in art. 340. repeated at the beginning of art. 345. to this:

$$\frac{1}{2}m\mu(\overline{u-v}^2 - \overline{W-w}^2) = (\mu f - m\varphi).(x+z) - \varrho(m+\mu)\int\beta\dot{z};$$

whence we get $\varrho = \frac{\frac{1}{2}m\mu(\overline{W-w}^2 - \overline{u-v}^2) + (\mu f - m\varphi).(x+z)}{(m+\mu)\int\beta\dot{z}}$;

and, in the caſe of the greateſt impreſſion,

$$\rho = \frac{\frac{1}{2} m \mu (W - w)^2 + (\mu f - m \varphi) . (x + z)}{(m + \mu) \int \beta \dot{z}}.$$

Though the firſt of theſe equations contains variables which do not enter the ſecond, they will, notwithſtanding, give always the ſame reſult, provided there be ſubſtituted the proper values of u, v, x, z, and $\int \beta \dot{z}$, at any inſtant of the ſhock. This naturally reſults from the hypotheſis of invariable hardneſs, which we ſee ſo much reaſon to adopt.

348. If *the body ſtruck be immoveable*, then $\mu = \infty$, and $w = 0$, and the preceding value of ρ will become $\rho = \frac{\frac{1}{2} m W^2 + f(x+z)}{\int \beta \dot{z}}$. Now if h be as before the height due to the velocity W, we have $W^2 = 64\frac{1}{3} h = 2 g h$ (art. 243.) g being $= 32\frac{1}{6}$. Subſtituting this value of W^2 for it, we have $\rho = \frac{g m h + f(x+z)}{\int \beta \dot{z}}$; and if the body falls vertically, then is $f = g m$; whence ariſes $\rho = \frac{g m (h + x + z)}{\int \beta \dot{z}} = \frac{f(h + x + z)}{\int \beta \dot{z}}$: thus *the hardneſs or reſiſting force* ρ *will be to gravity* f, as $h + x + z$ to $\int \beta \dot{z}$, or as 1 to $\frac{\int \beta z}{h + x + z}$. Here, however, we muſt not loſe ſight of the circumſtance that it is not the relation of theſe powers conſidered in their nature, but ſolely in their effects, that may be aſcertained; and that f is the product of a maſs and velocity.

After we have found the value of ρ, we may readily find that of r, from the equation $b r \dot{x} = \beta \rho \dot{z}$ (art. 335.): for this gives $r \int b \dot{x} = \rho \int \beta \dot{z}$, and conſequently $r = \rho \frac{\int \beta \dot{z}}{\int \beta \dot{x}} = \rho \int \frac{\beta \dot{z}}{\beta \dot{x}}$.

Formulæ ſuch as theſe may be rendered very uſeful in determining the relative hardneſs or reſiſting force of different bodies. A table of the relative hardneſs of materials united with a table of their relative denſities (or ſpecific gravities) would be of great utility; and particularly in the conſtruction of arches, ſluices, &c.

We have all along taken m as the *capacity* of a ſphere, becauſe in bodies of the ſame matter the capacities vary as the weights: in bodies of different matter in order to obtain a conſtant value of ρ, we muſt make $m \propto$ cube of diameter × denſity.

349. At the inſtant of the greateſt impreſſion the fluxion of

P or of its equal $\frac{\beta \varrho b r}{\beta \varrho + b r}$ (art. 334.) is equal to nothing; that is, $\frac{r \varrho (b \dot{\beta} + \beta \dot{b}) . (\beta \varrho + b r) - b r \beta \varrho (\varrho \dot{\beta} + r \dot{b})}{(\beta \varrho + b r)^2} = 0$. Performing the multiplications at length, ſuppreſſing the quantities which deſtroy each other, and dividing by $\frac{r \varrho}{(\beta \varrho + b r)^2}$, we have $\varrho \beta^2 \dot{b} + r b^2 \dot{\beta} = 0$: now this quantity cannot $= 0$ except the fluxions $\dot{b}$ and $\dot{\beta}$ are likewiſe $= 0$, that is, unleſs the amplitudes b and β, and conſequently the impreſſions, have received all the augmentation of which they are ſuſceptible. Therefore, *the greateſt force of percuſſion is that which obtains at the moment when the impreſſion is a maximum.*

350. The force of percuſſion in any inſtant of the ſhock is $P = \frac{\beta . \varrho b . r}{\beta \varrho + b r}$. Making then $\int \beta \dot{x} = I$, and $\int \beta \dot{z} = i$, which will make r (art. 349.) $= \varrho \frac{I}{i}$; and introducing this value of r for it in the value of P, it becomes $P = \frac{\varrho \varrho b \beta i}{I (\beta \varrho + b \varrho i \div I)} = \frac{\varrho b \beta i}{\beta I + b i}$. Laſtly, ſubſtituting for ϱ, in this equation, its final value in art. 347. we thence obtain

$$P = \frac{b \beta}{\beta I + b i} . \frac{\frac{1}{2} m \mu (W - w)^2 + (\mu f - m \varphi) . (x + z)}{m + \mu}.$$

Here it muſt be remarked, that although we have uſed that value of ϱ which ſuits the greateſt impreſſion, yet as ϱ is conſtant, the value of P would refer not leſs to any other inſtant of the ſhock. As x and z are introduced by the ſubſtitution of ϱ, they are conſtant, and expreſs the greateſt depths of impreſſion: thus there are no other variable quantities in the value of P than β and b.

351. When the body ſtruck is immoveable we have $\mu = \infty$, $w = 0$, and $\varphi = 0$. And if we ſuppoſe moreover that the body ſtruck is ſufficiently hard to receive no impreſſion, then will $I = 0$, and $x = 0$; whence we ſhall find in this caſe

$$P = \frac{\beta}{i} (\tfrac{1}{2} m W^2 + f z).$$

352. Here alſo, taking h the height due to the velocity W, or $W^2 = 2 g h$, and f for the gravity, or $f = g m$; then will the value of P become $P = \frac{\beta}{i} g m (h + z) = \frac{g \beta m}{i} (h + z)$: whence

flows this proportion, $P : f :: \frac{g \beta m}{i} (h+z) : g m :: \frac{\beta}{i} (h+z) : 1$.

And this is the relation of the percuſſive to the gravitating force.

353. Retaining always the immobility of the body ſtruck, if the impinging body is ſuſceptible of impreſſion, the equation which furniſhes the value of the force of percuſſion will be

$$P = g m \frac{b \beta}{\beta I + b i} (h+x+z)$$

whence it appears that $f : P :: 1 : \frac{b \beta}{\beta I + b i} (h+x+z)$.

If when the body ſtruck is immoveable we have very nearly $b = \beta$, $I = i$, and $x = z$, then will $P = \frac{\beta}{2 i} (\frac{1}{2} m W^2 + 2 f z)$; and when f is gravity, $P = \frac{g \beta m}{2 i} (h+2 z)$: ſo that, in this caſe, the percuſſion will be to gravity as $\frac{\beta}{2 i} (h+2 z)$ to unity.

354. When two very hard bodies, as two of iron, for example, are made to ſtrike one another, the impreſſion i which is made in each is nearly infinitely ſmall with reſpect to $\frac{1}{2} \beta (h+2 z)$; therefore, in this caſe, the force of percuſſion is almoſt infinite with reſpect to gravity. Take, for example, *the ſtroke of a hammer on an anvil:* ſince i repreſents the magnitude of the impreſſion, which muſt vary as the product of β into a quantity proportional to its depth, which we know by experience to be extremely ſmall; we may ſuppoſe $i = \beta \frac{1}{\omega}$, ω expreſſing any number whatever, ſuch that $\frac{1}{\omega}$ is leſs than the depth of the impreſſion, ſuppoſed leſs extended at the bottom than at the ſurface. This depth when it is at the greateſt cannot fairly exceed the $\frac{1}{15000}$ or $\frac{1}{12000}$ of a foot. Subſtituting, therefore, for i, its value $\frac{\beta}{\omega}$, the force of gravity will be to that of percuſſion as $1 : \beta (h+2 z) \div \frac{2 \beta}{\omega}$, or as $1 : \frac{\omega}{2} (h+2 z)$, or, finally (neglecting $2 z$ becauſe of its extreme minuteneſs), as $1 : \frac{1}{2} \omega h$. Now if the velocity of the hammer be equivalent to that which would be acquired by falling freely 10 feet, we ſhall have $h = 10$, and making $\omega = 12000$ only, there will reſult $\frac{1}{2} \omega h = 60000$; that is, the weight of the hammer will be to the force of percuſſion as 1 to 60000. Thus the effect of the hammer is at leaſt equivalent to what would be cauſed if at the ſame point where the blow was directed *a weight were laid ſixty thouſand*

times as great as that of the hammer. This will ſuffice to ſhow the prodigious effects of the force of percuſſion, even with ſo moderate a velocity as that due to a free deſcent through 10 feet.

355. *The duration of the ſhock* is an intereſting particular, to the determination of which we may now proceed: Now we found (art. 337.) $\dot{t} = \frac{\dot{x}+\dot{z}}{u-v}$, and (art. 340.) the equation $u - v = \pm\left(\overline{\mathrm{W}-w}^{2} + \frac{2(\mu f - m\varphi)\,.\,(x+z)}{m\mu} - \frac{2(m+\mu)\,\varrho\int\beta\dot{z}}{m\mu}\right)^{\frac{1}{2}}$. And if we put for $u-v$ in the firſt equation its value in the ſecond, at the ſame time making $\frac{2\varrho(m+\mu)}{m\mu} = \varepsilon$, $(\mathrm{W}-w)^2 = \gamma$, and $\frac{2(\mu f - m\varphi)}{m\mu} = \delta$, we ſhall have $\dot{t} = \frac{\dot{x}+\dot{z}}{\pm(\gamma + \delta\,.\,\overline{x+z} - \varepsilon\int\beta\dot{z})^{\frac{1}{2}}}$; an equation the integration of which will depend upon the value of β, and the relation of x and z.

We may at firſt ſuppoſe $x=0$, and $\dot{x}=0$; then will the equation become $\dot{t} = \frac{\dot{z}}{\pm(\gamma+\delta z-\varepsilon\int\beta\dot{z})^{\frac{1}{2}}}$. We may alſo imagine $x = z$, and then will $\dot{t} = \frac{2\dot{z}}{\pm(\gamma+2\delta z-\varepsilon\int\beta\dot{z})^{\frac{1}{2}}}$. Now it is manifeſt that by finding the fluents in this ſecond hypotheſis ſolely we may readily infer the value of t in the firſt; it will merely be neceſſary to take half the term which contains δ, and after that to take half the value of t.

The value of β depends on the figure, the diſpoſition, and the reciprocal hardneſs of the ſtriking bodies. So that we may conceive $\int\beta\dot{z}$ equal to any function of z with conſtant quantities; for although it is not poſſible that this ſuppoſition can ſuit all bodies, yet we may always determine to what bodies it may be accommodated.

Suppoſing then $\int\beta\dot{z}^2 = \mathrm{C}z^2$, C being a conſtant quantity, we have $\dot{t} = \frac{2\dot{z}}{\pm(\gamma+2\delta z-\varepsilon\mathrm{C}z^2)^{\frac{1}{2}}} = \frac{2\dot{z}\div\sqrt{\varepsilon\mathrm{C}}}{\pm\left(\frac{\gamma}{\varepsilon\mathrm{C}}+\frac{\delta^2}{\varepsilon^2\mathrm{C}^2}-\frac{\delta^2}{\varepsilon^2\mathrm{C}^2}+\frac{2\delta z}{\varepsilon\mathrm{C}}-z^2\right)^{\frac{1}{2}}}$: in which we have added $\frac{\delta 2}{\varepsilon^2\mathrm{C}^2}-\frac{\delta^2}{\varepsilon^2\mathrm{C}^2}=0$, to the denominator, in order that its three laſt terms with their ſigns changed ſhould form a complete ſquare. Making $\frac{\gamma}{\varepsilon\mathrm{C}}+\frac{\delta^2}{\varepsilon^2\mathrm{C}^2} = \mathrm{R}^2$, we obtain after a little reduction $\dot{t} = \frac{2}{R\sqrt{\varepsilon C}} \cdot \frac{R\dot{z}}{\pm\left[R^2-(z-\frac{\delta}{\varepsilon C})^2\right]^{\frac{1}{2}}} = \frac{2}{\sqrt{\varepsilon C}} \cdot \frac{\dot{z} \div R}{\pm\left[1-(\frac{z}{R}-\frac{\delta}{R\varepsilon C})^2\right]^{\frac{1}{2}}}$.

Now it is known that if x be an arc of a circle whoſe radius is unity, and s its ſine, we have univerſally $\dot{x} = \frac{\dot{s}}{\sqrt{1-s^2}}$; and that the fluxion of the ſupplement of x is equal to $\frac{\dot{s}}{-\sqrt{1-s^2}}$. Hence, the laſt value of $\dot{t}$ expreſſes the product of $\frac{2}{\sqrt{\varepsilon C}}$ into the fluxion of an arc of a circle whoſe radius $= 1$, and ſine $= \frac{z}{R}-\frac{\delta}{R\varepsilon C}$, or into the fluxion of the ſupplement of that arc, according as the denominator of the ſecond member has the poſitive or the negative ſign. Thus then, the fluents of this equation will be $t = \frac{2}{\sqrt{\varepsilon C}}$ arc ſin. $(\frac{z}{R}-\frac{\delta}{R\varepsilon C}) + M$.

To determine the conſtant quantity M it muſt be remarked that when $t=0$ the depth of impreſſion $z=0$ alſo; in this caſe, therefore, obſerving that negative ſines anſwer to negative arcs, we have $\frac{-2}{\sqrt{\varepsilon C}}$ arc ſin. $\frac{\delta}{R\varepsilon C}+M=0$, or $M=\frac{2}{\sqrt{\varepsilon C}}$ arc ſin. $\frac{\delta}{R\varepsilon C}$.

Thus the value of t is

$$t=\frac{1}{\frac{1}{2}\sqrt{\varepsilon C}} \cdot \left[\text{arc ſin. } (\frac{z}{R}-\frac{\delta}{R\varepsilon C}) + \text{arc ſin.} \frac{\delta}{R\varepsilon C}\right].$$

Hence, reſtoring the values of γ, δ, and ε, we have

$$t = \sqrt{\frac{2m\mu}{C\varrho(m+\mu)}} \cdot \left[\text{arc ſin. } (\frac{z}{R}-\frac{\mu f-m\varphi}{RC\varrho(m+\mu)}) + \text{arc ſin. } \frac{\mu f-m\varphi}{RC\varrho(m+\mu)}\right].$$

This is the value of t when the poſitive ſign is taken to the denominator of the ſecond member of the preceding fluxional equation; and it then expreſſes any time elapſed from the commencement of the ſhock to any inſtant previous to the greateſt impreſſion. To have the time which elapſes from the commencement of the ſhock up to the inſtant *after* the greateſt impreſſion, we muſt take, inſtead of the expreſſion arc ſin. $(\frac{z}{R}-\frac{\mu f-m\varphi}{RC\varrho(m+\mu)})$, its ſupplement, that is, π − arc. ſin. $(\frac{z}{R}-$

$\frac{\mu f - \mu \varphi}{R\,C\,\varrho\,(m+\mu)}$); π repreſenting, as heretofore, 3·141593 the ſemicircumference to radius 1. Hence, for the latter caſe, we have

$$t = \sqrt{\frac{2\,m\,\mu}{C\,\varrho\,(m+\mu)}} \cdot \Big[\pi - \text{arc ſin.}\,\Big(\frac{z}{R} - \frac{\mu f - m\varphi}{R\,C\,\varrho\,(m+\mu)}\Big) + \text{arc. ſin.}\,\frac{\mu f - m\varphi}{R\,C\,\varrho\,(m+\mu)}.$$

Theſe two expreſſions for the time ought to be equal for the inſtant of the greateſt impreſſion: we have, therefore, for that inſtant . . . . $\frac{1}{2}\pi = \text{arc ſin.}\,\Big(\frac{z}{R} - \frac{\mu f - m\varphi}{R\,C\,\varrho\,(m+\mu)}\Big)$.

Subſtituting this value in either of the preceding equations for t, we ſhall have for the *time from the commencement of the ſhock to the inſtant of greateſt impreſſion*

$$t = \sqrt{\frac{2\,m\,\mu}{C\,\varrho\,(m+\mu)}} \cdot \Big(\frac{1}{2}\pi + \text{arc. ſin.}\,\frac{\mu f - m\varphi}{R\,C\,\varrho\,(m+\mu)}\Big).$$

In *bodies perfectly elaſtic* the impreſſion diminiſhes till $z = 0$; ſubſtituting, therefore, this value of z in the ſecond equation for the time, we have for the *whole duration of the ſhock* in ſuch bodies

$$t = \sqrt{\frac{2\,m\,\mu}{C\,\varrho\,(m+\mu)}} \quad \Big(\pi + 2\ \text{arc ſin.}\,\frac{\mu f - m\varphi}{R\,C\,\varrho\,(m+\mu)}\Big).$$

Whence it appears that the whole time is *double* that which elapſes from the commencement of the ſhock to the inſtant of greateſt impreſſion: thus ſhewing the exactneſs of the laſt condition in the definition of perfect elaſticity (art. 329.)

In bodies *void of elaſticity*, the total duration of the ſhock is equal to the time which elapſes between the commencement of the ſhock and the inſtant of greateſt impreſſion.

If the bodies are not ſubjected to the operation of any active power, we have $f = 0$, and $\varphi = 0$: whence the *whole duration* of the ſhock for *non-elaſtic bodies* will be $t = \frac{1}{2}\pi \sqrt{\frac{2\,m\,\mu}{C\,\varrho\,(m+\mu)}}$; and for *elaſtic bodies* $t = \pi \sqrt{\frac{2\,m\,\mu}{C\,\varrho\,(m+\mu)}}$.

The quantity R which contains the initial velocities, not entering either of theſe expreſſions, we ſee that, when no active power animates the bodies, the time of *duration of the ſhock is always the ſame* (cæt. par.), *whatever are the initial velocities.*

The value of R is, as we have ſeen,

$$R = \sqrt{\frac{\gamma}{\varepsilon\,C} + \frac{\delta^2}{\varepsilon^2\,C^2}} = \Big(\frac{\frac{1}{2}\,m\,\mu\,(W - w)^2}{C\,\varrho\,(m+\mu)} + \Big(\frac{\mu f - m\varphi}{C\,\varrho\,(m+\mu)}\Big)^2\Big)^{\frac{1}{2}}$$

Now if W be $= 0$, and $w = 0$, that is, if the bodies are merely actuated by powers, the value of R will become $R = \frac{\mu f - m\varphi}{C\varrho(m+\mu)}$. Subſtituting this value in the expreſſions for the whole time, they will become,

For non-elastic bodies $t = \sqrt{\frac{2m\mu}{C\varrho(m+\mu)}}\left(\frac{1}{2}\pi + \text{arc sin. } (1)\right) = \pi\sqrt{\frac{2m\mu}{C\varrho(m+\mu)}}$:

For elastic bodies $t = 2\pi\sqrt{\frac{2m\mu}{C\varrho(m+\mu)}}$.

Here it will be obſerved that ſin. (1) = radius, and arc. ſin. (1) = 90°. Hence it follows, 1ſt, That the duration of the ſhocks when the bodies are only actuated by powers, or accelerating forces, without initial velocities, is double the duration of the ſhock when they only move with certain velocities. 2dly, That the value of theſe powers or accelerating forces, not entering the equations above, the duration of the ſhock is the ſame, whatever are the powers or accelerating forces which act upon the bodies. It muſt be recollected though, that the accelerating forces are ſuppoſed conſtant.

356. The quantities C and ϱ which enter the preceding equations being embarraſſing in the computation, it will not be improper to find values of t which do not contain thoſe quantities, and by means of which we may deduce the value of t immediately from experiments.

Let X be the greateſt depth of impreſſion; then, becauſe $\int \beta \dot{z} = C z^2$, we have $CX^2 = i$; and from the hypotheſis of $x = z$ we have $\varrho = \frac{\frac{1}{2}m\mu(W+w)^2 + (\mu f - m\varphi)\,2X}{(m+\mu)\,CX^2}$; whence we find

$(m+\mu)\,\varrho\,C = \frac{\frac{1}{2}m\mu(W-w)^2 + (\mu f - m\varphi)\,2X}{X\,2}$. We have alſo

$$R^2 = \frac{\frac{1}{2}m\mu(W-w)^2}{C\varrho(m+\mu)} + \left(\frac{\mu f - m\varphi}{C\varrho(m+\mu)}\right)^2.$$

Multiplying the two terms by $\overline{C\varrho(m+\mu)}\,\vert^2$, ſubſtituting for $C\varrho(m+\mu)$ its value, reducing the whole to a fraction, and extracting the roots, we ſhall have at length

$$RC\varrho(m+\mu) = \frac{\frac{1}{2}m\mu(W-w)^2 + X(\mu f - m\varphi)}{X}.$$

Subſtituting theſe values of $RC\varrho$, and of $C\varrho$, in the preceding values of t, we have, for *non-elaſtic bodies*,

$$t = \sqrt{\frac{2m\mu X^2}{\frac{1}{2}m\mu(W-w)^2 + 2X(\mu f - m\varphi)}}\cdot\left(\frac{1}{2}\pi + \text{arc ſin.}\ \frac{X(\mu f - m\varphi}{\frac{1}{2}m\mu(W-w^2 + X(\mu f - m\varphi)}\right).$$

And for *perfectly elaſtic bodies,*

$$t = \sqrt{\frac{2\,m\,\mu\,X^2}{\frac{1}{2}m\,\mu\,(W - w)^2 + 2\,X\,(\mu f - m\,\varphi)}} \cdot \left(\pi + 2\ \text{arc ſin.}\ \frac{X\,(\mu f - m\,\varphi)}{\frac{1}{2}m\,\mu\,(W - w)^2 + X\,(\mu f - m\,\varphi)}\right).$$

In theſe equations neither C, ϱ, nor R, is to be found; ſo that they compriſe merely ſuch quantities as reſult at once from experience.

When $f = 0$, and $\varphi = 0$, the preceding equations become, for *non-elaſtic bodies* $t = \frac{\pi X}{W - w}$; for *elaſtic bodies* $t = \frac{2\,\pi X}{W - w}$.

It muſt be obſerved that in the equation $t = \frac{\pi X}{W - w}$ the difference $W - w$ may remain the ſame, though the abſolute values of W and w vary; ſo that this equation by no means contradicts the obſervation made in the preceding article, that, when the accelerating forces vaniſh, the times are independent of the initial velocities.

If the bodies are only animated by powers, or only act by mutual preſſure, then are $W = 0$ and $w = 0$; and the values of t become $t = \pi\sqrt{\frac{m\,\mu\,X}{\mu f - m\,\varphi}}$, and $t = 2\,\pi\sqrt{\frac{m\,\mu\,X}{\mu f - m\,\varphi}}$: if the body ſtruck is immoveable, theſe equations become $t = \pi\sqrt{\frac{m\,X}{f}}$, and $t = 2\,\pi\sqrt{\frac{m\,X}{f}}$. If we ſuppoſe f to be gravity, then $f = g\,m$, and $t = \pi\sqrt{\frac{X}{g}}$, or $t = 2\,\pi\sqrt{\frac{X}{g}}$, in the reſpective caſes of non-elaſtic and elaſtic bodies.

Retaining the hypotheſis of the immobility of the body ſtruck, we have $\mu = \infty$, and of conſequence $w = 0$; then will the expreſſion for the time of greateſt impreſſion become

$$t = \sqrt{\frac{2\,m\,X^2}{\frac{1}{2}m\,W^2 + 2\,f\,X}} \cdot \left(\tfrac{1}{2}\,\pi + \text{arc ſin.}\ \frac{f\,X}{\frac{1}{2}m\,W^2 + f\,X}\right).$$

If the ſtriking body is moved by gravity, h being the height due to the velocity W, we have $W^2 = 2\,g\,h$, and $f = g\,m$. Subſtituting theſe values of W^2 and f for them, there will reſult

$$t = \frac{X}{\sqrt{\frac{1}{2}g\,(h + 2X)}} \cdot \left(\tfrac{1}{2}\,\pi + \text{arc ſin.}\ \frac{X}{h + X}\right).$$

When X is extremely ſmall with reſpect to h, this equation will become $t = \frac{\pi X}{\frac{1}{4}g\sqrt{h}}$. This theorem would apply to the caſe of a hammer ſtriking on an anvil.

357. In a ſimilar manner we may find the time in which the greateſt impreſſions are formed, on the ſuppoſition that $x=0$, and $\dot{x}=0$: for, we have ſeen (art. 355.) that in order to this we muſt take the half of the term multiplied by $\delta = \frac{\mu f - m\varphi}{\frac{1}{2} m\mu}$, and then to take half the value of t; thus we ſhall have

$$t = \sqrt{\frac{m\mu X^2}{m\mu(W-w)^2 + 2(\mu f - m\varphi)X}} \cdot \left(\tfrac{1}{2}\pi + \text{arc fin.} \frac{(\mu f - m\varphi)X}{m\mu(W-w)^2 + (\mu f - m\varphi)X}\right).$$

The method will be the ſame for every other caſe, by finding the fluents of the general equation,

$$\dot{t} = \frac{\dot{x} + \dot{z}}{\pm\left[(W-w)^2 + \frac{(\mu f - m\varphi)x + z) - (m+\mu)\varrho\int\beta\dot{z}}{\frac{1}{2}m\mu}\right]^{\frac{1}{2}}}.$$

We have found (art. 340.) the equation

$$m\mu\,(\dot{u} - \dot{v}) = (\mu f - m\varphi - \overline{m+\mu}\,.\,P)\,\dot{t};$$

if in this equation we ſuppoſe P conſtant, its fluent will be $m\mu(u-v) = (\mu f - m\varphi - \overline{m+\mu}\,.\,P)\,t + F$; when $t=0$, $u=W$, and $v=w$, whence we have then $m\mu\,(W-w) = F$, and conſequently $m\mu\,(\overline{u-v} - \overline{W-w}) = (\mu f - m\varphi - \overline{m+\mu}\,.\,P)\,t$, or, by diviſion, $t = \frac{m\mu\,(\overline{u-v} - \overline{W-w})}{\mu f - m\varphi - \overline{m+\mu}\,.\,P}$. At the inſtant of the greateſt impreſſion $u-v=0$, and the equation becomes $t = \frac{m\mu\,(W-w)}{\overline{m+\mu}\,.\,P - (\mu f - m\varphi)}$.

If the body ſtruck is immoveable, $\varphi = \infty$, $w=0$, and $\varphi=0$; wherefore $t = \frac{mW}{P-f}$.

When $f=0$, and $\varphi=0$, then $t = \frac{m\mu\,(W-w)}{(m+\mu)P}$: and if $\mu = \infty$, and $w=0$, $t = \frac{mW}{P}$.

In theſe equations the value of P may be found, as in arts. 350—355.

358. The application of the preceding calculus depends principally on the determination of the impreſſion: and this determination is ſometimes attended with difficulties. For, notwithſtanding that many authors have ſuppoſed that the figure of the impreſſion is generally the ſame as that of the impinging body, it is manifeſt that this ſuppoſition will not be accurate

with regard to hard and tenacious bodies. In a great number of theſe latter the amplitude of the impreſſion is always much the largeſt; for the parts contiguous to the ſurface of contact, being not eaſily detached from the neighbouring parts, yield to the impulſion at the ſame time that this ſurface carries before it thoſe which touch, and ſo on ſucceſſively. Hence the diameter of the actual impreſſion is always greater than the diameter of the ſurface of contact; ſo that it is difficult to obtain an exact meaſure of the real impreſſion.

This remark, however, true as it is, cannot be applied to all caſes without ſome modification; becauſe there are ſome forms ofthe impinging body, ſuch as the ſpherical, and convex forms in general, which greatly diminiſh this exceſs of the diameter of the impreſſion over that of the ſurface of contact: indeed this exceſs may often be conſidered as entirely vaniſhed, when the body ſtruck is not of an extreme hardneſs and tenacity.

359. It will appear, from the peruſal of this chapter, that the theory is made to flow from facts adverted to in art. 329. and ſerves for their explication. We hence ſee that when we know the phyſical circumſtances of a percuſſion we may always aſſign an equivalent preſſure, which will put it in equilibrio. It is not, therefore, abſurd to ſay that we can weigh or balance the blow of a maſs, or the ſtroke of a hammer; it is ſolely neceſſary to obſerve that the ſtroke of a hammer or other maſs is not an abſolute weight, but that ſuch weight depends on the hardneſs, the form, and the nature of the bodies ſtriking and ſtruck. Thus, in general, when a reſiſtance of preſſure is equivalent to a very conſiderable weight, it will reſiſt the ſhock of a maſs whoſe weight is ſmall in compariſon with the former, and of which the velocity is that due to a moderate height.

360. It has been all along ſuppoſed that the direction of the ſtroke is the ſame as that of the motion: but this is far from being always the caſe; and when it is not the *impact* is called *oblique*. To conſider this in its utmoſt extent would carry us very far indeed: but we have not room for a particular inveſtigation. We ſhall, therefore, juſt mention two or three general facts already proved; and from them deduce a few eaſy caſes. Theſe facts are: 1ſt, That the actions of bodies on each other depend upon their relative motions. 2dly, That the motion of the common centre is not changed by the colliſion. By theſe we can reduce all to the caſe of a body in motion ſtriking another at reſt. Now the relative motion may be determined by the conſtruction in art. 219. and to this muſt be ſuperadded the common motion which changes the relative into the true motions.

If, for example, two bodies A, B, (fig. 2. pl. XV.) deſcribe the lines AD, BD, and meet in D, the colliſion is the ſame as if B had remained at reſt, and A had ſtruck it with the direction and velocity AB. In the mean time, the common centre of inertia has deſcribed CD with a uniform velocity, which at the end of an equal portion of time will carry it to *c*, CD being =D*c*. If the bodies are non-elaſtic they remain together, and will proceed along D*c*; their common velocity being repreſented by D*c*, while AD, BD, repreſents the individual velocities of A and B previous to the impact. If the bodies are imperfectly elaſtic, draw through *c* the line *ab* parallel to AB; make AC to *ac*, as the force of compreſſion to the force of reſtitution, with regard to the body A; and make BC to *bc* as the force of compreſſion to the reſtitutive force, relatively to B; join D*a*, D*b*, and thoſe lines will be the paths of the bodies after colliſion. If the bodies are perfectly elaſtic, make *cb*=CB, *ac* =AC; and D*a*, D*b*, will be the paths of the bodies. If AB be perpendicular to C*c*, then will angle ADC=*a*D*c*, and BDC=*b*D*c*. And in a ſimilar way it naturally follows, that if a body A perfectly elaſtic impinges upon a perfectly hard plane C*c*, in the direction AD, it will rebound from D in ſuch a line D*a*, that *the angles of incidence and reflection* ADC, *a*D*c*, *ſhall be equal.*

361. On this latter principle depends the ſolution of the problems often propoſed relative to the game of billiards: the reader may take the following as a ſpecimen.

To find in what points C, D, *on the two ſides* KL, LI, *of a rectangular billiard table, an elaſtic ball placed at* B *muſt ſtrike, ſo that it may hit a ball placed at* A, *after a double reflexion.*

From the point A (fig. 3. pl. XV.) draw AH perpendicular to IL, and produce it till IH=IA: parallel to LI draw HF, and make it equal to 2GH, the point G being in KL produced. Join the points F, B, by a right line cutting KL in C, and join the points C, H, by a right line cutting LI in D; then ſhall C and D be the points required. For the angles ADI, CDL, are each equal to IDH, and the angles FCG, GCH, each equal to BCK; whence the truth of the conſtruction is obvious.

After methods ſlightly varied might ſeveral entertaining problems be ſolved: but ſuch great ſimplicity is not to be expected in many of the caſes of oblique colliſion which actually occur. We ſhall, therefore, conclude this chapter with a general propoſition (from Dr. Robiſon), by means of which the different motions, whether progreſſive or rotatory, may in general be determined without much trouble; and which, in fact, requires merely a combination of ſome of the principles already exhibited in the preceding and preſent chapters.

362. *Let the body* A (fig. 4. pl. XV.), *moving with the velocity* V *in the direction* AD, *ſtrike the body* B *at reſt; to determine the circumſtances reſulting from the ſhock.*

Let F be the point of mutual contact, and hFH a plane touching both bodies in F. Draw AFP perpendicular to this tangent plane, and through G, the centre of poſition of B, draw PGC perpendicular to FP, and GI parallel to FP. Let C, in the line PG, be the ſpontaneous centre of converſion correſponding to the point of percuſſion F. Join CF. Let the direction cut the tangent plane in H, and PF in A; and let AH repreſent the velocity V.

The impulſe is made at the point F, in the direction AF or FP; and the centre of inertia of the body B will advance in the direction GI, parallel to FP, the direction of the effective impulſe. But, becauſe this does not paſs through the centre G, the body will advance, and will alſo turn round an axis paſſing through G, perpendicular to the plane of the lines GP, PF, and the ſpontaneous axis of converſion will paſs through ſome point C of the line PG, and will alſo be perpendicular to the ſame plane. Complete the parallelogram AFHE. It is plain that the motion AH is equivalent to AE and AF. By the motion AE, A only ſlides along the ſurface of B without preſſing it, or cauſing any tendency to motion in that direction, except perhaps a little ariſing from friction. It is by the motion AF alone that the impulſe is made. Therefore let v be $= \mathrm{V} \times \frac{\mathrm{AF}}{\mathrm{AH}}$; and then $\mathrm{A} \times v$ may be called the *efficient impulſe* of the body A in the preſent circumſtances, and v the *efficient velocity*. This will be diminiſhed by the colliſion. Let x be the unknown velocity remaining in A after the colliſion, or rather in the inſtant of the greateſt compreſſion and common motion of the touching points of A and B, eſtimated in the direction FP. The effective momentum loſt by A muſt therefore be $\mathrm{A} \times (v-x)$: but the ſame muſt be gained by B, and its centre G muſt move in the direction GI, parallel to FP, with this momentum; and therefore with the velocity $\frac{\mathrm{A} \times (v-x)}{\mathrm{B}}$. That this may be the caſe, the point of percuſſion F muſt yield with the velocity x, becauſe the bodies are in contact. But becauſe C is the ſpontaneous axis of converſion, every particle is *beginning* to deſcribe an arch of a circle round this axis. Therefore F is beginning to move in the direction Fg, perpendicular to the momentary radius vector CF. Let Fg be a very minute arch, deſcribed in a moment of time. Draw gf perpendicular to FP. Then Ff is the motion Fg reduced to the direction FP, and will expreſs the yielding of B in the direction of the impulſe, while G deſcribes a ſpace equal to $\frac{A \times (v - x)}{B}$, and A deſcribes a ſpace x. Therefore F g will expreſs x. Let P p be the ſpace deſcribed in the ſame time that F g is deſcribed. Draw p C, cutting GK in the point I. GI is the yielding of the body B to the impulſe, and muſt therefore be equal to $\frac{A \times (v - x)}{B}$.

The triangles F f g and CPF are ſimilar; for the angle CFP is the complement of f F g to a right angle: it is alſo the complement of PCF to a right angle. Therefore F g : F f : : FC : CP. But F g : P p : : FC : CP; becauſe the little arches F g, P p, have the ſame angle at C. Therefore P p = F f, $= x$. It is plain, that CG : CP : : GI : P p. Therefore CG : CP : : $\frac{A \times (v - x)}{B} : x$, and $x = \frac{A \times (v - x) \times CP}{B \times CG}$, or $x = v \times \frac{A \times CP}{B \times CG} - x \times \frac{A \times CP}{B \times CG}$; wherefore $x \times B \times CG + x \times A \times CP = v \times A \times CP$, and $x \times \overline{B \times CG + A \times CP} = v \times A \times CP$, and $x = v \times \frac{A \times CP}{B \times CG + A \times CP}$, = the velocity remaining in A, eſtimated in the direction FP.

And u, the velocity with which G will advance, is $x \frac{CG}{CP}$; for CP : CG : : P p : GI : : x : u. It is evident that A will change its direction by the colliſion: for in the inſtant of greateſt compreſſion it was reacted on by a force $= A \cdot (v - x)$ in the direction FA. This muſt be compounded with A × V, in the direction AH, in order to obtain the new motion of A; or it may be found by compounding x, which is retained by A, with FH, which has ſuffered no change by the colliſion. *The bodies will, therefore, ſeparate, although they be unelaſtic.* If they are perfectly elaſtic we *double* theſe changes in each.

If B was alſo in motion before the colliſion, the motion of A muſt be reſolved into two, one of which is equal and parallel to the motion of B; and the other muſt be employed in the ſame manner as AH in the preceding part of this article.

CHAPTER VI.

On the Motion of Machines, and their Maximum Effects.

363. When forces acting in contrary directions, or in any ſuch directions as produce contrary effects, are applied to machines, there is, with reſpect to every ſimple machine (and of conſequence with reſpect to every combination of ſimple machines) a certain relation between the powers and the diſtances at which they act, which, if ſubſiſting in any ſuch machine when at reſt, will always keep it in a ſtate of reſt, or of *ſtatical* equilibrium (art. 28.); and for this reaſon, becauſe the efforts of theſe powers, when thus related, with regard to magnitude and diſtance, being equal and oppoſite, annihilate each other, and have no tendency to change the ſtate of the ſyſtem to which they are applied. So alſo, if the ſame machine have been put into a ſtate of *uniform* motion, whether rectilinear or rotatory, by the action of any power diſtinct from thoſe we are now conſidering, and theſe two powers be made to act upon the machine in ſuch motion in a ſimilar manner to that in which they acted upon it when at reſt, their ſimultaneous action will preſerve it in that ſtate of uniform motion, or of *dynamical* equilibrium (art. 28.); and this for the ſame reaſon as before, becauſe their contrary effects deſtroy each other, and have therefore no tendency to change the *ſtate* of the machine. But, if at the time a machine is in a ſtate of balanced reſt, any one of the oppoſite forces be increaſed while it continues to act at the ſame diſtance, this exceſs of force will diſturb the ſtatical equilibrium, and produce motion in the machine; and if the ſame exceſs of force continues to act in the ſame manner it will, like every conſtant force, produce an accelerated motion; or, if it ſhould undergo particular modifications when the machine is in different poſitions, it may occaſion ſuch variations in the motion as will render it alternately accelerated and retarded. Or the different ſpecies of reſiſtance to which a moving machine is ſubjected, as the rigidity of ropes, friction, reſiſtance of the air, &c. may ſo modify a

motion as to change a regular or irregular variable motion into one which is uniform.

364. Hence, then, the motion of machines may be conſidered as of *three* kinds. 1. That which is gradually accelerated, which obtains commonly in the firſt inſtants of the communication. 2. That which is entirely uniform. 3. That which is alternately accelerated and retarded. Pendulum clocks, and machines which are moved by a balance, are related to the third claſs. Moſt other machines, a ſhort time after their motion is commenced, fall under the ſecond. Now although the motion of a machine is alternately accelerated and retarded, it may, notwithſtanding, be meaſured by a uniform motion, becauſe of the periodical and regular repetition which may exiſt in the acceleration and retardation. Thus the motion of a ſecond pendulum, conſidered in relation to a ſingle oſcillation, is accelerated during the firſt half ſecond, and retarded during the ſecond: but the ſame motion taken for many oſcillations may be conſidered as uniform. Suppoſe, for example, that the extent of each oſcillation is five inches, and that the pendulum has made ten oſcillations: its total effect will be to have run over 50 inches in 10 ſeconds; and, as the ſpace deſcribed in each ſecond is the ſame, we may compare the effect to that produced by a moveable which moves for 10 ſeconds with a velocity of 5 inches per ſecond. We ſee, therefore, that the theory of machines whoſe motions are uniform conduces naturally to the eſtimation of the effects produced by machines whoſe motion is alternately accelerated and retarded: ſo that the problems compriſed in this chapter will be directed to thoſe machines whoſe motions fall under the firſt two heads; ſuch problems being of far the greateſt utility in practice.

365. DEFS. 1. When in a machine there is a ſyſtem of forces or of powers mutually in oppoſition, thoſe which produce or tend to produce a certain effect are called *movers* or *moving powers;* and thoſe which produce or tend to produce an effect which oppoſes thoſe of the moving powers are called *reſiſtances.* If various movers act at the ſame time, their equivalent (found by the theorems in book I. chap. i.) is called individually *the moving force;* and, in like manner, the reſultant of all the reſiſtances reduced to ſome one point *the reſiſtance.* This reduction in all caſes ſimplifies the inveſtigation.

2. The *impelled point* of a machine is that to which the action of the moving power may be conſidered as immediately applied; and the *working point* is that where the reſiſtance ariſing from the work to be performed immediately acts, or to which it ought all to be reduced. Thus in the wheel and axle (fig. 6. pl. V.) where the moving power P is to overcome the

weight or reſiſtance W, by the application of the cords to the wheel and to the axle, A is the impelled point, and B the working point.

3. The *velocity of the moving power* is the ſame as the velocity of the impelled point; the *velocity of the reſiſtance* the ſame as that of the working point.

4. The *performance* or *effect* of a machine, or the *work done*, is meaſured by the product of the reſiſtance into the velocity of the working point; the *momentum of impulſe* is meaſured by the product of the moving force into the velocity of the impelled point.

Theſe definitions being eſtabliſhed, we may now exhibit a few of the moſt uſeful problems, giving as much variety in their ſolutions as may render one or other of the methods of eaſy application to any other caſes which may occur.

366. Prop. *If* R *and* r *be the diſtances of the power* P, *and the weight or reſiſtance* W *from the fulcrum* F *of a ſtraight lever* (fig. 1. pl. IV.), *then will the velocity of the power and of the weight at the end of any time* t *be* $\frac{R^2P - RrW}{R^2P + r^2W}gt$, *and* $\frac{RrP - r^2W}{R^2P + r^2W}gt$, *reſpectively, the weight and inertia of the lever itſelf not being conſidered.*

If the effort of the power balanced that of the reſiſtance, P would be equal to $\frac{rW}{R}$. Conſequently, the difference between this value of P and its actual value, or $P - \frac{r}{R}W$ will be the force which tends to move the lever. And becauſe this power applied to the point A accelerates the maſſes P and W, the maſs to be ſubſtituted for W in the point A muſt be $\frac{r^2}{R^2}W$ (art. 310. cor. 4.) in order that this maſs at the diſtance R may be equally accelerated with the maſs W at the diſtance R. Hence the power $P - \frac{r}{R}W$ will accelerate the quantity of matter $P + \frac{r^2}{R^2}W$; and the accelerating force $F = (P - \frac{r}{R}W) \div (P + \frac{r^2}{R^2}W) = \frac{PR^2 - RrW}{PR^2 + r^2W}$. But (art. 228.) $v \propto Ft$ or is $= gtF$; which in this caſe $= \frac{R^2P - RrW}{R^2P + r^2W} \cdot gt$, the velocity of P. And, becauſe veloc. of P : veloc. of W :: R : r, we have veloc. of $W = \frac{r}{R}$ veloc. of $P = \frac{r}{R} \times \frac{R^2P - RrW}{R^2P + r^2W}gt = \frac{RrP - r^2W}{R^2P + r^2W} \cdot gt$.

COR. 1. The ſpace deſcribed by the power in the time t will be $= \frac{R^2P - RrW}{R^2P + r^2W} \cdot \frac{1}{2}gt^2$; the ſpace deſcribed by W in the ſame time will be $= \frac{RrP - r^2W}{R^2P + r^2W} \cdot \frac{1}{2}gt^2$.

COR. 2. If R : r : : n : 1, then will the force which accelerates A be $= \frac{Pn^2 - Wn}{Pn^2 + W}$.

COR. 2. If at the ſame time the inertia of the moving force P be = 0, as in muſcular action, the force accelerating A will be $= \frac{Pn^2 - Wn}{W}$.

COR. 3. If the maſs moved have no weight, but poſſeſſes inertia only, as when a body is moved along a horizontal plane, the force which accelerates A will be $= \frac{Pn^2}{Pn^2 + W}$. And either of theſe values may be readily introduced into the inveſtigation.

COR. 4. The work done in the time t, if we retain the original notation, will be $= \frac{RrP - r^2W}{R^2P + r^2W} gt \times W = \frac{RrPW - r^2W^2}{R^2P + r^2W} \cdot gt$.

COR. 5. When the work done is to be a maximum, and we wiſh to know the weight when P is given, we muſt make the fluxion of the laſt expreſſion = 0. Then we ſhall have $rR^3P^2 - 2r^2R^2PW - r^4W^2 = 0$ and $W = P \times \left(\sqrt{\frac{R^4}{r^4} + \frac{R^3}{r^3}} - \frac{R^2}{r^2}\right)$.

COR. 6. If R : r : : n : 1, the preceding expreſſion will become $W = P \times (\sqrt{n^4 + n^3} - n^2)$.

COR. 7. When the arms of the lever are equal in length, that is, when $n = 1$, then is $W = P \times (\sqrt{2} - 1) = \cdot414214\,P$, or nearly $\frac{5}{12}$ of the moving force.

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367. If we compare the values of s and v in this propoſition and firſt corollary with thoſe in the fourth example, art. 267. which relates to motion on the axis in peritrochio, it will be ſeen that the expreſſions correſpond exactly. Hence it follows, that when it is required to proportion the power and weight ſo as to obtain a maximum effect on the wheel and axle (the weight of the machinery not being conſidered), we may adopt the concluſions of cors. 5 and 6 of this prop. And in the extreme caſe where the wheel and axle becomes a pulley,

the expreſſion in cor. 7. may be adopted. The like concluſions may be applied to machines in general, if R and r repreſent the diſtances of the impelled and working points from the axis of motion; and if the various kinds of reſiſtance ariſing from friction, ſtiffneſs of ropes, &c. be properly reduced to their equivalents at the working points, ſo as to be comprehended in the character W for reſiſtance overcome.

368. Prop. *Given* R *and* r, *the arms of a ſtraight lever* M *and* m, *their reſpective weights, and* P *the power acting at the extremity of the arm* R, *to find the weight raiſed at the extremity of the other arm when the effect is a maximum.*

In this caſe $\frac{m}{2}$ is the weight of the ſhorter end reduced to B (fig. 1. pl. IV.), and conſequently $\frac{mr}{2R}$ is the weight which, applied at A, would balance the ſhorter end: therefore, $\frac{mr}{2R}+\frac{r}{R}W$, would ſuſtain both the ſhorter end and the weight W in equilibrio. But $P+\frac{1}{2}M$ is the power really acting at the longer end of the lever; conſequently $P+\frac{1}{2}M-\left(\frac{mr}{2R}+\frac{r}{R}W\right)$, is the abſolute *moving* power. Now by art. 312. the diſtance of the centre of gyration of the beam from F is $=\sqrt{\frac{R^3+r^3}{3(R+r)}}$, which let be denoted by ϱ; then (by cor. 4. art. 310.) $\frac{\varrho^2}{R^2}\cdot(M+m)$ will repreſent the maſs equivalent to the beam or lever when reduced to the point A; while the weight equivalent to W, when referred to that point, will be $\frac{r^2}{R^2}W$. Hence, proceeding as in the laſt prop. we ſhall have $\frac{\varrho^2}{R^2}\cdot(M+m)+P+\frac{r^2}{R^2}W$ for the inertia to be overcome; and $(P+\frac{1}{2}M-\frac{mr}{2R}-\frac{r}{R}W)\div\frac{\varrho^2}{R^2}(M+m)+P+\frac{r^2}{R^2}W=$ the accelerating force of P, or of W reduced to A. Multiply this by W, and, for the ſake of ſimplifying the proceſs, put q for $P+\frac{1}{2}M-\frac{rm}{2R}$, and n for $P+\frac{\varrho^2}{R^2}(M+m)$ then we ſhall have $\frac{qW-\frac{rW^2}{R}}{n+\frac{r^2}{R^2}W}$, a quantity which varies as the effect varies, and which, indeed, when multiplied by $g\,t$, denotes the effect itſelf. Putting the fluxion of this equal to nothing, and reducing, we at length find $W = \frac{R}{s}\sqrt{\frac{n\,q\,R}{r}+\frac{n^2R^2}{r^2}} - \frac{n\,R^2}{r^2}$.

COR. When $R=r$, and $M=m$, if we reſtore the values of n and q the expreſſion will become $W = \sqrt{2\,P^2+2\,m\,P+\frac{4}{9}m^2} - (P+\frac{2}{3}m)$.

369. PROP. *Given the length* l *and angle* e *of elevation of an inclined plane* BC (fig. 5. pl. XV.), *to find the length* L *of another inclined plane* AC, *along which a given weight* W *ſhall be raiſed from the horizontal line* AB *to the point* C, *in the leaſt time poſſible, by means of another given weight* P *deſcending along the given plane* CB: *the two weights being connected by an inextenſible thread* PCW *running always parallel to the two planes.*

Let the angle of elevation CAD be denoted by E, and let the expreſſion for the ſpace paſſed over in a given time, found art. 267. II. be accommodated to the preſent notation; it will then be $s = \frac{P.\sin.e - W.\sin.E}{P+W}\,\frac{1}{2}g\,t^2$. This gives $t^2 = \frac{s\,(P+W)}{\frac{1}{2}g\,(P\sin.e - W\sin.e)}$. But in the triangle ABC we have AC : BC :: ſin. B : ſin. A, that is, $L : l :: \text{ſin.}\,e : \text{ſin.}\,E$; hence $\frac{1}{m}L = \text{ſin.}\,e$, and $\frac{1}{m}l = \text{ſin.}\,E$; m being a conſtant quantity always determinable from the data given. And t^2 becomes $\frac{s\,(P+W)}{\frac{1}{2}g\,\frac{1}{m}(P\,L - W\,l)}$. Now when any quantity, as t, is a minimum, its ſquare is manifeſtly a minimum: ſo that ſubſtituting for s its equal L, and ſtriking out the conſtant factors, we have $\frac{L^2}{PL - Wl} =$ a min. or its fluxion $\frac{2\,L\dot{L}\,(P\,L - W\,l)\ P\,L^2\dot{L}}{(PL - W\,l)^2} = 0$. Here, as in all ſimilar caſes, ſince the fraction vaniſhes, its numerator muſt be equal to 0; conſequently $2\,PL^2 - 2\,W\,l\,L - PL^2 = 0$, $PL = 2\,W\,l$, or $L : l :: 2\,W : P$.

COR. 1. Since neither ſin. e nor ſin. E enters the final equation, it follows, that if the elevation of the plane BC is not given the problem is unlimited.

COR. 2. When ſin. $e = 1$, BC coincides with the perpendicular CD, and the power P acts with all its intenſity upon the weight W. This is the caſe of the preſent problem which has commonly been conſidered.

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370. This propoſition admits of a neat *geometrical* demonſtration. Thus, let CE (fig. 5. pl. XV.) be the plane upon

which, if W were placed, it would be ſuſtained in equilibrio by the power P upon the plane CB, or the power P′ hanging freely in the vertical CD; then (art. 154. cor. 5, 6.) BC : CD : CE :: P : P′ : W. But W is to the force with which it tends to deſcend along the plane CA as CA to CD; conſequently, the weight P′ is to that force as CA : CE; or the weight P upon the plane BC is to the ſame force in the ſame ratio; becauſe either of theſe weights in their reſpective poſitions would ſuſtain W on CE. Therefore the exceſs of P above that force (which exceſs is the power accelerating the motions of P and W) is to P as CA − CE to CA; or, taking CH = CA, as EH to CA. Now, the motion being uniformly accelerated, we have $S \propto FT^2$, or $T^2 \propto \frac{S}{F}$: conſequently, the ſquare of the time in which AC is deſcribed by W will be as AC directly, and as $\frac{EH}{AC}$ inverſely; and will be leaſt when $\frac{CA^2}{EH}$ is a *minimum*; that is, when $\frac{CE^2}{EH} + EH + 2\,CE$, or (becauſe 2 CE is invariable) when $\frac{CE^2}{EH} + EH$ is a minimum. Now, as when the ſum of two quantities is given their product is a *maximum* when they are equal to each other; ſo it is manifeſt that when their product is given their ſum muſt be a *minimum* when they are equal. But the product of $\frac{CE^2}{EH}$ and EH is CE^2, and conſequently given; therefore, the ſum of $\frac{EC^2}{EH}$, and EH is leaſt when thoſe parts are equal; that is, when EH = CE, or CA = 2 CE. So that the length of the plane CA is double the length of that on which the weight W would be kept in equilibrio by P acting along CB.

When CD and CB coincide the caſe becomes the ſame as that conſidered by that admirable mathematician Mr. Maclaurin, in his *View of Newton's Philoſophical Diſcoveries*, pa. 183, 8vo. ed. Whence, with a very ſlight variation, the method in this ſcholium was deduced.

371. Prop. *Let the given weight* P (fig. 5.) *deſcend along* CB, *and by means of the thread* PCW (*running parallel to the planes*) *draw a weight* W *up the plane* AC: *it is required to find the value of* W, *when its momentum is a maximum, the lengths and poſitions of the planes being given.*

The general expreſſion for the velocity is $v = \frac{P.\sin. e - W \sin. E}{P + W} gt$

(art. 267. ex. II.); which, by ſubſtituting $\frac{1}{m}L$ for ſin. e, and $\frac{1}{m}l$ for ſin. E, becomes $v = \frac{\frac{1}{m}(PL - Wl)}{P+W} g t$. This multiplied into W gives $\frac{\frac{1}{m}(PWL - W^2 l)}{P+W} g t$; which, by the prop. is to be a maximum. Or, ſtriking out the conſtant factors, $\frac{1}{m}$, $g t$, we have $\frac{PWL - W^2 l}{P+W} =$ a max. Putting this into fluxions, and reducing, we have $P^2 L - 2PWl - W^2 l = 0$, or $W = P\sqrt{\frac{L}{l} + 1} - P$.

COR. When the inclinations of the planes are equal, L and l are equal, and $W = P\sqrt{2} - P = P \times (\sqrt{2} - 1) = \cdot 4142\,P$; agreeing with the concluſion of the lever of equal arms, or the extreme caſe of the wheel and axle, *i. e.* the pulley. Art. 366. cor. 7. art. 367.

372. PROP. *Given the radius* R *of a wheel, and the radius* r *of its axle, the weight of both,* w, *and the diſtance of the centre of gyration from the axis of motion,* ϱ; *alſo a given power* P *acting at the circumference of the wheel; to find the weight* W *raiſed by a cord folding about the axle, ſo that its momentum ſhall be a maximum.*

The force which abſolutely impels the point A (fig. 6. pl. V.) is P, while W acts in a direction contrary to P, with a force $= \frac{rW}{R}$; this, therefore, ſubducted from P, leaves $P - \frac{rW}{R} = \frac{RP - rW}{R}$, for the reduced force impelling the point A. And the inertia which reſiſts the communication of motion to the point A will be the ſame as if the maſs $\frac{\varrho^2 w + r^2 W + R^2 P}{R^2}$ were concentrated in the point A (art. 310. cor. 4. 5.). If the former of theſe be divided by the latter, the quotient $\frac{R(RP - r^2 W)}{\varrho^2 w + r^2 W + R^2 P}$ is the force accelerating A: multiplying this by $\frac{r}{R}$, we have $\frac{RrP - r^2 W}{\varrho^2 w + r^2 W + R^2 P}$ for the force which accelerates the weight W in its aſcent. Conſequently the velocity of W will be $= \frac{RrP - r^2 W^2}{\varrho^2 w + r^2 W + R^2 P} g t$: which multiplied into W gives $\frac{RrPW - r^2 W^2}{\varrho^2 w + r^2 W + R^2 P} g t$ for the momentum. As this is to be a

maximum, its fluxion will $= 0$; whence we ſhall obtain $W = \frac{\sqrt{R^4 P^2 + 2 R^2 P \varrho^2 w + \varrho^4 w^2 + P w R r \varrho^2 + P^2 R^3 r} - R^2 P - \varrho^2 w}{r^2}$.

Cor. 1. When $R = r$, as in the caſe of the ſingle fixed pulley, then $W = \sqrt{2 P^2 R^3 + 2 RP \varrho^2 w + \frac{\varrho^4}{R} w^2 + P w R \varrho^2} - \frac{\varrho^2}{R^2} w - P$.

Cor. 2. When the pulley is a cylinder of uniform matter $\varrho^2 = \frac{1}{2} R^2$, and the expreſſion becomes $W = \sqrt{R^3 (2P^2 + \frac{1}{2} P w + \frac{1}{4} w^2)} - \frac{1}{2} w - P$.

Cor. 3. If in the firſt general expreſſion for the momentum of W we put $Q = R^2 P + \varrho^2 w$, we ſhall have $\frac{R r PW - r^2 W^2}{Q + r^2 W} =$ a max. Which fluxed and reduced gives $W = \frac{1}{r^2} \sqrt{Q . (Q + R r P)} - \frac{1}{r^2} Q$.

Cor. 4. If the moving force be deſtitute of inertia, then will $Q = \varrho^2 w$, and W, as in the laſt corollary.

373. *Let a given power* P *be applied to the circumference of a wheel, its radius* R, *to raiſe a weight* W *at its axle, whoſe radius is* r, *it is required to find the ratio of* R *and* r *when* W *is raiſed with the greateſt momentum; the characters* w *and* ϱ *denoting the ſame as in the laſt propoſition.*

Here we ſuppoſe r to vary in the expreſſion for the momentum of W, $\frac{W R r P - r^2 W^2}{\varrho^2 w + r^2 W + R^2 P} g t$. And we ſuppoſe that by the conditions of any ſpecified inſtance we can aſcertain what quantity of matter q ſhall make $r^2 q = \varrho^2 w$, which, in fact, may always be done as ſoon as we can determine ϱ. The expreſſion for the work will then become $\frac{R r PW - r^2 W^2}{R^2 P + r^2 (q + W)} g t$. The fluxion of which being made $= 0$, gives, after a little reduction, $r = \frac{R \sqrt{P^2 W^2 + P^3 (q + W)} - P W}{P (q + W)}$.

Cor. When the inertia of the machine is evaneſcent, with reſpect to that of P + W, then is $r = R \sqrt{1 + \frac{P}{W}} - 1$.

374. Prop. *In any machine whoſe motion accelerates, the weight will be moved with the greateſt velocity when the velocity of the power is to that of the weight as* $1 + P \sqrt{1 + \frac{P}{W}}$ *to* 1; *the inertia of the machine being diſregarded.*

For any ſuch machine may be conſidered as reduced to a lever, or to a wheel and axle whoſe radii are R and r; in which the velocity of the weight $\frac{RrP - r^2W}{R^2P + r^2W}gt$ (art. 366.) is to be a maximum, r being conſidered as variable. Hence, then, following the uſual rules we ſhall find $PR = r\,(W + \sqrt{W^2 + PW})$. From which, ſince the velocities of the power and weight are reſpectively as R and r, the ratio in the propoſition immediately flows.

COR. *When the weight moved is equal to the power, then is* $R : r :: 1 + \sqrt{2} : 1 :: 2{\cdot}4142 : 1$ *nearly.*

375. PROP. *If in any machine whoſe motion accelerates, the deſcent of one weight cauſes another to aſcend, and the deſcending weight be given, the operation being ſuppoſed continually repeated, the effect will be greateſt in a given time when the aſcending weight is to the deſcending weight, as* 1 *to* 1·618, *in the caſe of equal heights; and in other caſes, when it is to the exact counterpoiſe in a ratio which is always between* 1 *to* $1\frac{1}{2}$ *and* 1 *to* 2.

Let the ſpace deſcended be 1, that aſcended s; the deſcending weight 1, the aſcending weight $\frac{1}{w}$: then would the equilibrium require $w = s$ (art. 363.); and $1 - \frac{s}{w}$ will be the force acting on 1. Now the maſs $\frac{1}{w}$ reduced to the point at which the maſs 1 acts will be $= \frac{1}{w}s^2 = \frac{s^2}{w}$; conſequently the whole maſs moved is equivalent to $1 + \frac{s^2}{w}$, and the relative force is $(1 - \frac{s}{w}) \div (1 + \frac{s^2}{w}) = \frac{w - s}{w + s^2}$. But, the ſpace being given, the time is as the root of the accelerating force inverſely, that is, as $\sqrt{\frac{w + s^2}{w - s}}$: and the whole effect in a given time being directly as the weight raiſed, and inverſely as the time of aſcent, will be as $\frac{1}{w}\sqrt{\frac{w - s}{w + s^2}}$; which muſt be a maximum. Conſequently its ſquare $\frac{w - s}{w^3 + s^2w^2}$ muſt be a maximum likewiſe. This latter expreſſion fluxed and reduced gives $w = \frac{s}{4}(\sqrt{s^2 + 10S + 9} - a + 3)$.

Here if $s = 1$, $w = \frac{1 + \sqrt{5}}{2}$: but if s be diminiſhed without limit, $w = \frac{3}{2}s$; if it be augmented without limit, then will $\sqrt{s^2 + 10s + 9}$ approach indefinitely near to $s + 5$, and conſequently $w = 2e$. Whence the truth of the propoſition is manifeſt.

376. Prop. *Let* φ *denote the abſolute effort of any moving force, when it has no velocity, and ſuppoſe it not capable of any effort when the velocity is* W; *let* F *be the effort anſwering to the velocity* V, *then, if the force be uniform,* F *will be* $= \varphi\,(1 - \frac{V}{W})^2$.

For it is the difference between the velocities W and V which is efficient, and the action being conſtant will vary as the ſquare of the efficient velocity. Hence we ſhall have this analogy, $\varphi : F :: (W - 0)^2 : (W - V)^2$: conſequently, $F = \varphi \left(\frac{W - V}{W}\right)^2 = \varphi \left(1 - \frac{V}{W}\right)^2$.

Although the preſſure of an animal is not actually uniform during the whole time of its action, yet it is nearly ſo: ſo that in general we may adopt this hypotheſis in order to approximate to the true nature of animal action. On which ſuppoſition the preceding prop. as well as the remaining one, in this chapter, will apply to animal exertion.

Cor. Retaining the ſame notation we have $W = \frac{V\sqrt{\varphi}}{\sqrt{\varphi} - \sqrt{F}}$. This, applied to the motion of animals, gives this theorem: *The utmoſt velocity with which an animal not impeded can move, is to the velocity with which it moves when impeded by a given reſiſtance, as the ſquare root of its abſolute force, to the difference of the ſquare roots of its abſolute and efficient forces.*

377. Prop. *To inveſtigate expreſſions by means of which the maximum effect, in machines whoſe motion is uniform, may be determined.*

I. It follows, from the obſervations made in art. 363. and the definitions in art. 365. that when a machine, whether ſimple or compound, is put into motion, the velocities of the impelled and working points are inverſely as the forces which are in equilibrio when applied to thoſe points in the direction of their motion. Conſequently, if f denote the reſiſtance when reduced to the working point, and v its velocity; while F and V denote the force acting at the impelled point, and its velocity; we ſhall have $FV = fv$, or introducing t the time, $FVt = fvt$. Hence, *in all working machines which have acquired a uniform motion, the performance of the machine is equal to the momentum of impulſe.*

II. Let F be the effort of a force upon the impelled point of a machine when it moves with the velocity V, the velocity being W when $F = 0$, and let the relative velocity $W - V = u$

Then ſince (art. 376.) $F = \varphi \left(\frac{W-V}{W}\right)^2$, the momentum of impulſe FV will become $V \varphi \left(\frac{u}{W}\right)^2 = \varphi \cdot \frac{u^2}{W^2} (W-u)$; becauſe $V = W - u$. Making this expreſſion for FV a maximum, or, ſuppreſſing the conſtant quantities, and making $u^2 (W-u)$ a max. or its fluxion $= 0$, when u is variable, we find $2 W = 3 u$, or $u = \frac{2}{3} W$. Whence $V = W - u = W - \frac{2}{3} W = \frac{1}{3} W$.

Conſequently, *when the ratio of V to v is given by the conſtruction of the machine, and the reſiſtance is ſuſceptible of variation, we muſt load the machine more or leſs till the velocity of the impelled point is one-third of the greateſt velocity of the force; then will the work done be a maximum.*

Or, the work done by an animal is greateſt when the velocity with which it moves is one-third of the greateſt velocity with which it is capable of moving when not impeded.

II. Since $F = \varphi \frac{u^2}{W^2} = \varphi \left(\frac{\frac{4}{9} W^2}{W^2}\right) = \frac{4}{9} \varphi$, in the caſe of the maximum, we have $FV = \frac{4}{9} \varphi V = \frac{4}{9} \varphi \cdot \frac{1}{3} W = \frac{4}{27} \varphi W$, for the momentum of impulſe, or for the work done, when the machine is in its beſt ſtate. *Conſequently when the reſiſtance is a given quantity we muſt make* $V : v :: 9 f : 4 \varphi$; *and this ſtructure of the machine will give the maximum effect* $= \frac{4}{27} \varphi W$.

III. If we enquire the greateſt effect on the ſuppoſition that φ only is variable, we muſt make it infinite in the above expreſſion for the work done, which would then become WF, or $W \frac{V}{v} f$, or $W \frac{V}{v} f t$, including the time in the formula. Hence we ſee, *that the ſum of the agents employed to move a machine may be infinite, while the effect is finite:* for the variations of φ which are proportional to this ſum do not influence the above expreſſion for the effect.

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378. The propoſitions now delivered contain, it is hoped, the moſt material principles in the theory of machines. The manner of applying ſeveral of them is very obvious: the application of ſome, being leſs manifeſt, may be briefly illuſtrated, and the chapter concluded with two or three obſervations.

The laſt theorem may be applied to the action of men and of horſes, with more accuracy than might at firſt be ſuppoſed. Obſervations have been made on men and horſes drawing a lighter along a canal, and working ſeveral days together. The force exerted was meaſured by the curvature and weight of the track-rope, and afterwards by a ſpring ſteelyard. The product of the force thus aſcertained, into the velocity per hour, was conſidered as the momentum. In this way the action of *men* was found to be very nearly as $(W-V)^2$: the action of horſes loaded ſo as not to be able to trot was nearly as $(W-V)^{1\cdot 7}$, or as $(W-V)^{\frac{9}{5}}$. Hence the hypotheſis we have adopted may in many caſes be ſafely aſſumed.

According to the beſt obſervations, the force of a man at reſt is on the average about 70 pounds; and the utmoſt velocity with which he can walk is about 6 feet per ſecond, taken at a medium. Hence, in our theorems, $\varphi = 70$, and $W=6$. Conſequently $F=\frac{4}{9}\varphi = 31\frac{1}{9}$ lbs. the greateſt force a man can exert when in motion: and he will then move at the rate of $\frac{1}{3}W$, or 2 feet per ſecond, or rather leſs than a mile and a half per hour.

The ſtrength of a horſe is generally reckoned about ſix times that of a man; that is, nearly 420 lbs. at a dead pull. His utmoſt walking velocity is about 10 feet per ſecond. Therefore his maximum action will be $\frac{4}{9}$ of $420=186\frac{2}{3}$ lbs. and he will then move at the rate of $\frac{1}{3}$ of 10, or $3\frac{1}{3}$ feet, per ſecond, or nearly $2\frac{1}{3}$ miles per hour. In both theſe inſtances we ſuppoſe the force to be exerted in drawing a weight along a horizontal plane; or by raiſing a weight by a cord running over a pulley which makes its direction horizontal. Other methods of exerting animal force will be ſpoken of in the ſecond volume, art. 66, &c.

379. The theorems juſt given may ſerve to ſhew in what points of view machines ought to be conſidered by thoſe who would labour beneficially for their improvement.

The firſt object of the utility of machines conſiſts in furniſhing the means of *giving to the moving force the moſt commodious direction*; and, when it can be done, of cauſing its action to be applied immediately to the body to be moved. Theſe can rarely be united: but the former can be accompliſhed in moſt inſtances; of which the uſe of the ſimple lever, pulley, and wheel and axle, furniſh many examples. The ſecond object gained by the uſe of machines is *an accommodation of the velocity of the work to be performed to the velocity with which alone a natural power can act*. Thus, whenever the natural power acts with a certain velocity which cannot be changed, and the work muſt be performed with a greater velocity, a machine is interpoſed moveable round a fixed ſupport, and the diſtances of the impelled and working points are taken in the proportion of the two given velocities.

But the eſſential advantage of machines, that, in fact, which properly appertains to the *theory* of mechanics, conſiſts in augmenting, or rather in modifying, the energy of the moving power, in ſuch manner that it may produce effects of which it would have been otherwiſe incapable. Thus a man might carry up a flight of ſteps twenty pieces of ſtone, each weighing 30 pounds (one by one) in as ſmall a time as he could (with the ſame labour) raiſe them all together by a piece of machinery, that would have the velocities of the impelled and working points as 20 to 1; and, in this caſe, the inſtrument would furniſh no real advantage, except that of ſaving his ſteps. But if a large block of 20 times 30, or 600lbs. weight, were to be raiſed to the ſame height, it would far ſurpaſs the utmoſt efforts of the man, without the intervention of ſome ſuch contrivance.

The ſame purpoſe may be illuſtrated ſomewhat differently; confining the attention all along to machines whoſe motion is uniform. The product fv repreſents, during the unit of time, the effect which reſults from the motion of the reſiſtance; this motion being produced in any manner whatever. If it be produced by applying the moving force immediately to the reſiſtance, it is neceſſary not only that the products FV and fv ſhould be equal; but that at the ſame time $F = f$, and $V = v$: if, therefore, as moſt frequently happens, f be greater than F, it will be abſolutely impoſſible to put the reſiſtance in motion by applying the moving force immediately to it. Now machines furniſh the means of diſpoſing the product FV in ſuch a manner that it may always be equal to fv, however much the factors of FV may differ from the analogous factors in fv; and, conſequently, of putting the ſyſtem in motion, whatever is the exceſs of f over F.

Or, generally, as M. Prony remarks (Archi. Hydraul. art. 504.), machines enable us to diſpoſe the factors of FVt in ſuch a manner, that while that product continues the ſame its factors may have to each other any ratio we deſire. Thus, to give another example: Suppoſe that a man exerting his ſtrength immediately upon a maſs of 25lbs. can raiſe it vertically with a velocity of 4 feet per ſecond; the ſame man acting upon a maſs of 1000lbs. cannot give it any vertical motion though he exerts his utmoſt ſtrength, unleſs he has recourſe to ſome machine. Now he is capable of producing an effect equal to $25 \times 4 \times t$: the letter t being introduced becauſe if the labour is continued the value of t will not be indefinite, but compriſed within aſſignable limits. Thus we have $25 \times 4 \times t = 1000 \times v \times t$; and conſequently $v = \frac{1}{10}$ of a foot. This man may, therefore, with a machine as a lever, or axis in peritrochio, cauſe a maſs of 1000lbs. to riſe $\frac{1}{10}$ of a foot, in the ſame time that he could raiſe 25 lbs. four feet without a machine; or he may raiſe the greater weight as far as the leſs, by employing 40 times as much time.

From what has been ſaid on the extent of the effects which may be attained by machines, it will be ſeen that ſo long as a moving force exerciſes a determinate effort with a velocity likewiſe determinate, or ſo long as the product of theſe is conſtant, the effect of the machine will remain the ſame: thus, under this point of view, ſuppoſing the preponderance of the effort of the moving power, and abſtracting from inertia and friction of materials, the convenience of application, &c. all machines are equally perfect. But, from what has been ſhewn, (arts. 376. 377.) a moving force may, by diminiſhing its velocity, augment its effort, and reciprocally. There is, therefore, a certain effort of the moving force, ſuch that its product by the velocity which comports to that effort is the greateſt poſſible. Admitting the truth of the law aſſumed in the articles juſt referred to, we have, when the effect is a *maximum*, $V = \frac{1}{3} W$, or $F = \frac{4}{9} \phi$; and theſe two values obtaining together their product $\frac{4}{27} \phi W$ expreſſes the value of the greateſt effect with reſpect to the unit of time. In practice it will always be adviſeable to approach as nearly to theſe values as circumſtances will admit; for it cannot be expected that they can always be exactly attained. But a ſmall variation will not be of much conſequence: for, by a well-known property of thoſe quantities which admit of a proper maximum and minimum, a value aſſumed at a moderate diſtance from either of theſe extremes will produce no ſenſible change in the effect.

If the relation of F to V followed any other law than that which we have aſſumed, we ſhould find from the expreſſion of *that law* values of F, V, &c. different from the preceding. The general method, however, would be nearly the ſame.

With reſpect to practice, the grand object in all caſes ſhould be to procure a *uniform motion*, becauſe it is that from which *(cæteris paribus)* the greateſt effect always reſults. Every irregularity in the motion waſtes ſome of the impelling power; and it is the greateſt only of the varying velocities which is equal to that which the machine would acquire if it moved uniformly throughout: for, while the motion accelerates, the impelling force is greater than what balances the reſiſtance at that time oppoſed to it, and the velocity is leſs than what the machine would acquire if moving uniformly; and when the machine attains its greateſt velocity, it attains it becauſe the power is not then acting againſt the whole reſiſtance. In both theſe ſituations, therefore, the performance of the machine is leſs than if the power and reſiſtance were exactly balanced; in which caſe it would move uniformly (art. 363.). Beſides this, when the motion of a machine, and particularly a very ponderous one, is irregular, there are continual repetitions of ſtrains and jolts which ſoon derange and ultimately deſtroy the whole ſtructure. Every attention ſhould, therefore, be paid to the removal of all cauſes of irregularity. Some of the moſt ſucceſsful methods of enſuring a uniformity of motion will be given in the ſecond volume. We muſt now turn to other ſubjects.

HYDROSTATICS.

BOOK III.

Introductory Definitions and Remarks.

380. According to the general diviſion marked out in art. 20, we now proceed to the ſubject of Hydrostatics, which compriſes the doctrine of the preſſure and the equilibrium of non-elaſtic fluids, as water, mercury, &c. and that of the weight and preſſure of ſolids immerſed in them.

Def. A *fluid* is a body whoſe parts are very minute, yield to any force impreſſed upon it (however ſmall), and by ſo yielding are eaſily moved among themſelves.

This is nearly the ſame as the definition given by Newton, in the *Principia*, book 2. ſect. 5. and is adopted here becauſe, in conjunction with two or three eſtabliſhed facts, it may ſerve as a baſis for all which diſtinguiſhes the doctrines of hydroſtatics from thoſe of pure mechanics. The writers on the continent, however, though they admit that the minuteneſs of fluid moleculæ, and their exceſſive mobility, are characteriſtics common to all ſuch bodies, yet they have recourſe to a different definition. Thus the celebrated Euler in the *New Commentaries of St. Peterſburgh*, vol. 13. takes for the baſis of his analyſis the following conſideration: "The diſtinguiſhing nature of fluids conſiſts in this property, namely, that when it is ſubjected to any preſſure whatever that preſſure is ſo diſtributed throughout the maſs, that while it remains in equilibrio all its parts are equally preſſed." And M. D'Alembert in his *Traite de l'Equilibre et du Mouvement des Fluides*, as well as M. Prony in his *Architecture Hydraulique*, adopt the ſame property as a definition. It is ſtrictly conſiſtent with experiment (though, as will ſoon be ſeen, it is rather a propoſition, capable of proof, than a definition), and furniſhes a natural foundation for an *algebraical* calculus, by which the whole doctrine of hydroſtatics may be exhibited in a few equations. But this method, though it poſſeſſes ſome advantages, is not entirely purſued here, from a firm conviction that a judicious combination of the geometrical and algebraical methods is far more likely to convey diſtinct ideas to the ſtudent than the modern analyſis merely.

Perfect fluidity, according to the Newtonian ſyſtem, ariſes from a want of any ſenſible coheſion between the conſtituent particles of the fluid; and this want of coheſion is commonly attributed to the ſpherical figure of the particles. The nature of this work does not require that we ſhould enter into minute diſquiſitions on the formal cauſe of fluidity. We ſhall merely ſtate that the late Dr. Black, of Edinburgh, ſpeaks of fluidity as an effect of heat: and before him Boerhaave pleaded ſtrenuouſly for the ſame opinion. According to this view of the matter, fluidity may be cauſed by a certain degree of fire, which, when employed for this purpoſe, ſeldom manifeſts itſelf by any other perceptible effect: not dilating the volume, but reſiſting the particular attachment of the parts. Some ſtrive to give mechanical ideas of a fluid body, by comparing it to a heap of ſand: but the impoſſibility of giving fluidity by any kind of mechanical comminution will appear by conſidering two of the circumſtances neceſſary to conſtitute a fluid body: 1. That the parts, notwithſtanding any compreſſion, *may* be moved in relation to each other, with the ſmalleſt conceivable force, or will give no *ſenſible reſiſtance* to motion within the maſs in any direction. 2. That the parts ſhall gravitate to each other, whereby there is a conſtant tendency to arrange themſelves about a common centre, and form a ſpherical body; which, as the parts do not reſiſt motion, is eaſily executed in ſmall bodies. Hence the appearance of drops always takes place when a fluid is in proper circumſtances. It is obvious that a body of ſand can by no means conform to theſe circumſtances.

Different fluids have different degrees of fluidity, according to the facility with which the particles may be moved amongſt each other. Water and mercury are claſſed among the moſt perfect fluids. Many fluids have a very ſenſible degree of tenacity, and are therefore called *viſcous* or *imperfect* fluids.

381. Def. Fluids may be divided into *compreſſible* and *incompreſſible*, or *elaſtic* and *non-elaſtic* fluids. A *compreſſible* or *elaſtic* fluid is one whoſe apparent magnitude is diminiſhed as the preſſure upon it is increaſed, and increaſed by a diminution of preſſure. Such is air, and the different vapours. An *incompreſſible* or *non-elaſtic* fluid is one whoſe dimenſions are not, at leaſt as to ſenſe, affected by any augmentation of preſſure. Water, mercury, wine, &c. are generally ranged under this claſs.

Although the uſe of that well-known inſtrument the *thermometer* is founded upon the circumſtance of different degrees of heat and cold cauſing a correſponding dilatation or condenſation in ſpirits of wine, mercury, and ſome other fluids; a fact which it might be ſuppoſed would have led to the opinion that ſuch fluids were compreſſible by other means; yet has it been univerſally believed and aſſerted till within the laſt half century, that after the fluid was freed from all air no art or violence could preſs it into leſs ſpace. This opinion has been grounded chiefly, if not altogether, on a groſs and inadequate experiment made by the *Academi del Cimento*, at Florence; in which water when violently ſqueezed made its way through the fine pores of a globe of gold, rather than yield to the compreſſion. Even ſo lately as 1790, ſo ſkilful a mathematician and philoſopher as M. Prony ſpeaks of the incompreſſibility of water with an obvious alluſion to the Florentine experiment, as though he was not at all conſcious of its inſufficiency, and ſeeming quite ignorant of any contrary experiments. For, ſays he, "Si une quantité d'eau eſt renfermée dans un vaſe de capacité et de forme quelconque, et qu'on l'y comprime avec toute la force qu'on voudra, jamais on ne pourra la reduire à occuper un eſpace moindre que celui qu'elle occupoit d'abord. Tout le monde connoît les experiences qu'on a faites pour conſtater cette propriété; on ſait que l'eau etant renfermée dans des globes de métal, quelque percuſſion ou quelque preſſion qu'on emploie pour le faire diminuer de volume, on n'y parvient jamais, et que lorſque la réſiſtance qu'elle oppoſe à de pareils efforts ne lui fait pas briſer, ſon enveloppe, elle ſe fait jour à travers les pores du métal, d'où elle ſort en forme de roſée."

But our ingenious countryman, Mr. Canton, attentively conſidering this experiment, found that it was not ſufficiently accurate to juſtify the concluſion which had always been drawn from it; ſince the Florentine philoſophers had no method of determining that the alteration of figure in their globe of gold occaſioned ſuch a diminution of its internal capacity as was exactly equal to the quantity of water forced into its pores. To bring this matter therefore to a more accurate and deciſive trial, he procured a ſmall glaſs tube of about two feet long, with a ball at one end, of an inch and a quarter in diameter. Having filled the ball and part of the tube with mercury, and brought it exactly to the heat of 50° of Fahrenheit's thermometer, he marked the place where the mercury ſtood in the tube, which was about ſix inches and a half above the ball; he then raiſed the mercury by heat to the top of the tube, and there ſealed the tube hermetically; then upon reducing the mercury to the ſame degree of heat as before, it ſtood in the tube $\frac{32}{100}$ of an inch higher than the mark. The ſame experiment was repeated with water exhauſted of air, inſtead of mercury, and the water ſtood in the tube $\frac{43}{100}$ of an inch above the mark. Since the weight of the atmoſphere on the outſide of the ball, without any counterbalance from within, will compreſs the ball, and equally raiſe both the mercury and water, it appears that the water expands $\frac{11}{100}$ of an inch more than the mercury by removing the weight of the atmoſphere. Having thus determined that water is really compreſſible, he proceeded to eſtimate the degree of compreſſion correſponding to any given weight. For this purpoſe he prepared another ball, with a tube joined to it; and finding that the mercury in $\frac{23}{100}$ of an inch of the tube was the hundred thouſandth part of that contained in the ball, he divided the tube accordingly. He then filled the ball and part of the tube with water exhauſted of air; and leaving the tube open, placed this apparatus under the receiver of an air-pump, and obſerved the degree of expanſion of the water anſwering to any degree of rarefaction of the air: and again by putting it into the glaſs receiver of a condenſing engine, he noted the degree of compreſſion of the water correſponding to any degree of condenſation of the air. He thus found, by repeated trials, that, in a temperature of 50°, and when the mercury has been at its mean height in the barometer, the water expands one part in 21740; and is as much compreſſed by the weight of an additional atmoſphere; or the compreſſion of water by twice the weight of the atmoſphere, is one part in 10870 of its whole bulk. Should it be objected that the compreſſibility of the water was owing to any air which it might be ſuppoſed to contain, he anſwers, that more air would make it more compreſſible; he therefore let into the ball a bubble of air, and found that the water was not more compreſſed by the ſame weight than before.

In ſome further experiments of the ſame kind, Mr. Canton found that water is more compreſſible in winter than in ſummer; but he obſerved the contrary in ſpirit of wine, and oil of olives.

The following table was formed, when the barometer was at 29 inches and a half, and the thermometer at 50 degrees.

Compreſſion of	Millionth parts.	Spec. grav.
Spirit of wine	66	846
Oil of olives	48	918
Rain water	46	1000
Sea water	40	1028
Mercury	3	13595

See *Phil. Tranſac.* for 1762 and 1764.

Indeed it ſeems reaſonable to conclude, independent of all experiments, that no fluids are abſolutely incompreſſible: for all bodies being porous, their parts may be brought nearer to each other; and a liquid being an aſſemblage of ſolid bodies, ſhould, therefore, be compreſſible. Hence, then, the uſual diſtinction of fluids into compreſſible and incompreſſible is, ſtrictly ſpeaking, inaccurate. Nevertheleſs, as the compreſſion of the liquids in the preceding table is *very ſmall* compared with their maſs, it may ſafely be neglected in moſt practical caſes, ſo that the fluids uſually conſidered as incompreſſible may ſtill be reckoned ſo in the inveſtigations we are about to enter upon; and the conſideration of air, and other eaſily compreſſible and elaſtic fluids, may be properly referred to the ſeparate head of Aeroſtatics or Pneumatics.

382. We know ſo little of the eſſential nature and conſtitution of fluids, that it would be by no means adviſeable to apply to them the principles of equilibrium and of motion, as they have been ſtated in the preceding part of this work, without firſt enquiring whether there is not ſome other general law which appertains to fluids only, and from which, in conjunction with the principles juſt adverted to, the doctrines of hydroſtatics may readily be deduced. For the action of fluids upon each other differs ſo eſſentially in ſome particulars from the mutual actions of ſolid bodies, that ſome diſtinct principle muſt be ſought, to account for ſuch varying effects. The parts of a ſolid are ſo connected together as to form but one and the ſame whole; their effort is, according to its nature, concentrated into one point (as the centre of gravity, centre of gyration, &c.); which is by no means the caſe with fluids, their particles being extremely moveable, and entirely independent of each other. Again: no ſtatical equilibrium can take place between two bodies of different weights, unleſs the lighter body acts at ſome mechanical advantage; whereas a very ſmall weight of fluid may, without acting in ſo advantageous a poſition, be made to balance any weight however large. Solid bodies, again, when left to themſelves, preſs only in the direction of gravity; while fluids preſs equally in all directions. This property indeed is one of the moſt extraordinary which we meet with in fluids, and from it moſt of the other properties may be readily inferred; on which account the continental philoſophers aſſume it as a kind of definition. The Newtonian definition is more ſimple, and naturally leads to this property, which can only be conceived to ariſe from the extreme freedom with which the particles move amongſt each other. But the moſt ſatisfactory proof reſults from experiment; to which it is proper to have recourſe in the eſtabliſhment of the firſt principles of Hydroſtatics, and which will at once furniſh the general law neceſſary to be combined with the received principles of proper mechanics.

383. Def. The *ſpecific gravity* of any ſolid or fluid body is the abſolute weight of a known volume of that ſubſtance, namely, of that which we take for unity in meaſuring the capacities of bodies.

Comparing this definition with that of denſity (art. 10.) it will appear that the two terms *denſity* and *ſpecific gravity* expreſs the ſame thing under different aſpects; the former being more accurately reſtrained to the greater or leſs vicinity of particles, the latter to a greater or leſs weight in a given volume; hence as weight depends upon the cloſeneſs of particles, the denſity varies as the ſpecific gravity, and the terms may in moſt caſes be indiſcriminately uſed. The ſpecific gravities of fluids are uſually conſidered without any regard to the empty ſpaces between the particles; though, if the particles of fluids are ſpherical, the vacuities make at leaſt $\frac{1}{4}$ of the whole bulk. But it is ſufficient that we know preciſely in what ſenſe the ſpecific gravity of fluids is underſtood.

CHAP. I.

On the Preſsure of non-elastic Fluids.

384. PROP. *The upper ſurface of a homogeneous heavy fluid in any veſſel, or any ſyſtem of communicating veſſels, is horizontal.*

This is a matter of univerſal experience; and, as it is eaſily obſerved, may be taken for the diſtinguiſhing property of fluids. Thus, if ABCDEF (fig. 7. pl. XV.) be a veſſel in which the branches CDH, EFG, have a free communication with the part AB; then, if water, or mercury, or wine, or any other fluid commonly reckoned non-elaſtic, be poured in, either at A, C, or E, and when the whole is at reſt, the ſurface of the fluid ſtands at IK in the larger trunk; if the line LIKM be drawn parallel to the horizon the ſurface of the fluid will ſtand at L in the branch EF, and at M in the branch CD; and this whatever are the inclinations of thoſe branches, or the angles at F and D, G and H.

Remark. This is uſually explained by ſaying, that, ſince the parts of a fluid are eaſily moveable in any direction, the higher particles will deſcend by reaſon of their ſuperior gravity, and raiſe the lower parts till the whole comes to reſt in a horizontal plane. Now what is called the horizontal plane is, in fact, a portion of a ſpherical ſurface whoſe centre is the centre of the earth: hence it will follow, that *if a fluid gravitate towards any centre it will diſpoſe itſelf into a ſpherical figure, the centre of which is the centre of force.*

385. PROP. *If a fluid, conſidered without weight, is contained in any veſſel whatever, and an orifice being made in the veſſel, any preſſure whatever be applied thereto, that preſſure will be diſtributed equally in all directions.*

Through any point N (fig. 7.) taken at pleaſure below the ſurface of the fluid LIKM, imagine the horizontal plane PNOQ to paſs. It is obvious the *weight* of the fluid contained in the veſſel below PNOQ contributes nothing to the ſupport of the columns LP, IO, MQ; ſo that the equilibrium would obtain in like manner if the fluid contained in that part of the veſſel below PNOQ had loſt its weight entirely. We may, therefore, regard this fluid as being ſolely a mean of communication between the columns LP, IO, and MQ; in ſuch manner that it will tranſmit the preſſure reſulting from the columns LP, MQ, to the column IO, and reciprocally. If, now, inſtead of the columns LP, IO, MQ, of the fluid, piſtons were applied to the ſurfaces P, NO, and Q, and were ſeparately urged by preſſures reſpectively equal to the preſſures of the columns LP, IO, MQ, the equilibrium would manifeſtly obtain in like manner. Or, if a preſſure equal to that of the column MQ be applied at Q, while the columns LP, IO, remain, the equilibrium will ſtill obtain; and this whatever are the directions of the ſeveral branches, and their ſinuoſities at D, F, &c. whence the propoſition is evident.

Cor. Not only is the preſſure tranſmitted equally in all directions, but *it acts perpendicularly upon every point of the ſurface of the veſſel which contains the fluid.*

For if the preſſure which acts upon the ſurface were not exerted perpendicularly, it is eaſy to ſee that it could not be entirely annihilated by the reaction of that ſurface; the ſurplus of force would, therefore, occaſion freſh action upon the particles of the fluid, which muſt of conſequence be tranſmitted in all directions, and thus neceſſarily occaſion a motion in the fluid: that is, the fluid could not be at reſt in the veſſel, which is contrary to experience.

Cor. 2. Hence, alſo, *if the parts of a fluid contained in any veſſel* ABCD (fig. 6. pl. XV.), *open towards the part* AB, *are ſolicited by any forces whatever, and remain notwithſtanding in equilibrio, theſe forces muſt be perpendicular to the ſurface* AB. For the equilibrium would obtain in like manner if a cover or a piſton of the ſame figure as the ſurface AB were applied to it; and it is manifeſt that, in this latter caſe, the forces which act at the ſurface, or their reſultant, muſt be perpendicular to that ſurface.

Cor. 3. If, therefore, the forces which act upon the particles of the fluid are thoſe of gravity, we ſhall ſee that the direction of gravity is neceſſarily perpendicular to the ſurface of a tranquil fluid: conſequently, *the ſurface of a heavy fluid muſt be horizontal to be in equilibrio, whatever may be the figure of the veſſel in which it is contained.*

Thus, again, we ſee the reaſon of the fact ſtated in art. 383.

Cor. 3. If a veſſel, as ABCD (fig. 6. pl. XV.) cloſed throughout, except at a ſmall orifice O, is full of a fluid without weight; then if any preſſure be applied at O, the reſulting preſſure on the plane ſurface or bottom CD will neither depend upon the quantity of fluid in the veſſel nor on its ſhape; but, ſince the preſſure applied at O is tranſmitted equally in all directions, the actual preſſure upon CD will be to the preſſure at O as the area of CD to that of the orifice.

Cor. 4. In the ſame manner will the preſſure applied at O be exerted in raiſing the top AB of the veſſel; ſo that if the top be a plane, of which O forms a part, the vertical preſſure tending to force AB upwards will be to the force applied at O as the ſurface AB to the area O.

386. Prop. *The preſſure of a fluid on the horizontal baſe of a veſſel in which it is contained is as the baſe and perpendicular altitude, whatever be the figure of the veſſel that contains it: the upper ſurface of the fluid being ſuppoſed horizontal.*

Let any horizontal plane GH (fig. 10. pl. XV.) be ſuppoſed drawn, and conceive the fluid contained in the part GCDH of the veſſel to be void of weight; then is it evident from cor. 3. of the foregoing propoſition, that any vertical filament whatever, EI of the heavy fluid ABHG, exerts at the point I a preſſure which is diſtributed equally through the fluid GCDH; and that this preſſure acts equally upwards, to oppoſe the action of each of the other filaments which ſtand vertically above GH; therefore, the filament EI alone keeps in equilibrium all the other filaments of the maſs AGHB: conſequently, the maſs GCDH being ſtill ſuppoſed without weight, there will not reſult any other preſſure on the bottom CD than that of a ſingle filament EI, which, being tranſmitted equally to all the points of CD, will make the preſſure upon CD to that upon the baſe I of the filament EI as the area of CD to the area I. If, therefore, we imagine (fig. 9. pl. XV.) a heavy fluid contained in ACB to be divided into horizontal laminæ, the upper lamina will communicate to the bottom CD no other action than would be communicated by the ſingle filament ab: and the ſame thing obtaining with reſpect to each lamina, the bottom, therefore, is preſſed in the ſame degree as it would be by the combined operation of the filaments ab, bc, cd, &c. Whence, as this preſſure is tranſmitted equally to all the points of CD, it will be equal to the product of CD into the ſum of the preſſures which the filaments ab, bc, cd, are capable of exerciſing on the ſame point, or it will be proportional to $\mathrm{CD} \times (ab+bc+cd+$ &c.)

Cor. 1. Hence, *if the fluid contained in the veſſel* ABDC *is homogeneous, the preſſure on the bottom* CD *will be expreſſed by* $\mathrm{CD} \times \mathrm{EC}$; *and will be meaſured by the weight of the priſm or cylinder whoſe baſe is* CD *and height* EC.

Cor. 2. Hence, alſo, *when the heights are equal, the preſſures (of the ſame fluid) are as the baſes: when the baſes are equal the preſſures are as the heights: when both heights and baſes are equal* *the preſſures on the horizontal bottoms are equal in all, however irregular the ſhape and different the capacities of the veſſels may be.*

Cor. 3. *In different veſſels containing different fluids, the preſſures are as the areas of the bottoms × depths × ſpecific gravities.*

Cor. 4. *If the laminæ* AH, GK, &c. *be of different denſities, or ſpecific gravities,* D, d, δ, &c. *then will the preſſure on the bottom* CD *be equal to* $CD \times (ab . D + bc . d + cd . \delta +$ &c.).

SCHOLIUM.

387. Upon the two principles that fluids preſs equally in all directions, and in proportion to their perpendicular depths, depends the explanation of the circumſtance known by the title of the *hydroſtatic parad x*, which is this: *any quantity of water or other fluid, how ſmall ſoever, may be made to balance and ſupport any quantity or any weight, however great:* a circumſtance which has been converted to a uſeful purpoſe in the conſtruction of ſome machines. (See Bramah's *Machine*, vol. ii.) A well-known contrivance to illuſtrate this principle is the *hydroſtatic bellows.* It conſiſts of two thick boards EF, CD, (fig. 8. pl XV.) about 16 or 18 inches diameter, covered or connected firmly with pliable leather round the edges, to open and ſhut like common bellows, but without valves; but there is a pipe AB about 3 feet high fixed into the bellows at B. Now let water be poured into the pipe at A, and it will run into the bellows, gradually ſeparating the boards by raiſing the upper one. Then, if ſeveral weights (three hundred weights, for inſtance) be laid upon the upper board, the water being poured in at the pipe till it be full, will ſuſtain all the weights, though the water in the pipe ſhould not weigh a quarter of a pound. For the narrower the pipe the better (beyond certain limits), provided we make it long enough, the proportion being always this:

As the area of the orifice or ſection of the pipe,
To the area of the bellows board, FE:
So is the weight of water in the pipe, AG,
To the weight it will ſuſtain on the board.

For the fluid at B, the bottom of the tube, is preſſed with a force varying as its altitude AB: and this preſſure is communicated horizontally to all the particles in the ſpace FE, and then diſtributed equally throughout the fluid in the bellows: conſequently, the preſſure upward at FE is equal to the weight of a cylinder of the fluid whoſe baſe is FE and altitude AB; while the actual weight of water borne up is only that of the cylinder whoſe baſe is FE and height BG; and hence no weights laid upon CD that do not exceed the weight of a cylinder of the fluid whoſe baſe is EF and altitude AG will diſturb the equilibrium.

388. **PROP.** *If two immiſceable fluids are included in a bent tube, and balance each other, their perpendicular altitudes, eſtimated from an horizontal plane drawn through the common ſurface where they are in contact, will be reciprocally as their ſpecific gravities.*

Let ABCD (fig. 1. pl. XVI.) be ſuch a bent tube, its form and dimenſions being arbitrary; and let the common ſurface of the two immiſceable fluids be GH; one fluid occupying the ſpace EFHG, the other the ſpace GHBCKI. Let the ſpecific gravity of the fluid in EFHG be s, that of the other S. Through the ſurface GH draw the horizontal plane GHLOM, then it is manifeſt (art. 384.) that the part GHBCML is naturally in equilibrio: in order, therefore, that the equilibrium may exiſt in the whole, the preſſures exerted upon GH by the fluids contained in EFHG, IKML, muſt be equal. Now (art. 386. cor. 3.) the former of theſe preſſures is denoted by $GH \times FH \times s$, and the latter by $GH \times NO \times S$. Conſequently $GH \times FH \times s = GH \times NO \times S$, or $FH \times s = NO \times S$; whence flows the propoſition, *i. e.* $FH : NO :: S : s$.

SCHOLIUM.

389. Before we commence the inveſtigation of the preſſure of fluids on oblique and curvilinear ſurfaces, we may juſt remark with reſpect to preſſures upon the horizontal bottoms of veſſels, that it is neceſſary to diſtinguiſh between the preſſure which the plane CD (fig. 6. pl. XV.) would ſuſtain as ariſing from the fluid, and that which it would have to ſuſtain if it carried the veſſel. If the bottom CD were detached from the veſſel, in order to prevent the eſcape of the water there, the bottom CD muſt be preſſed upwards with a force equal to the weight of the cylinder CDEF of the fluid: but if we would ſupport the veſſel, it will require a force equal to the weight both of the veſſel and the fluid it contains. Thus, when the veſſel is narroweſt at bottom it will require more force to ſupport the veſſel than to keep its bottom from falling: while, if the veſſel is wideſt at bottom, it may be ſupported with a leſs effort than would be neceſſary to prevent the bottom from ſeparating from the ſides of the veſſel. But the preſſure of the fluid on the bottom of an upright priſmatic veſſel is *equal* to its weight.

390. **PROP.** *Any plane ſurface immerſed in a heavy fluid, of which the upper ſurface is horizontal, is perpendicularly preſſed with a force equal to the weight of a column of that fluid, having the ſurface preſſed for its baſe, and the depth of its centre of gravity under the ſurface of the fluid for its altitude.*

Let ABCD (fig. 10. 11. pl. XV.) be a vertical ſection of a veſſel terminated by ſurfaces either plane or curved, and any way inclined to the horizon; and let the veſſel be filled with a fluid whoſe upper ſurface interſects the ſection ABCD in the horizontal line AB. If $GHhg$ be an indefinitely thin lamina of the fluid, we may conſider it abſtractedly from its weight, and then conceive this lamina as preſſed by the ſuperior fluid. Now this preſſure is diſtributed equally through all the particles of the lamina, and acts perpendicularly and equally upon all the points of the faces Gg, Hh: hence, becauſe this force is the ſame as would be occaſioned by the filament EI alone, the preſſure which is exerted perpendicularly upon Gg will be expreſſed by $Gg \times EI$: and the ſame will manifeſtly obtain, if, inſtead of regarding Gg as an evaneſcent right line, we conſider it as an evaneſcent ſurface. Therefore, in general, the preſſure which is exerted perpendicularly upon any evaneſcent ſurface, by a heavy homogeneous fluid, is eſtimated by the continual product of that ſurface, its diſtance from the horizontal ſurface, and the ſpecific gravity of the fluid.

Hence it will follow, that the total preſſure exerted upon any plane ſurface whatever, whether vertical or oblique, is equal to the product of the ſpecific gravity into the ſum of the products of the evaneſcent parts of this ſurface into their reſpective diſtances from the upper ſurface of the fluid: but by the nature of the centre of gravity (art. 108.) the ſum of theſe latter products is equal to the product of the whole ſurface into the diſtance of its centre of gravity from the horizontal ſurface of the fluid: ſo that the whole preſſure will be denoted by the continual product of the ſurface preſſed, the diſtance of its centre of gravity from the upper ſurface, and the ſpecific gravity of the fluid; which is the propoſition in other words.

Cor. 1. *The entire lateral preſſure of a veſſel whoſe ſides are perpendicular to the baſe is equal to the weight of the fluid contained in a rectangular priſm, whoſe altitude is that of the fluid, and baſe is a parallelogram, one ſide of which is equal to the altitude of the fluid, and the other to the ſemiperimeter of the veſſel.*

Cor. 2. *The preſſure againſt one ſide of a cubical veſſel filled with a fluid is equal to half the preſſure againſt the bottom. And the whole preſſure againſt the ſides and bottom is equal to three times the weight of the fluid in the veſſel.*

Cor. 3. *If* ABCD, CDEF (fig. 2. pl. XVI.), *are two rectangles whoſe common breadth is* CD, *ſtanding vertically in a fluid, whoſe upper ſurface is* SS′, *then will the preſſures upon the rectangles* ABCD *and* CDEF *be as* AC^2 *and* $AE^2 - AC^2$.

For if G and g be the reſpective centres of gravity of the two rectangles, we ſhall have preſſure upon ABCD : preſſure upon CDEF :: $ABCD \times IG : CDEF \times Ig :: AC \times \frac{1}{2}AC : CE \times$

$(\mathrm{AC} + \tfrac{1}{2}\mathrm{CE}) :: \mathrm{AC} \times \tfrac{1}{2}\mathrm{AC} : (\mathrm{AE} - \mathrm{AC}) \times \tfrac{1}{2}(\mathrm{AE} + \mathrm{AC}) :: \mathrm{AC}^2 : \mathrm{AE}^2 - \mathrm{AC}^2$.

COR. 4. *Hence, if* AE *be to* AC *as* $\sqrt{2}$ *to* 1, *the preſſures upon* ABCD *and* CDEF *will be equal.*

391. **DEF.** The *centre of preſſure* is that point of a ſurface againſt which any fluid preſſes, through which the reſultant of all the individual preſſures paſſes, or to which, if a force equal to the whole preſſure were applied in a contrary direction, it would keep the ſurface at reſt.

392. PROP. *If a plane ſurface which is preſſed by a fluid be produced to the horizontal ſurface of it, and their common interſection be made the axis of ſuſpenſion, the centre of percuſſion will be the centre of preſſure.*

Let ABCD (fig. 3. pl. XVI.) be the horizontal ſurface of the fluid which preſſes upon the plane EIF: produce this plane till it meets the ſurface of the fluid in the line MN; and let O be the centre of preſſure. From any point p of the ſurface preſſed draw the vertical pm, meeting the horizontal ſurface in m; and in the plane CB draw from m the line mM perpendicular to MN. The preſſure upon p (art. 390.) is as $p \cdot pm$, and its effect to turn the plane about MN is as $p \cdot pm \cdot p\mathrm{M}$, by the nature of the lever: alſo, its effect to turn the plane about NI is as $p \cdot pm \cdot \mathrm{MN}$. In like manner, if the plane EIF be ſuppoſed to revolve about the axis MN, and to ſtrike an obſtacle at O, the percuſſive force of the particle p, by which it endeavours to move the plane about MN, will be as $p \cdot pm^2$, or as $p \cdot pm \cdot p\mathrm{M}$; and its force to turn the plane about NI will be as $p \cdot p\mathrm{M} \cdot \mathrm{MN}$, or as $p \cdot pm \cdot \mathrm{MN}$. And the like correſpondence between the percuſſive and the preſſive forces, of any other particles in the plane EF, may be ſhewn in the ſame manner. Conſequently, the percuſſive forces of the whole of the particles, whereby they endeavour to move the plane in the two directions, have the ſame relation as the forces of preſſure, and therefore the centres of preſſure and percuſſion are coincident.

COR. 1. Hence, the theorems given for the centre of percuſſion in arts. 318. 319. may be applied to the determination of the centre of preſſure.

COR. 2. Hence alſo appears the miſtake of thoſe who aſſert that the centres of percuſſion and of preſſure do not coincide. They are the centres of *oſcillation* and of *preſſure* which do not coincide univerſally. See art. 320.

SCHOLIUM.

393. To adopt the general formulæ for the centre of percuſſion to the inſtance of the centre of preſſure, it will be proper to make a ſlight change in the notation. Let d be the

diſtance from MN of any particle, or of any horizontal lamina of the fluid in contact with the plane EIF; let l be the length of ſuch lamina, and $\dot{d}$ its depth (being conſidered as evaneſcent), then will $l\dot{d}$ be its area; alſo, let δ be the diſtance of the centre of gravity of the plane EIF from the line MN, where that plane interſects the ſurface; and let the horizontal diſtance of $l\dot{d}$ from the line NI be denoted by h: then, with reſpect to the line MN the formula (art. 318.) will become $\frac{\int l\,d^2\,\dot{d}}{\int l\,d\,\dot{d}}$ or $\frac{\int l\,d^2\,\dot{d}}{\text{EIF}\times\delta}$; and with reſpect to the line NI, the formulæ (art. 319.) will become $\frac{\int l\,\text{H}\,d\,\dot{d}}{\int l\,d\,\dot{d}}$ or $\frac{\int l\,\text{H}\,d\,\dot{d}}{\text{EIF}\times\delta}$.

A few examples are here added to illuſtrate the uſe of theſe theorems.

I. Let a reſervoir which contains water, or any other fluid (its ſpecific gravity being s), have one of its ſides plane and vertical: if we imagine a right line drawn vertically upon this plane, its length being λ, and the diſtance of its centre of gravity from the ſurface of the fluid δ, the preſſure exerted upon this line will be $s\delta\lambda$ (art. 390.). Let the diſtance of the ſuperior extremity of the line λ from the ſurface of the fluid be a, and make $a+\lambda=e$, ſo ſhall e be the diſtance of the lower extremity of the line λ from the horizontal ſurface of the fluid. Then, to find the centre of preſſure of the line λ, we take the complete fluent of the expreſſion $\frac{\int l\,d^2\dot{d}}{\int l\,d\,\dot{d}}$ (l being conſtant) which is $\frac{2(d^3-a^3)}{3(d^2-a^2)}$; or, when the variable d becomes equal to e we have $\frac{2(e^3-a^3)}{3(e^2-a^2)}$, for the diſtance of the centre of preſſure from the horizontal ſurface of the fluid. When one of the extremities of the line λ coincides with this ſurface we have $a=0$, and $e=\lambda$, and the diſtance of the centre of preſſure becomes $\frac{2}{3}\lambda$.

II. If upon the vertical line λ we conſtruct a rectangle, of which the horizontal baſe is h, the whole preſſure upon it will be $sh\delta\lambda$, and the diſtance of the centre of preſſure from the ſurface of the fluid will be $\frac{2(e^3-a^3)}{3(e^2-a^2)}$ the ſame as we have juſt found. And this centre muſt evidently be found upon the vertical line which divides the parallelogram into two equal parts.

If the upper horizontal ſide of the parallelogram coincide with the ſurface of the fluid (as the ſide AB of the parallelogram ABCD, fig. 2. pl. XVI.) its tendency to turn about its baſe will be $\frac{1}{2} s h \lambda^2 \times \frac{1}{3}\lambda = \frac{1}{6} s h \lambda^3$, and its tendency to turn about one of its vertical ſides will be $\frac{1}{2} s h \lambda^2 \times \frac{1}{2} h = \frac{1}{4} s h^2 \lambda^2$: thus the firſt of theſe efforts will be to the ſecond as $\frac{1}{6} s h \lambda^3$: $\frac{1}{4} s h^2 \lambda^2$, or as $2\lambda : 3b$; which reduces to 2 : 3, when the rectangle becomes a ſquare.

III. To determine the centre of preſſure in the triangle CAB (fig. 4. pl. XVI.) whoſe ſide AB is horizontal, and which is placed vertically in a fluid whoſe horizontal ſurface is SS′. Through C, the ſummit of the triangle, draw the vertical line SCP, alſo the line CQ biſecting the baſe AB, and any line TR parallel to AB. Make $CP = \lambda$, $AB = h$, $CS = a$, $SP = a + \lambda = e$, the diſtance of the horizontal line in which lies the centre of gravity of the triangle from $S = \delta$ (that is, if $Cg = \frac{2}{3} CQ$, the diſtance between SS′ and $g = \delta$), the angle $PCQ = k$, $SM = d$, $TR = l$, $CM = c = d - a$.

The whole preſſure upon this triangle will be repreſented by $s \times \frac{1}{2} b \lambda \times \delta = \frac{1}{2} s \delta \lambda b$. And to find the depth of the centre of preſſure below SS′ we muſt find the fluent of the expreſſion $\frac{\int l d^2 \dot{d}}{ABC \times \delta}$, or $\frac{\int l d^2 \dot{d}}{\frac{1}{2} \delta \lambda h}$. In order to this we have CP : AB :: CM : TR, or $\lambda : b :: c : l = \frac{ch}{\lambda} = \frac{h}{\lambda}(d - a)$, which ſubſtituted for l in $\int l\, d^2 \dot{d}$, gives $\frac{h}{\lambda} \int (d^3 - a d^2)\, \dot{d} = \frac{h}{\lambda}(\frac{1}{4} d^4 - \frac{1}{3} a d^3) + C$. The conſtant quantity C may be determined by conſidering that the fluent muſt vaniſh at the point C, that is, when $l = 0$, or when $d - a = 0$, that is, when $d = a$. Hence, making the ſubſtitution, we have

$$\frac{h}{\lambda}(\tfrac{1}{4} a^4 - \tfrac{1}{3} a^4) + C = 0, \text{ whence } C = \frac{h}{12\lambda} a^4$$

and the correct fluent is $\frac{h}{\lambda} \frac{1}{4} d^4 - \frac{1}{3} a d^3 + \frac{1}{12} a^4)$.

Hence then, for the whole triangle, in which SM = SP, or $d = e$, we ſubſtitute e for d in the fluent, and divide by the denominator $\frac{1}{2} \delta \lambda b$ of the general expreſſion, which gives for the depth of the centre of preſſure of the whole triangle the value $\frac{\frac{1}{4} e^4 - \frac{1}{3} a e^3 + \frac{1}{12} a^4}{\frac{1}{2} \delta \lambda^2}$, which is evidently independent of the baſe b of the triangle.

When the vertex of the triangle is at the horizontal ſurface of the fluid $e=\lambda$, $a=0$, $\delta=\frac{2}{3}\lambda$, and the expreſſion becomes $\frac{3}{4}\lambda$.

IV. To determine the diſtance of the centre of preſſure from the vertical line CP. Here we muſt take the formulæ $\frac{\int l\,\mathrm{H}\,d\,\dot{d}}{\int l\,d\,\dot{d}}$, in which H repreſents $\mathrm{MN}=c$ tang. $k=(d-a)$ tang. k. Subſtituting this value of H, and $\frac{h}{\lambda}(d-a)$ for l, its value found above, we have $\int l\,\mathrm{H}\,d\,\dot{d}=\frac{h\,\text{tang.}\,k}{\lambda}\int(d^3-2\,a\,d^2+a^2d)\,\dot{d}=\frac{h\,.\,\text{tang.}\,k}{\lambda}(\frac{1}{4}d^4-\frac{2}{3}a\,d^3+\frac{1}{2}a^2d^2)+\mathrm{C}$. In this caſe, alſo, the fluent vaniſhes when $d=a$; whence we have $\frac{h\,.\,\text{tang.}k.}{12\,\lambda}a^4+\mathrm{C}=0$, and $\mathrm{C}=-\frac{h\,.\,\text{tang.}\,k}{12\,\lambda}a^4$:

ſo that the correct fluent is $\frac{h\,.\,\text{tang.}\,k}{\lambda}(\frac{1}{4}d^4-\frac{2}{3}a\,d^3+\frac{1}{2}a^2d^2-\frac{1}{12}a^4)$.

Hence, making $d=e$, and dividing by $\int l\,d\,\dot{d}=\frac{1}{2}\delta\,\lambda\,h$, we have for the diſtance ſought, tang. k × $\frac{\frac{1}{4}e^4-\frac{2}{3}a\,e^3+\frac{1}{2}a^2e^2e-\frac{1}{12}a^4}{\frac{1}{2}\delta\,\lambda^2}$.

1. When the triangle is iſoſceles the angle $k=0$, and the preceding value vaniſhes, as it obviouſly ought to do; for in that caſe the triangle is ſymmetrical with reſpect to CP, and the preſſures are in equilibrio about that line.

2. When the triangle is right-angled, and has its baſe or horizontal ſide $\mathrm{AB}=h$ one of the ſides about the right-angle, that is, when CA coincides with CP, then is $h=2\,\lambda\,.$ tang. k, or tang. $k=\frac{h}{2\,\lambda}$, which transforms the expreſſion for the centre of preſſure's diſtance from SP

to $\frac{h}{\delta\,\lambda^3}(\frac{1}{4}e^4-\frac{2}{3}a\,e^3+\frac{1}{2}a^2e^2-\frac{1}{12}a^4)$.

3. Finally, when the vertex of the triangle is at the ſurface SS′ of the fluid, $a=0$, $e=\lambda$, $\delta=\frac{2}{3}\lambda$, and the expreſſion becomes $\frac{3}{4}\lambda\,.$ tang. k; which, for the right-angled triangle, reduces to $\frac{3}{8}h$.

4. When the triangle has its vertex at the horizontal ſurface of the fluid, the tendency to turn about the baſe is to that to turn about the perpendicular let fall from the vertex upon the baſe as 1 to 3 tang. k; and, in the caſe of the right-angled triangle, as $2\,\lambda$ to $3\,h$: which, when the legs of the triangle are

equal, reduces to the ratio of 2 to 3. As in the caſe of the rectangle and ſquare.

If C and AB lie in different ſides of SS′, that is, if part of the triangle is out of the fluid, no other change will be neceſſary in the preceding expreſſions than a change of ſigns in thoſe terms which contain *uneven* powers of a. So that this ſimple transformation will accommodate the preceding general theorems to the caſe of trapezoids.

V. If the radius of a circle be r, and δ the diſtance of its centre below the ſurface of the fluid in the plane of the circle, then is the diſtance of the centre of preſſure from the upper ſurface, in the ſame plane, expreſſed by $\delta + \frac{r^2}{4\delta}$; which when the upper part of the circle juſt touches the ſurface becomes $\frac{5}{4}r$. The inveſtigation of this is left for the ſtudent's exerciſe.

It is hardly neceſſary to remark, that the reſults of the operations in this ſcholium may be ſafely applied in the analogous enquiries relative to the centre of percuſſion.

394. PROP. *To enquire generally into the reſults of all the preſſures, upon any ſurface plane or curved, regular or irregular, both in the vertical and the horizontal direction.*

This is uſually performed by foreign authors, by means of the calculus of partial differences. But another mode of inveſtigation is purſued here, as we think it likely to carry more conviction to the mind of a learner. Had the vertical preſſure alone been the object of inveſtigation, it might be determined far more conciſely.

I. Whatever the figure of a body may be, we may always imagine it to conſiſt of an aſſemblage of an infinite number of indefinitely ſmall laminæ reſpectively parallel, and the ſurface of each lamina as an aſſemblage of many trapezoids, their number indeed being infinite likewiſe, when the ſurface in contact with the fluid is curved. Hence, to eſtimate the reſult of the preſſures of a fluid, whether upon the interior ſurface of a veſſel which contains it, or upon the exterior ſurface of a body immerſed in it, we muſt firſt eſtimate the reſult of the preſſure upon the ſurface of a trapezoid whoſe height is evaneſcent.

Conceive, therefore, *a b c d* (fig. 5. pl. XVI.) a trapezoid whoſe two parallel ſides are *a b*, *c d*, and whoſe height HK is infinitely ſmall with regard to thoſe ſides. To reſiſt the preſſure upon this ſurface we muſt apply to the centre of gravity *g* of the trapezoid a force P perpendicularly to its plane, the value of which is expreſſed by the product of the ſurface of the trapezoid into the diſtance Gg of its centre of gravity from the horizontal ſurface ABCD of the fluid.

To determine the effect of this force P both in the vertical and the horizontal direction, conceive a vertical plane cdFE to paſs through the line cd, and a horizontal plane abEF through the line ab; the common interſection of theſe planes being EF: then, having drawn the vertical lines cE, dF, meeting the horizontal plane in E and F, join bE and aF: again, through the direction Pg of the force P, conceive a plane KIH perpendicular to cd, and of which HgK and HI are the interſections with the two planes $abcd$ and FEcd: this plane will be perpendicular to the planes $abcd$ and FEcd, becauſe cd is their common interſection: finally, from the point K, where ab and Hg meet, draw KI perpendicular to the plane FEcd, this line muſt neceſſarily be perpendicular to HI.

The conſtruction effected, reſolve the force P (repreſented by gN) into two others which are alſo in the plane KIH, of which the horizontal one is gL, and the vertical one gM. Calling theſe component forces L and M, we have, by the nature of the parallelogram of forces $P : L : M :: gN : gL : gM :: gN : gL : LN :: HK : HI : IK$, the triangles gLN, HIK, being evidently ſimilar. Multiplying the three latter terms by $\frac{ab+cd}{2}\times Gg$, which will not change the ratio, we ſhall have $P : L : M :: HK\times\frac{ab+cd}{2}\times Gg : HI\times\frac{ab+cd}{2}\times Gg : IK\times\frac{ab+cd}{2}\times Gg$.

Now it may be obſerved, 1ſt. That $HK\times\frac{ab+cd}{2}$ is the ſurface of the trapezoid $abcd$. 2dly. That ſince cE and dF are parallel, as likewiſe cd and EF, we have $cd = FE$, therefore $IK\times\frac{ab+cd}{2} = IK\times\frac{ab+EF}{2}$, which, of conſequence, is the ſurface of the trapezoid abEF. 3dly. That, becauſe the height of the trapezoid $abcd$ is evaneſcent with reſpect to the ſides ab and cdEF, which is equal to cd, may be taken both for cd and for ab; ſo that $HI\times\frac{ab+cd}{2}$ reduces to $HI\times EF$, which is the ſurface of the rectangle E$cd$F. We have, therefore, $P : L : M :: abcd\times Gg : EcdF\times Gg : aFEb\times Gg$. But we have ſuppoſed that the force P is expreſſed by $abcd\times Gg$; conſequently, the horizontal force L is denoted by $EcdF\times Gg$, and the vertical force M by $aFEb\times Gg$.

As the triangle may be conſidered as a trapezoid, of which one of the parallel ſides vaniſhes, the ſame thing, therefore, obtains for any evaneſcent triangle.

Conceiving, now, that from the angles *a*, *d*, *c*, *b*, lines are drawn to fall perpendicularly upon the plane ABCD, theſe perpendiculars will be the edges of a priſmatic fruſtrum, of which the horizontal baſe is equal to *a* FE *b*, and the inclined baſe *a b c d*; or, as *a b* and *c d* are ſuppoſed indefinitely near, the ſolidity of the priſmatic fruſtrum will not differ ſenſibly from that of the priſm which has the ſame horizontal baſe, and whoſe height is G*g*: but this latter is equal to *a* FE *b* × G*g*, which is preciſely the expreſſion above found for the vertical force M. Hence it appears, that *this force is equal to the weight of a priſmatic fruſtrum of the fluid whoſe inclined baſe is* a b c d, *and horizontal baſe the projection of* a b c d *upon the horizontal ſurface* ABCD.

II. Let us next conſider any ſolid whatever cut into an indefinite number of horizontal laminæ, ſuch as ABDE *a b d e* (fig. 6. pl. XVI.), and that perpendicularly to the centre of gravity of the ſurface of each trapezoid into which the contour of the laminæ is divided, forces are applied, each repreſented by the product of the ſurface of the correſponding trapezoid into the diſtance of its centre of gravity from the horizontal ſurface A′D′. Theſe forces are the preſſures of a heavy fluid, ſuſtained by the interior ſurface of the laminæ ABDE *a b d e* of a veſſel which contains it; they are alſo the preſſures of ſuch a fluid which would be ſuſtained by the exterior ſurface of a ſolid whoſe contour is the ſame, and which is immerſed to the ſame depth. But it is manifeſt that if each of thoſe forces P, P′ P″, &c. were decompoſed into two others, the one vertical, the other horizontal, each vertical force would be repreſented by the weight of a priſmatic fruſtrum of the fluid whoſe inclined baſe is one of the trapezoids in the contour of the lamina, and its horizontal baſe the projection of that trapezoid upon the upper ſurface of the fluid. Therefore the ſum of theſe vertical forces, or the ſingle vertical reſulting force, will be repreſented by the ſum of the weights of all thoſe priſmatic fruſtrums: and the ſame property may obviouſly be extended to every other horizontal lamina, we may conclude,

1. That, *if a veſſel, of any figure, be full of a fluid, and have over every part of the ſides and bottom a perpendicular column of the fluid reaching to the ſurface, the whole vertical preſſure of the fluid upon the bottom and ſides of that veſſel will be equal to the weight of the whole fluid.*

2. That, *if a body, as* AEDBM (fig. 7. pl. XVI.) *of which* AIBF *is the greateſt horizontal ſection, is immerſed in a fluid to any depth whatever, and if we drop the conſideration of the preſſure*

ſuſtained by the upper part AMB, *the vertical effort of the fluid to raiſe the body is equal to the weight of the volume of fluid which is compriſed between the ſurface* A′D′, *the ſurface* AIBFE, *and the convex ſurface formed by perpendiculars let fall from all the points of the perimeter* AIBF *upon the plane* A′D′; *that is, equal to the ſum of the weights of fluid in the priſm* a i b f AIBF, *and the ſpace* AIBFCE.

III. If we would now eſtimate the preſſure ſuſtained by the ſuperior ſurface AMBFI of the body, we ſhall ſee, by the ſame kind of reaſoning, that the reſult in the vertical direction tending to force the body downwards, is an effort equal to the weight of the volume compriſed between the horizontal projection *a i b f*, and the upper ſurface AMBFI of the body. If, then, from the firſt of theſe efforts we deduct the ſecond, it will appear that the body is puſhed vertically upwards, with an effort equal to the weight of a volume of the fluid equal to that of the body immerſed. We conclude therefore that, *if a body is immerſed in any fluid whatever, it will loſe (relatively) as much of its weight as is equal to the weight of the quantity of fluid it diſplaces.*

IV. With regard to the reſultant of all the vertical forces whoſe magnitude we have juſt determined, it is eaſy to ſee that it muſt paſs through the centre of gravity of the volume of fluid diſplaced. For, if we conceive this volume decompoſed into an infinite number of evaneſcent vertical filaments, the effort made by the fluid to puſh each filament vertically will be expreſſed by the weight of a quantity of fluid equal to that filament. Therefore, to obtain the diſtance of the reſultant from any vertical plane whatever, we muſt multiply the maſs of each filament (conſidered as of the ſame nature with the fluid) by its diſtance from this plane, and divide the ſum of the products by the ſum of the filaments; which is preciſely the rule that muſt be followed to find the centre of gravity of the volume diſplaced. Therefore, univerſally, *a body immerſed either wholly or in part in a heavy fluid, and at reſt, receives from the fluid preſſures which are together equivalent to a vertical force directed upwards through the centre of gravity of the fluid diſplaced by the body, and equal to the weight of a quantity of the fluid ſo diſplaced by the immerſed part of the body.*

Indeed we may readily aſſign a reaſon, *à priori*, of this: for, ſuppoſing a force acting on a body without heavineſs retains it in equilibrio when immerſed either wholly or partly in a heavy fluid; if we ſubſtitute for the immerſed part of the body, that is, for the fluid it diſplaces, an equal and ſimilar portion of the ſame fluid become ſolid (as ice, and the denſity unchanged), the equilibrium will ſtill obviouſly ſubſiſt: conſequently, the preſſure of the fluid upon the immerſed body will be altogether equal and directly oppoſed to the weight of this ſolid; and muſt, therefore, paſs through its centre of gravity in order to ſuſtain it in equilibrium.

V. It now remains for us to conſider how the horizontal forces are diſpoſed of.—If we take any one of the horizontal laminæ into which either the fluid, or the ſolid immerſed in the fluid, may be imagined to be divided, and through the ſides *a b*, *b c*, *c d*, &c. (fig. 6. pl. XVI.) of the inferior ſection conceive vertical planes to paſs, and to be terminated by the ſuperior ſection; theſe planes will form the contour of a priſm whoſe height is that of the lamina, and each face of the priſm will have (I) the meaſure of its ſurface proportional to the value of the horizontal force to which it is perpendicular. But, as all theſe faces are of the ſame altitude, their ſurfaces are proportional to their baſes *a b*, *b c*, &c. and conſequently the horizontal forces are reſpectively in the ratio of the ſides *a b*, *b c*, &c. And as the altitudes of theſe faces are evaneſcent, we may regard all theſe forces as applied in the ſame horizontal plane *a b c d e f*, and to be each reſpectively proportional to the length of the ſide, on the middle of which it acts perpendicularly. Now it has been ſhewn (art. 53.), that if any number of forces repreſented in magnitude and direction by the ſides of a polygon taken in order, act ſimultaneouſly upon the ſame point, they will be mutually deſtroyed, and the point continue at reſt: alſo (art. 85.), that when any number of forces are in equilibrio when applied to different points of a body, they are the ſame as would be in equilibrio about a ſingle point; and, ſince the directions of the ſeveral forces P, P′, P″, &c. in the preſent caſe would, if produced, form a polygon ſimilar to *a b c d e f*, the conſequences juſt referred to will apply to them likewiſe: and, in like manner, to the preſſures upon any other horizontal laminæ. Conſequently, *the efforts which reſult in the horizontal direction, from the preſſure of a heavy fluid upon the ſurface of any body immerſed in it, are mutually deſtroyed.*

SCHOLIUM.

395. From the preceding doctrine of the preſſure of fluids, an important practical maxim may be deduced. We have ſeen that in any veſſel containing a heavy fluid, the parts that are deepeſt below the ſurface ſuſtain a proportionally greater preſſure. If, therefore, we have to conſtruct an aſſemblage of vertical pipes or tubes, to elevate water or any other fluid, we may run into a ſuperfluous expence, by giving the ſame thickneſs to the material in every part. For, if the ſubſtance be uniformly thick, and the lower parts are ſufficiently ſtrong, the upper parts are, of conſequence, much thicker than neceſſary. The method ſuggeſted by theory is, while we give to

the whole aſſemblage the ſame interior diameter, to give a ſafe and ſufficient thickneſs to the material at the loweſt part, and let it gradually diminiſh to the top, in the ſame ratio nearly as the diminution in the depth of the fluid. The ſame maxim may alſo find an application in the conſtruction of ſluice-gates, dams, banks, &c. And in all ſuch caſes it is adviſeable to determine, firſt, the adequate ſtrength to reſiſt the preſſure at the greateſt depth; as, by this means, ſafety may always be enſured without any waſte of materials.

CHAPTER II.

On the Determination of the Specific Gravities of Solid and Fluid Bodies.

396. THE term *ſpecific gravity* has already been defined in art. 383. The object of the preſent chapter is to explain the principles on which the different methods of aſcertaining the ſpecific gravities of ſolids and fluids are founded, and to give ſome account of the beſt of thoſe methods. Previous to which it may be proper to make a few obſervations, naturally deduced from the definition of ſpecific gravity and the nature of bodies in general.

1. The ſpecific gravities of bodies are in the ſame proportion as their weights, when the magnitudes of the bodies are equal.
2. Where the weights of the bodies are equal, the ſpecific gravities are inverſely as their magnitudes.
3. When the ſpecific gravities are equal, their weights are directly as their magnitudes.
4. When neither the magnitudes nor the ſpecific gravities are equal, the weights of bodies are as their magnitudes and ſpecific gravities conjointly.

To expreſs theſe relations algebraically, let $\frac{W}{w}$ repreſent the ratio of the weights of two ſubſtances, $\frac{M}{m}$ the ratio of their magnitudes, and $\frac{S}{s}$ that of their ſpecific gravities; then will the general relation of theſe quantities be expreſſed by the equation $\frac{W}{w}=\frac{M}{m}\times\frac{S}{s}$. In eſtimating the weights, magnitudes, and ſpecific gravities of ſubſtances, ſome ſtandard quantities are always aſſumed to which other bodies are referred: the letters w, m, and s, may repreſent theſe ſtandards, and each of them might be aſſumed equal to 1; but ſuch aſſumption would not correſpond with the meaſures and weights now in uſe. It will, therefore, be more eligible that m ſhould repreſent the magnitude of ſome known meaſure which may be aſſumed for unity, as a cubic inch, a cubic foot, &c. and s any convenient number in the geometrical progreſſion 1, 10, 100, 1000, &c. Now it has been found, that the denſity of rain-water is more nearly uniform in different circumſtances of time and place, than any other ſubſtance, whether ſolid or fluid; and by a fortunate coincidence it happens, that the weight of a cubic foot of rain-water is exactly 1000 ounces avoirdupoiſe. If, then, we make $w = 1000$, $m = 1$, and $s = 1000$, we ſhall obtain $W = M \times S$, that is, *the weight of any body in avoirdupoiſe ounces will be equal to the product of the magnitude in cubic feet, into the ſpecific gravity taken from that ſcale in which the ſpecific gravity of rain-water is* 1000. Hence it appears, that a knowledge of the ſpecific gravities of homogeneous bodies will enable us to determine their weight, without actually weighing them, provided we can aſcertain their magnitudes: and converſely, however irregular the ſhape of bodies may be, if we know their weights and ſpecific gravities we may readily determine their magnitudes in feet, namely, *by dividing the weight in avoirdupoiſe ounces, by the ſpecific gravity, or by the weight of a cubic foot in avoirdupoiſe ounces.*

But in philoſophical ſubjects the weights of bodies being for the moſt part ſmall, are eſtimated in Troy ounces or grains, the magnitudes being referred to a cubic inch as a ſtandard. Now a troy ounce is to an avoirdupoiſe ounce as 480 to $437\frac{1}{2}$: conſequently $\frac{1000}{1728} \times \frac{437\frac{1}{2}}{480} = \cdot52746$ of an ounce troy, or $253\cdot181$ grains, the weight of a cubic inch of water. And hence the magnitude of a ſolid, eſtimated in cubic inches, is $= \frac{W}{253\cdot181\,S}$, when the weight W is in grains; or $= \frac{W}{\cdot52746\,S}$, when the weight is known in troy ounces. And converſely, the weight eſtimated in grains $= 253\cdot181 \times M \times S$, when the magnitude M is in cubic inches; and, if eſtimated in troy ounces, $W = \cdot52746 \times M \times S$. In all theſe caſes S being the ſpecific gravity, in terms correſponding to 1000, for that of rain-water.

Hence alſo we may ſhew how to determine very accurately the diameter D of any ſmall ſphere, whoſe ſpecific gravity is S (to that of water 1000), its weight W being known in grains. For the content of a ſphere whoſe diameter is 1 being $\cdot523598$, we have $1 : 0\cdot523598 :: 253\cdot181$ gr. (the weight of a cubic inch of water) : $132\cdot5648$ gr. the weight of a ſphere of water whoſe diameter is 1 inch. Therefore, ſince the weights are as the magnitudes and ſpecific gravities conjointly; and the magnitudes of ſpheres are as the cubes of their diameters, we have $132\cdot5648 \,.\, D^3 \,.\, \frac{S}{1000} = W$; whence we find $D = 1\cdot961208 \sqrt[3]{\frac{W}{S}}$.

After a manner not very widely different, may various other uſeful rules and theorems, applicable to the admeaſurement of bodies either regular or irregular, be eaſily deduced. What is done above is intended chiefly as a ſpecimen of the method; but it ſerves at the ſame time to ſhew the importance of an acquaintance with the ſpecific gravities of different ſubſtances. We now proceed to exhibit the moſt uſeful propoſitions in this branch of our ſubject.

397. PROP. *A body immerſed in a fluid will, when left to itſelf, ſink, if its ſpecific gravity be greater than that of the fluid: it will riſe to the ſurface and float there, if its ſpecific gravity be leſs than that of the fluid: but, if the ſpecific gravities of the ſolid and fluid be equal, the body will reſt in any part wherever it happens to be placed.*

1. For the body endeavours to deſcend by its own weight, and is ſupported by a force equivalent to the weight of an equal bulk of fluid, or of as much fluid as will fill the ſpace occupied by the body. If, therefore, the body be heavier than the fluid, bulk for bulk, its weight will be greater than the upward preſſure of the fluid which is to counteract it, and, conſequently, this latter preſſure is not ſufficient to prevent the body from ſinking.

2. If the body be ſpecifically lighter than the fluid, its preſſure downwards will be leſs than the upward preſſure of the water at the ſame depth; conſequently, in this caſe alſo the greater force will overcome the leſs, and the ſurplus cauſe the body to riſe.

3. When the body and the fluid are of the ſame ſpecific gravity, equal maſſes of each are of the ſame weight, and conſequently the force with which the body endeavours to deſcend, and the force which oppoſes the deſcent, are equal to each other; and as they act in contrary directions, the body will reſt between them, ſo as neither to ſink by its own weight, nor to aſcend by the upward preſſure of the fluid.

COR. 1. If by any contrivance the ſpecific gravity of a ſolid can be ſo varied, as to be one while *greater*, another *leſs*, and then *equal* to the ſpecific gravity of the fluid wherein it is immerſed, the body will ſink, or riſe, or remain ſuſpended, according to the variations of its ſpecific gravity.

This is the caſe in the experiment with thoſe little glaſs images (which ſome philoſophers exhibit in their lectures) which are made to aſcend or deſcend, or remain ſuſpended, at pleaſure.

Cor. 2. If a ſolid ſpecifically heavier than a fluid be immerſed to a depth which is to its thickneſs, as the ſpecific gravity of the ſolid to that of the fluid, and the preſſure of the fluid from above be removed, the body will be ſuſtained by the fluid. For the preſſure from above being removed, the body is in the ſame ſtate with reſpect to the contrary preſſure, as though with the ſame weight it filled the whole ſpace to the ſurface of the fluid; that is, as though its ſpecific gravity and that of the fluid were equal.

This ſerves for the explication of the common experiment of making lead *ſwim*, in conſequence of being fitted to the bottom of a glaſs tube.

Cor. 3. Hence alſo we ſee the meaning of the propoſition, that *all bodies when immerſed in a fluid loſe the weight of an equal bulk of that fluid* (art. 394. III.). The weight is not otherwiſe loſt than as it is ſuſtained by the action of a contrary force. And it therefore becomes obvious, why the weight of a bucket of water is not perceived while it is in the water; it is not becauſe that weight is *deſtroyed*, but becauſe it is *ſupported;* not becauſe fluids do not gravitate when they are in fluids of the ſame ſort, but becauſe there is a preſſure in a contrary direction which is preciſely equal to their gravity.

Cor. 4. The weights thus loſt, by immerging the ſame body in different fluids, are as the ſpecific gravities of the fluids.

Cor. 5. Bodies of equal weight, but different bulk, loſe in the ſame fluid weights which are reciprocally as the ſpecific gravities of the bodies, or directly as their bulks.

Cor. 6. The whole weight of a body which will float in a fluid, is equal to the weight of as much of the fluid as the immerſed part of the body diſplaces.

Cor. 7. Hence the magnitude of the whole body is to that of the part immerſed, as the ſpecific gravity of the fluid to that of the body. And, if the body be any priſm with its baſe kept horizontal, the altitude of the priſm will be to the depth immerſed, as the ſpecific gravity of the fluid to that of the body.

Cor. 8. And becauſe, when the weight of a body taken in a fluid is ſubtracted from its weight out of the fluid, the difference is the weight of an equal volume of the fluid; this, therefore, is to its weight in the air, as the ſpecific gravity of the fluid is to that of the body.

Conſequently, if W be the weight of a body in air,
W' its weight in water, or any other fluid;
S the ſpecific gravity of the body,
s the ſpecific gravity of the fluid,

we ſhall have $W-W' : W :: s : S$; whence

$$S=\frac{W}{W-W'}s, \text{ the ſpecific gravity of the body,}$$

$$\text{and } s=\frac{W-W'}{W}S, \text{ the ſpecific gravity of the fluid.}$$

So that *the ſpecific gravities of bodies are as their weights in the air directly, and their loſs in one and the ſame fluid inverſely.*

COR. 9. Hence, for two bodies connected together, or mixed together into one compound of different ſpecific gravities, we may, ſuppoſing there is no *penetration of dimenſions*, eaſily deduce the neceſſary equations.

Let the reſpective weights and ſpecific gravities be denoted thus;

$$\left.\begin{array}{l}H=\text{weight of the heavier body in air,}\\ H'=\text{weight of the ſame in water,}\end{array}\right\}S=\text{its ſpec. gravity;}$$

$$\left.\begin{array}{l}L=\text{weight of the lighter body in air,}\\ L'=\text{weight of the ſame in water,}\end{array}\right\}S'=\text{its ſpec. gravity;}$$

$$\left.\begin{array}{l}C=\text{weight of the compound in air,}\\ C'=\text{weight of the ſame in water,}\end{array}\right\}ſ=\text{its ſpec. gravity;}$$

$s=$the ſpecific gravity of water. Then,

1ſt. $(H-H')S=Hs$.
2d. $(L-L')S'=Ls$.
3d. $(C-C')ſ=Cs$.
4th. $H+L=C$.
5th. $H'+L'=C'$.
6th. $\frac{H}{S}+\frac{L}{S}=\frac{C}{ſ}$.

From which equations any of the above quantities may be found in terms of the reſt.

If the body L be of leſs ſpecific gravity than water, then L' muſt be conſidered as negative, and to find its ſpecific gravity we muſt have recourſe to a compound maſs, as C: thus, becauſe from equa. 4 and 5, $L-L'=(C-C')-(H-H')$; and from equa. 2. $S'=\frac{Ls}{L-L'}$; conſequently, ſubſtituting for $L-L'$ its value, we have $S'=\frac{Ls}{(C-C')-(H-H')}$. Or, if we deduce the value of S' from the laſt equa. we ſhall thence find $S'=\frac{SſL}{CS-Hſ}$.

398. PROP. *If a veſſel contain two immiſceable fluids (ſuch as water and mercury); and a ſolid of ſome intermediate ſpecific gravity, be immerſed under the ſurface of the lighter fluid and float on the heavier; the part of the ſolid immerſed in the heavier fluid is to the whole ſolid, as the difference between the ſpecific gravities of the ſolid*

and the lighter fluid to the difference between the ſpecific gravities of the two fluids.

Let the ſpecific gravity of the heavier fluid be s, the part of the body immerſed in it $= B$; the ſpecific gravity of the lighter fluid $= s'$, the part of the body immerſed in it $= B'$; and let the ſpecific gravity of the body be S. Let alſo the area of the horizontal ſection of the ſolid coinciding with the contiguous ſurfaces of the two fluids be $= A$, and its perpendicular diſtance from the upper ſurface of the lighter fluid be $= d$. Then (art. 394.) the preſſure againſt the ſection A from the lighter fluid will be $A\,d - B'\,s'$; which, added to the weight of the ſolid $(B+B')\,S$, will give the whole force by which this ſection of the ſolid is urged downwards. And the preſſure upward againſt the ſame ſection is $A\,d + B\,s$. But as the ſolid is ſuſtained in equilibrio by theſe contrary forces, they muſt be equal; that is, $A\,d - B'\,s' + (B+B')\,S = A\,d + B\,s$: whence we find $BS + B'S = B\,s + B'\,s'$, or $B : B' :: S - s' : s - S$; or laſtly, $B : B+B' :: S - s' : s - s'$.

Cor. 1. The analogy $B : B :: S - s' : s - S$, may be thus expreſſed in words: *As the part of the ſolid within the heavier fluid, to the part within the lighter; ſo is the difference between the ſpecific gravities of the ſolid and lighter fluid to the difference between the ſpecific gravities of the heavier fluid and the ſolid.*

Cor. 2. When the ſpecific gravity of the lighter fluid is very ſmall, compared with that of the heavier or that of the ſolid, then we may, without any ſenſible error, uſe the proportion $B : B + B :: S : s$. Thus, if the lighter fluid be air, the heavier water, and the ſolid elm-wood, their ſpecific gravities being $1\frac{2}{9}$, 1000, and 600 reſpectively; then would the ratio of $600 - 1\frac{2}{9} : 1000 - 1\frac{2}{9}$ or $588\frac{7}{9} : 988\frac{7}{9}$ be very nearly equal to that of $600 : 1000$.

Cor. 3. Hence alſo it appears, that the common rule for aſcertaining the ſpecific gravities of a fluid and a lighter ſolid by the ratio of the part immerſed to the whole, is not *accurately*, though nearly, true; becauſe the air is a heavy fluid, and therefore every ſolid floating on a fluid and in air, is in fact a ſolid of intermediate ſpecific gravity floating between two immiſceable fluids. We may, however, render the rule accurate, by *ſubducting the number expreſſing the ſpecific gravity of air from the two numbers expreſſing the ſpecific gravity of the ſolid and the fluid on which it floats;* the remainders will expreſs the actual ratio between thoſe ſpecific gravities, and may be reduced to the uſual ſtandard by a ſimple and obvious analogy.

399. Prop. *To find the ſpecific gravity of a body.*

This may be done generally by means of the *hydroſtatic balance*, which is a kind of balance contrived for the exact and eaſy determination of the weight of bodies, either in the air, or when immerſed in water; (ſee vol. II. p. 92, for a deſcription of this inſtrument). The problem may be divided into three caſes, as below.

I. *When the body is heavier than water:* weigh it both in water, and out of water, and the difference of theſe weights will expreſs the weight loſt in water. Then if B repreſent the weight of the body out of water, B' its weight in water, S its ſpecific gravity, and s the ſpecific gravity of water, the firſt equation in cor. 9. art. 397, will become $(B-B')\ S=B\ s$, whence we find $S=\frac{B\ s}{B-B'}$; conſequently the general rule in words at length is this:

As the weight loſt in water
Is to the whole or abſolute weight;
So is the ſpecific gravity of water
To the ſpecific gravity of the body.

II. *When the body will not ſink in water, being ſpecifically lighter.* In this caſe attach to it a piece of another body heavier than water, ſo that the maſs compounded of the two may ſink together. Weigh the denſer body and the compound body ſeparately, both out of the water and in it; and find how much each loſes in water, by ſubtracting its weight in water from its weight in air; and ſubtract the leſs of theſe remainders from the greater. Then uſe this proportion:

As the laſt remainder
Is to the weight of the light body in air;
So is the ſpecific gravity of water
To the ſpecific gravity of the body.

This alſo follows from cor. 9. art. 397, where it was ſhewn that $S'=\frac{L\ s}{(C-C')\mp(H-H')}$.

III. *When the ſpecific gravity of a fluid is required.* Take a piece of ſome body of known ſpecific gravity; weigh it both in and out of the fluid, and find the loſs of weight by taking the difference of theſe two; then ſay,

As the whole or abſolute weight
Is to the loſs of weight;
So is the ſpecific gravity of the ſolid
To the ſpecific gravity of the fluid.

This rule flows evidently from cor. 4. art. 397.

On the ſame principle, likewiſe, depends the uſe of the hydrometer, as will ſoon be ſeen (art. 401.).

400. PROP. *To find the reſpective weights of two known ingredients in a given compound.*

If we adopt the notation in cor. 9. art. 397. and make uſe of the 4th and 6th equations there given, namely, $H+L=C$, and $\frac{H}{S}+\frac{L}{S'}=\frac{C}{\text{ſ}}$, we ſhall thence find $H=\frac{(\text{ſ}-S')S}{(S-S')\text{ſ}}C$, and $L=\frac{(S-\text{ſ})S'}{(S-S')\text{ſ}}$. From whence we deduce the following rule in words at length:

Take the three differences of every pair of the three ſpecific gravities, viz. the ſpecific gravities of the compound, and of each ingredient; and multiply each ſpecific gravity by the difference of the other two: then ſay,

As the greateſt product
Is to the whole weight of the compound;
So is each of the other two products
To each reſpective weight of the two ingredients.

Cor. If, inſtead of finding the weights, we were to find the magnitudes M and M′ of the two ingredients, the ſpecific gravities being as above; we ſhould have the weight of $M=SM$, and the weight of $M'=S'M'$, while the weight of the compound would be $\text{ſ}\times(M\times M')$. Hence we ſhould have $SM+S'M'=\text{ſ}.M+\text{ſ}.M'$, an equation ſimilar to that in art. 398. and, conſequently, ſimilar analogies, viz. $M:M'::\text{ſ}-S':S-\text{ſ}$; and $M:M+M'::\text{ſ}-S':S-S'$.

It is here ſuppoſed that the magnitude $M+M'$ of the compound is equal to the ſum of the magnitudes of the two parts when ſeparate. But it very frequently happens that the magnitude of the mixture is leſs than this ſum; a circumſtance which is probably occaſioned by two cauſes, the different magnitudes of the conſtituent particles of the two bodies, and their chemical affinity. But this rule is, notwithſtanding, of uſe in determining the quantity of penetration or rarefaction, by comparing the computed magnitudes or denſities with thoſe which are diſcovered by obſervation.

On the Hydrometer.

401. The *Hydrometer*, or *Areometer*, is an inſtrument contrived to meaſure the ſpecific gravity of water and other fluids; and, indeed, has ſometimes been made uſe of to determine the ſpecific gravity of ſolid bodies. The general principle on which the conſtruction and uſe of the hydrometer depends is contained in cor. 4. art. 397. from which it follows that a body ſpecifically lighter than ſeveral fluids will ſerve to find out their ſpecific gravities, becauſe it will ſink deepeſt in the fluids whoſe ſpecific gravity is the leaſt. Thus, if AB (fig. 9. pl. XVI.) be a

ſmall uniform glaſs tube, hermetically ſealed, having a ſcale of equal diviſions marked upon it, with a hollow ball of about an inch in diameter at bottom, and a ſmaller ball C under it, communicating with the firſt; into the little ball is put mercury or ſmall ſhot, before the tube is ſealed, ſo that it may ſink in water below the ball, and float or ſtand upright, the diviſions on the ſtem ſhewing how far it ſinks. If this inſtrument be dipped in common water, and ſink to D, it will ſink only to ſome lower point E in ſalt water; but in port wine it will ſink to ſome higher point F, and in brandy perhaps to B.

It is evident that an hydrometer of this kind will only ſhew that one liquid is ſpecifically heavier than another; but the true ſpecific weight of any liquid cannot be determined without a calculation for this particular inſtrument, the tube of which ſhould be truly cylindrical. Beſides, theſe inſtruments will not ſerve for fluids whoſe denſities are much different.

402. Mr. Clarke conſtructed a new hydrometer, ſhewing whether any ſpirits be proof, or above or below proof, and in what degree. This inſtrument was made of a ball of copper (becauſe ivory imbibes ſpirituous liquors, and glaſs is apt to break), to which is ſoldered a braſs wire about a quarter of an inch thick; upon this wire is marked the point to which it exactly ſinks in proof ſpirits; as alſo two other marks, one above and one below the former, exactly anſwering to one-tenth above proof and one-tenth below proof. There are alſo a number of ſmall weights made to add to it, ſo as to anſwer to the other degrees of ſtrength beſides thoſe above, and for determining the ſpecific gravities of different fluids. *Philoſ. Tranſ. Abr.* vol. vi. p. 326.

403. Dr. Deſaguliers contrived an hydrometer for determining the ſpecific gravities of different waters to ſuch a degree of nicety, that it would ſhew when one kind of water was but the 40,000th part heavier than another. It conſiſts of a hollow glaſs ball of about 3 inches in diameter, charged with ſhot to a proper degree, and having fixed in it a long and very ſlender wire, of only the 40th part of an inch in diameter, and divided into tenths of inches, each tenth anſwering to the 40,000 part, as above. See his *Exper. Philoſ.* vol. 2. p. 234.

404. Mr. Nicholſon has made an improvement by which the hydrometer is adapted to the general purpoſe of finding the ſpecific gravity both of ſolids and fluids, (fig. 8. pl. XVI.). A is a hollow ball of copper; B a diſh affixed to the ball by a ſhort ſlender ſtem D; C is another diſh affixed to the oppoſite ſide of the ball by a kind of ſtirrup. In the inſtrument actually made, the ſtem D is of hardened ſteel, $\frac{1}{40}$ of an inch in diameter, and the diſh C is ſo heavy as in all caſes to keep the ſtem vertical

when the inſtrument is made to float in any liquid. The parts are ſo adjuſted that the addition of 1000 grains in the upper diſh B, will juſt ſink it in diſtilled water (at the temperature of 60° of Fahrenheit's thermometer) ſo far that the ſurface ſhall interſect the middle of the ſtem D. Let it now be required to find the ſpecific gravity of any fluid. Immerſe the inſtrument in it, and by placing weights in the diſh B cauſe it to float, ſo that the middle of its ſtem D ſhall be cut by the ſurface of the fluid. Then, as the known weight of the inſtrument, added to 1000 grains, is to the ſame known weight added to the weights uſed in producing the laſt equilibrium, ſo is the weight of a quantity of diſtilled water diſplaced by the floating inſtrument, to the weight of an equal bulk, of the fluid under examination. And theſe weights are in the direct ratio of the ſpecific gravities.

Again, let it be required to find the ſpecific gravity of a ſolid body whoſe weight is leſs than 1000 grains. Place the inſtrument in diſtilled water, and put the body in the diſh B. Make the adjuſtment of ſinking the inſtrument to the middle of the ſtem, by adding weights in the ſame diſh. Subtract thoſe weights from 1000 grains, and the remainder will be the weight of the body. Place now the body in the lower diſh C, and add more weight in the upper diſh B, till the adjuſtment is again obtained. The weight laſt added will be the loſs the ſolid ſuſtains by immerſion, and is the weight of an equal bulk of water. Conſequently the ſpecific gravity of the ſolid is to that of water, as the weight of the body to the loſs occaſioned by the immerſion. Mr. Nicholſon ſays, "This inſtrument was found to be ſufficiently accurate to give weights true to leſs than one-twentieth of a grain." *Nicholſon's Philoſophy*, vol. II. p. 16. See alſo the art. *Gravimeter* in our ſecond volume.

405. The Areometer invented by *M. de Parcieux* in 1766, and preſented to the Academy of Sciences, is very ſimple in its ſtructure, and at the ſame time very convenient in application, and furniſhing very accurate reſults. This inſtrument conſiſts of a ſmall glaſs phial, about two inches, or at moſt two inches and a half, in diameter, and ſeven or eight inches long. Its bottom muſt not be bent inwards (as bottles and phials commonly are), leſt air ſhould be lodged in the cavity when it is immerſed in any liquid. The mouth is cloſed with a very tight cork ſtopper, into which is fixed, without paſſing through it, a very ſtraight wire (of iron or of braſs) about a line in diameter and twenty-eight or thirty inches long. The bottle is then loaded in ſuch a manner, by introducing into it ſmall grains of ſhot, that the inſtrument when immerſed in the lighteſt of the liquids to be compared, ſinks ſo as to leave only the end of the wire above its ſurface, while in the heavieſt the wire is immerſed

ſome inches. This may be properly regulated either by augmenting or diminiſhing the weight in the bottle; or by attaching a little diſh or baſin to the top of the wire, and changing the weights in it; or by varying the thickneſs of the wire To perform experiments of this kind it will be neceſſary to have a veſſel for the reception of the fluid to be tried; a convenient ſize would be a cylinder of 3 or 4 inches in diameter, and as many feet in length: it may be made of tin plate; and a ſcale divided into inches and lines may be carried up above one ſide of the veſſel, which will ſerve to meaſure the different depths of immerſion in the fluid, by noting the diviſion on the ſcale againſt which the top of the wire ſtands. M. de Parcieux recommends the uſe of *two* inſtruments of this kind at once, in order that two fluids may be compared at the ſame inſtant.

The areometer, when thus conſtructed, will exhibit very ſenſibly the leaſt difference in the ſpecific gravities of different liquors, or the changes which the ſame liquor may experience, in this reſpect, under different circumſtances. Indeed M. Montucla ſays, "we have ſeen an inſtrument of this kind, the motion of which was ſo ſenſible, that when immerſed in water cooled to the uſual temperature; it ſunk ſeveral inches while the rays of the ſun fell upon the water, and immediately roſe on the rays of that luminary being intercepted. A very ſmall quantity of ſalt or ſugar thrown into the water made it alſo riſe ſome inches." *Hutton's Montucla's Recreations*, vol. ii. p. 26.

406. We here add a few articles reſpecting the theory of this areometer; obſerving, by the bye, that the ſame theory may, with ſlight and obvious modifications, be applied to the examination of any other inſtrument of a nearly ſimilar kind:

Let s denote the ſpecific gravity of the fluid,
c the capacity or volume of the phial,
w the total weight of the areometer,
l the length of the part of the wire immerſed,
r its ſemidiameter,
π the multiplier 3·141593.

Then will πr^2 repreſent a ſection of the wire parallel to its axis, and $\pi r^2 l$ the content of the wire. And when the areometer is in equilibrio, if we conſider that it diſplaces a portion of the fluid whoſe weight is equal to its own, we ſhall have $w = s\,(c + \pi r^2 l)$; whence we deduce

$$s = \frac{w}{c + \pi r^2 l}, \text{ and } l = \frac{w - cs}{\pi r^2 s}.$$

407. Let us firſt conſider the equation $l = \frac{w - cs}{\pi r^2 s}$. Here the

quantity $w-cs$ expreſſes the difference between the weight of the areometer, or that of the liquor diſplaced, and the weight of the quantity diſplaced by the phial only: it is therefore the weight of the fluid diſplaced by the part of the wire which is immerſed in it. But, conſidering the minuteneſs of the diameter, and conſequently of the radius r, the difference $w-cs$ is a very ſmall quantity, equal perhaps to a few grains; ſo that a very inconſiderable variation in s or in w muſt cauſe a great one in l, which is the length of the portion of the wire plunged in the fluid.

The variations of l are more eſpecially ſenſible when r varies, becauſe it is the ſquare r^2 which is a diviſor in its value: thus, then, we may ſee how it happens that this areometer is ſo well calculated to render manifeſt the ſlighteſt differences in ſpecific gravities.

Suppoſe that when the ſpecific gravity s becomes s', the length l of wire immerſed is changed to l', and we ſhall have $l'=\frac{w-cs'}{\pi r^2 s'}$: ſubtracting the value of l' from that of l, and making the neceſſary deductions, we ſhall have $l-l'=\frac{w(s'-s)}{\pi s s' r^2}$; this is the difference in the depth immerſed, occaſioned by the difference of denſity, which, *ceteris paribus*, is proportional to w.

Thus we may augment the ſenſibility of the inſtrument, relative to the variation of ſpecific gravity, either by augmenting w, the weight of fluid diſplaced by the areometer (which may be accompliſhed by enlarging the phial, or by putting in more ſhot, or by loading the diſh at the top), or by diminiſhing the diameter of the wire; and, in general, all other things being equal, $\frac{w}{r^2}$ will expreſs the ſenſibility of the inſtrument with regard to the ſpecific gravity of the fluid.

408. But when the ſpecific gravity remains the ſame, the ſenſibility of the inſtrument, the quantity by which it ſinks by the addition of a little weight, *depends only upon that little weight and* r^2. To prove this, reſume the equation $l=\frac{w-cs}{\pi r^2 s}$: if the areometer be charged with the little weight ω, l becomes l', and we have $l'=\frac{w+\omega-cs}{\pi r^2 s}$. Taking the firſt of theſe equations from the ſecond, there will remain $l'-l=\frac{\omega}{\pi r^2 s}$: ſo that the vari-

ation in the immerſion $l' - l$ is proportional to $\frac{w}{r^2}$ when s is conſtant *.

We ſee, therefore, that two areometers may be equally ſenſible when the ſpecific gravity varies, and that the one of them may have need of an additional weight to deſcend an equal quantity when the ſpecific gravity remains the ſame. Thoſe, however, which have the little baſin at top are preferable, becauſe they are more ſolid, the wire being made thicker; but chiefly becauſe by means of them we may aſcertain more preciſely the *difference* of ſpecific gravities.

409. If the ſpecific gravity s becomes s', and, in order to

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* In uſing the areometer it is often requiſite to know the diameter of the wire: to thoſe who have not a micrometer M. *Prony* recommends the following method:

Meaſure the length of the wire exactly; weigh it both in water and in the air, and take the difference of theſe two weights. Calling L the whole length of the wire, d the difference between its weight in air and in water, s the ſpecific gravity of water, and D the diameter of the wire; then is $D = 2\sqrt{\frac{d}{\pi s L}}$. This may be done very expeditiouſly by means of logarithms.

To find the thickneſs of the wire in an areometer already conſtructed, we muſt charge the little baſin at the top of the wire with a certain weight, and examine how far this cauſes the inſtrument to deſcend: then we muſt put in the formulæ $D = 2\sqrt{\frac{d}{\pi s L}}$, for d the little weight with which the areometer was charged, and for L the depth of immerſion it occaſioned. This follows from the equation $l' - l = \frac{w}{\pi r^2 s}$, which gives $2r = D = 2\sqrt{\frac{w}{\pi s (l' - l)}}$.

This may not be an improper place to deſcribe a method of finding the radii of ſmall cylinders (ſuch as axles in fine machinery), firſt given, we believe, by Mr. Atwood; which is this.—Having fixed to the extremity of a very fine and flexible line a weight ſufficient to keep it ſtretched, faſten the other extremity to the axle whoſe radius is required; the line being ſtretched by this weight, meaſure by a ſcale of equal parts any convenient length, 6 inches, for example, and mark the extremities of the length ſo meaſured. Then holding the axle horizontally, let the meaſured part of the line be wound round it in the form of a helix, the circumferences being every-where contiguous. Count the number of revolutions, and parts of a revolution, and repreſent this number by n; alſo meaſure the length of the cylinder occupied by the helix, and let this $= a$; and the length of the helix, or line firſt mentioned, $= \lambda$; then will the radius of the cylinder $= \frac{\sqrt{\lambda^2 - a^2}}{2\pi n}$. "The exactneſs of this method may be known," ſays Mr. Atwood, "by obſerving that, if the cylinder be truly made, and the proceſs carefully repeated with different values of λ, n, and a, the radius deduced will, however, always come out the ſame, to the fourth or even the fifth decimal place."

retain the areometer at the ſame depth, it be neceſſary that the weight w become $w+\omega$, we have (art. 406.) $s'=\frac{w+\omega}{c+\pi r^2 l}$. Dividing this equation by the equation $s=\frac{w}{c+\pi r^2 l}$, there reſults $\frac{s'}{s}=\frac{w+\omega}{w}$, a formula expreſſing *the ratio of* the ſpecific gravities.

This equation ſtated as an analogy gives $s':s::w+\omega:w$, whence $s'-s:s::\omega:w$, and $s'-s=\frac{\omega s}{w}$, a formula which gives *the difference of the ſpecific gravities.*

When this theorem is applied to different waters, we may always ſuppoſe one of them, as $s=1000$ ounces averd. without apprehending any ſenſible error in the value of $s'-s$: for the quantity ω being only a few grains, s may undergo variations of 6 or 7 units, without the product $\frac{s}{w}\omega$ undergoing any ſuch change as to affect the accuracy of the reſult. So that in finding the difference between the ſpecific gravities of two waters it is not abſolutely neceſſary to know one of them with great preciſion.

When we would make experiments with an areometer whoſe weight is exactly known, it will be proper, by means of the equations $\frac{s'}{s}=\frac{w+\omega}{w}$, and $s'-s=\frac{\omega s}{w}$, to conſtruct a table which may contain the ratios and the differences of ſpecific gravities, anſwering to the different values of ω between 1 and 40 or 50 grains: ſuch a table, but little extended, will be eaſy to calculate, and will ſave the labour of a diſtinct computation for every individual experiment.

410. Previous to giving a table of medium ſpecific gravities, we ſhall juſt remark, that all bodies of homogeneous or unorganiſed texture expand by heat, and contract by cooling. The expanſion and contraction by the ſame change of temperature is very different in different bodies. Thus, water when heated from 60° to 100° increaſes its volume nearly $\frac{1}{167}$ of its bulk; mercury only $\frac{1}{243}$, and many ſubſtances much leſs. Hence it follows, that an experiment determines the ſpecific gravity only in that very temperature in which the bodies are examined. It will therefore be proper always to note this temperature; and it will be convenient to adopt ſome very uſeful temperature for ſuch trials in general: perhaps about 60° of Fahrenheit's thermometer is as convenient as any. It may al-

ways be procured in theſe climates without inconvenience. A temperature near to freezing would have ſome advantages, becauſe water changes its bulk very little between the temperature 32° and 45°. But this temperature cannot always be obtained. It will much conduce to the facility of the compariſon to know the variation which heat produces on pure water. The following table, taken from the obſervations of Dr. Blagden and Mr. Gilpin (Phil. Tranſ. 1792), will anſwer this purpoſe.

Temperature of Water.	Bulk of Water.	Specific Gravity.
30		
35	99910	1,00090
40	99070	1,00094
45	99914	1,00086
50	99932	1,00068
55	99962	1,00038
60	100000	1,00000
65	100050	0,99950
70	100106	0,99894
75	100171	0,99830
80	100242	0,99759
85	100320	0,99681
90	100404	0,99598
95	100501	0,99502
100	100602	0,99402

Thoſe gentlemen obſerved the expanſion of water to be very anomalous between 32° and 45°. This is diſtinctly ſeen during the gradual cooling of water to the point of freezing. It contracts for a while, and then ſuddenly expands. But we ſeldom have occaſion to meaſure ſpecific gravities in ſuch temperature.

Liquor-merchants often avail themſelves of this circumſtance of the expanſion of fluids by heat, by contriving to make their chief purchaſes in the winter. M. *Homberg* and M. *Eiſenſchmid* found the abſolute weight of a cubic inch of brandy to be 4 drs. 42 grs. in the winter, and only 4 drs. 32 grs. in the ſummer; and in ſpirits of nitre the difference was greater ſtill. And, taking the average with reſpect to ſpirits in general, it is found that 32 gallons in the winter will expand to 33 gallons in the height of ſummer.

A TABLE

Of the Specific Gravities of different Bodies.

Metals.

Antimony, crude,	4064
——, glass of,	4946
——, molten,	6702
Arsenic, glass of, natural,	3594
——, molten,	5763
——, native orpiment,	5452
Bismuth, molten,	9823
——, native	9020
——, ore of, in plumes	4371
Brass, cast, not hammered,	8396
——, ditto, wire-drawn,	8544
——, cast, common,	7824
Cobalt, molten,	7812
——, blue glass of,	2441
Copper, not hammered,	7788
——, the same wire-drawn,	8878
——, ore of soft copper, or natural verdigr.	3572
Gold, pure, of 24 carats, melted, but not hammered,	19258
——, the same hammered,	19362
——, Parisian standard, 22 car. not hammered,	17486
——, the same hammered,	17589
——, guinea of Geo. II.	17150
——, guinea of Geo. III.	17629
——, Spanish gold coin,	17655
——, Holland ducats,	19352
——, trinket standard, 20 car. not hammered,	15709
——, the same hammered,	15775
Iron, cast,	7207
——, bar, either hardened or not,	7788
Steel, neither tempered nor hardened,	7833
——, hardened, but not tempered,	7840
——, tempered and hardened,	7818
——, ditto not hardened,	7816
Iron, ore prismatic,	7355
—— ditto specular,	5218
——, ditto lenticular,	5012
Lead, molten,	11352
Lead, ore of, cubic,	7587
——, ditto horned,	6072
——, ore of black lead,	6745
——, ditto white lead,	4059
——, ditto ditto vitreous,	6558
——, ditto red lead,	6027
——, ditto saturnite,	5925
Manganese, striated,	4756
Molybdena,	4738
Mercury, solid, or congealed,	15632
——, fluent,	13568
——, natural calx of,	9230
——, precipitate, *per se*,	10871
——, precipitate red,	8399
——, brown cinabar,	10218
——, red cinabar,	6902
Nickel, molten,	7807
——, ore of, called Kupfer-nickel of Saxe,	6648
——, Kupfer-nickel of Bohemia,	6607
Platina, crude, in grains,	15602
——, purified, not hammered,	19500
——, ditto hammered,	20337
——, ditto wire-drawn,	21042
——, ditto rolled,	22069
Silver, virgin, 12 deniers, fine, not hammered,	10474
——, ditto hammered,	10511
——, Paris standard,	10175
——, shilling of Geo. II.	10000
——, shilling of Geo. III.	10534
——, French coin,	10408
Tin, pure Cornish, melted, and not hardened,	7291
——, the same hardened,	7299
——, of Malacca, not hardened,	7296
——, the same hardened,	7307
——, ore of, red,	6935
——, ore of, black,	6901
——, ore of, white,	6008
Tungstea,	6066
Uranium,	6440
Wolfram,	7119
Zinc, molten,	7191

Precious Stones.

Beryl, or aqua-marine oriental,	3549
———, ditto occidental,	2723
Chrysolite, of the jewellers,	2782
———, of Brazil,	2692
Chrystal, pure rock of Madagascar	2653
———, of Brazil,	2653
———, European,	2655
———, rose coloured,	2670
———, yellow,	2654
———, violet, or amethyst,	2654
———, white amethyst,	2651
———, Carthaginian,	2657
———, black,	2654
Diamond, white oriental,	3521
———, rose-coloured orient.	3531
———, orange ditto,	3550
———, green ditto,	3524
———, blue ditto,	3525
———, Brazilian,	3444
———, yellow,	3519
Emerald, of Peru,	2775
Garnet, of Bohemia,	4189
———, of Syria,	4000
———, dodecaedral,	4063
———, volcanic, 24 faces,	2468
Girasol,	4000
Hyacinth, common,	3687
Jargon of Ceylon,	4416
Quartz, crystallised,	2655
———, in the mass,	2647
———, brown crystallised,	2647
———, fragile,	2640
———, milky,	2652
———, fat, or greasy,	2646
Ruby, oriental,	4283
———, Spinell,	3760
———, Ballas,	3646
———, Brazilian,	3531
Sapphire, oriental,	3994
———, ditto white,	3991
———, of Puys,	4077
———, Brazilian,	3131
Spar, white sparkling,	2595
———, red ditto,	2438
———, green ditto,	2704
———, blue sparkling,	2693
———, green and white ditto,	3105
———, transparent ditto,	2564
———, adamantine,	3873
Topaz, oriental,	4011
———, pistachio ditto,	4061
———, Brazilian,	3536
———, of Saxe,	3564
———, white ditto,	3554
Vermilion,	4230

Silicious Stones.

Agate, oriental,	2590
———, onyx,	2638
———, cloudy,	2625
———, speckled,	2607
———, veined,	2667
———, stained,	2632
Calcedony, common,	2616
———, transparent,	2664
———, veined,	2606
———, reddish,	2665
———, bluish,	2587
———, onyx,	2615
Carnelian, pale,	2630
———, speckled,	2612
———, veined,	2623
———, onyx,	2623
———, stalactite,	2598
———, simple,	2613
Flint, white,	2594
———, black,	2582
———, veined,	2612
———, Egyptian,	2565
Jade, white,	2950
———, green,	2966
———, olive,	2983
Jasper, clear green,	2359
———, brownish green,	2681
———, red,	2661
———, brown,	2691
———, yellow,	2710
———, violet,	2711
———, cloudy,	2735
———, veined,	2696
———, onyx,	2816
———, red and yellow,	2750
———, bloody,	2628
Opal,	2114
Pearl, virgin oriental,	2684
Pebble onyx,	2664
———, of Rennes,	2654
———, English,	2609
———, veined,	2612
———, stained,	2587
Prasium,	2581
Sardonyx, pure,	2603
———, pale,	2606
———, speckled,	2622
———, veined,	2595
———, onyx,	2595
———, blackish,	2628
Schorl, black prism. hexaedral,	3364
———, octaedral,	3226
———, tourmalin of Ceylon,	3054
———, antique basaltes,	2923
———, Brazilian emerald,	3156
———, cruciform,	3286
Stone, paving,	2416

Stone, cutlers,	2111
——, grind,	2143
——, mill,	2484

Various Stones, Earths, &c.

Alabaster, oriental white,	2730
——, ditto semi-transparent,	2762
——, yellow,	2699
——, stained brown,	2744
——, veined,	2691
——, of Piedmont,	2693
——, of Malta,	2699
——, Spanish saline,	2713
——, of Valencia,	2638
——, of Malaga,	2876
Amber, yellow transparent,	1078
Ambergris,	926
Amianthus, long,	909
——, short,	2313
Asbestos, ripe,	2578
——, starry,	3073
Basaltes from Giants' Causeway,	2864
Bitumen, of Judea,	1104
Brick,	2000
Chalk, Spanish,	2790
——, Coarse Briançon,	2727
——, British,	2784
Gypsum, opaque,	2168
——, semi-transparent,	2306
——, fine ditto,	2274
——, rhomboidal,	2311
——, ditto 10 faces,	2312
——, cuneiform crystallised,	2306
Glass, green,	2642
——, white,	2892
——, bottle,	2733
——, Leith crystal,	3189
——, fluid,	3329
Granite, red Egyptian,	2654
Hone, white razor,	2876
Lapis nephriticus,	2894
—— Lazuli,	3054
—— Hæmatites,	4360
—— Calaminaris,	5000
—— Judaicus,	2500
—— Manati,	2270
Limestone,	3179
——, white fluor,	3156
——, green,	3182
Marble, green Campanian,	2742
——, red,	2724
——, white Cassara,	2717
——, white Parian,	2838
——, Pyrenean,	2726
——, black Biscayan,	2695
——, Brocatelle,	2650
——, Castilian,	2700
——, Valencian,	2710
——, white Grenadan,	2705
Marble, Siennien,	2678
——, Roman violet,	2755
——, African,	2708
——, violet Italian,	2858
——, Norwegian,	2728
——, Siberian,	2718
——, green Egyptian,	2668
——, Swisserland,	2714
——, French,	2649
Obsidian stone,	2348
Peat, hard,	1329
Phosphorus,	1714
Porcelaine, Seves,	2146
——, Limoges,	2341
——, China,	2385
Porphyry, red,	2765
——, green,	2676
——, red, from Dauphiny,	2793
——, red, from Cordone,	2754
——, green, from ditto,	2728
Pyrites, coppery,	4954
——, feruginous cubic,	3900
——, ditto round,	4101
——, ditto of St. Domingo,	3440
Serpentine, opake, green Italian,	2430
——, ditto, veined black and olive,	2594
——, ditto, red and black,	2627
——, semi-transpar. grained,	2586
——, ditto fibrous,	3000
——, ditto, from Dauphiny,	2669
Slate, common,	2672
——, new,	2854
——, black stone,	2186
——, fresh polished,	2766
Stalactite, transparent,	2324
——, opake,	2478
Stone, pumice,	915
——, prismatic basaltes,	2722
——, touch,	2415
——, Siberian blue,	2945
——, oriental ditto,	2771
——, common,	2520
——, Bristol,	2510
——, Burford,	2049
——, Portland,	2496
——, rag,	2470
——, rotten,	1981
——, hard paving,	2460
——, rock of Chatillon,	2122
——, clicard, from Brachet,	2357
——, ditto, from Ouchain,	2274
——, Notre-Dame,	2378
——, St. Maur,	2034
——, St. Cloud,	2201
Sulphur, native,	2033
——, molten,	1991
Talc, of Muscovy,	2792
——, black crayon,	2089
——, ditto German,	2246

Talc, yellow,	2655
——, black,	2900
——, white,	2704

Liquors, Oils, &c.

Acid, sulphuric,	1841
——, ditto, highly concentrated,	2125
——, nitric,	1271
——, ditto, highly concentrated,	1580
——, muriatic,	1194
——, red acetous,	1025
——, white acetous,	1014
——, distilled ditto,	1010
——, fluoric,	1500
——, acetic,	1063
——, phosphoric,	1558
——, formic,	994
Alcohol, commercial,	837
———, highly rectified,	829
Alcohol, mixed with water	
15-16ths alcohol	853
14-16ths ditto	867
13-16ths ditto	882
12-16ths ditto	895
11-16ths ditto	908
10-16ths ditto	920
9-16ths ditto	932
8-16ths ditto	943
7-16ths ditto	952
6-16ths ditto	960
5-16ths ditto	967
4-16ths ditto	973
3-16ths ditto	979
2-16ths ditto	985
1-16th ditto	992
Ammoniac, liquid,	897
Beer, pale,	1023
——, brown,	1034
Cyder,	1018
Ether, sulphuric,	739
———, nitric,	909
———, muriatic,	730
———, acetic,	866
Milk, woman's,	1020
——, cow's,	1032
——, ass's,	1036
——, ewe's,	1041
——, goat's	1035
——, mare's,	1034
——, cow's clarified,	1019
Oil, essential, of turpentine,	870
—, essential, of lavender,	894
—, ditto, of cloves,	1036
—, ditto, of cinnamon,	1044
—, of olives,	915
—, of sweet almonds,	917
—, of filberts,	916
—, linseed,	940
—, of walnuts,	923
Oil of whale,	923
— of hempseed,	926
— of poppies,	924
—, rapeseed,	919
Spirit of wine. See Alcohol.	
Turpentine, liquid,	991
Urine, human,	1011
Water, rain,	1000
———, distilled,	1000
———, sea, (average)	1026
———, of Dead-sea,	1240
Wine, Burgundy,	992
——, Bourdeaux,	994
——, Madeira,	1038
——, Port,	997
——, Canary,	1033

Resins, Gums, and Animal Substances, &c.

Aloes, socotrine,	1380
——, hepatic,	1359
Asafœtida,	1328
Bees-wax, yellow,	965
————, white,	969
Bone of an ox,	1656
Butter,	942
Calculus humanus,	1700
——— ditto,	1240
——— ditto,	1434
Camphor,	989
Copal, opake,	1140
——, Madagascar,	1060
——, Chinese,	1063
Crassamentum. human blood,	1126
Dragon's blood,	1205
Elemi,	1018
Fat, beef,	923
—, hog's,	937
—, mutton,	924
—, veal,	934
Galbanum,	1212
Gamboge,	1222
Gum, ammoniac,	1207
——, Arabic,	1452
——, Euphorbia,	1124
——, seraphic,	1201
——, tragacanth,	1316
——, bdellium,	1372
——, scammony of Smyrna,	1274
——, ditto of Aleppo,	1235
Gunpowder, shaken,	932
—————, in a loose heap,	836
—————, solid,	1745
Honey,	1450
Indigo,	769
Ivory,	1826
Juice of liquorice,	1723
——of Acacia,	1515

Labdanum,	1186
Lard,	948
Mastic,	1074
Myrrh,	1360
Opium,	1336
Scammony. See Gum.	
Serum of human blood,	1030
Spermaceti,	943
Storax,	1110
Tallow,	942
Terra Japonica,	1398
Tragacanth. See Gum.	
Wax. See Bees'-wax.	
——, shoemaker's,	897

Woods.

Alder,	800
Apple-tree,	793
Ash, the trunk,	845
Bay-tree,	822
Beech,	852
Box, French,	912
——, Dutch,	1328
——, Brazilian red,	1031
Campechy wood,	913
Cedar, wild,	596
——, Palestine,	613
——, Indian,	1315
——, American,	561
Citron,	726
Coco-wood,	1040
Cherry-tree,	715
Cork,	240
Cypress, Spanish,	644
Ebony, American,	1331
——, Indian,	1209
Elder-tree,	695
Elm, trunk of,	671
Filbert-tree,	600
Fir, male,	550
——, female,	498
Hazel,	600
Jasmin, Spanish,	770
Juniper-tree,	556
Lemon-tree,	703
Lignum vitæ,	1333
Linden-tree,	604
Logwood. See Campechy.	
Mastick-tree,	849
Mahogany,	1063
Maple,	750
Medlar,	944
Mulberry, Spanish,	897
Oak, heart of, 60 years old,	1170
Olive-tree,	927
Orange-tree,	705
Pear-tree,	661
Pomegranate-tree,	1354
Poplar,	383
——, white, Spanish,	529
Plum-tree,	785
Quince-tree,	705
Sassafras,	482
Vine,	1327
Walnut,	671
Willow,	585
Yew, Dutch,	788
——, Spanish,	807

Weight and Specific Gravities of different Gases.

Fahrenheit's Thermom. 55° Barometer 30 inches.

	Spec. Grav.	Wt. Cub. Foot.
Atmospheric air,	1·2	525·0 grs.
Hydrogenous gas,	0·1	43·75
Oxygenous gas,	1·435	627·812
Azotic gas,	1·182	517·125
Nitrous gas,	1·4544	636·333
Ammoniac gas,	·7311	319·832
Sulphureous acid gas,	2·7611	1207·978

In this table the weights and specific gravities of the principal gases are given, as they correspond to a state of the barometer and thermometer which may be chosen for a medium. The specific gravity of any one gas to that of another will not conform to exactly the same ratio under different degrees of heat and other pressures of the atmosphere, because the various expansions by no means follow the same law.

CHAPTER III.

Equilibrium, Stability, and Oscillations of Floating Bodies.

411. Among the different bodies which float on the ſurface of a fluid, the moſt important are ſhips and other veſſels employed on ſeas, rivers, and canals, in commerce. It is of much conſequence to determine the ſtability of ſuch veſſels, and the poſitions they aſſume when they float freely on the water. The principles on which the ſtability of the different poſitions depend flow naturally from what has been already done, and may be here illuſtrated and explained. Two or three definitions are premiſed.

Defs. 1. *The plane of floatation* is the horizontal ſurface of the fluid in which the veſſel floats.

2. *The line of ſupport* is the vertical line paſſing through the centre of gravity of the part of the ſolid or veſſel immerſed.

3. *The metacentre* is the point of interſection of the axis paſſing through the centre of gravity (round which the body revolves through an indefinitely ſmall angle) and the line of ſupport.

412. That a body floating upon a quieſcent fluid may be in equilibrio, ſomething more is neceſſary than that the weight of the body ſhould be equal to that of the fluid diſplaced: for this condition only enſures an equality of the upward puſh of the fluid and the weight of the body; and theſe two vertical forces cannot deſtroy each other unleſs they are directly oppoſed: they muſt therefore paſs through the ſame point; that is to ſay, *the right line which joins the centres of gravity of the floating body and of the fluid diſplaced muſt be vertical.* Without this condition the two vertical forces, though equal, cannot annihilate each other's effects; and, of conſequence, the body will have about its centre of gravity a rotatory motion, as if that point were fixed.

Indeed, poſitions may be aſſumed in which the circumſtances juſt recited concur, and yet the ſolid will take ſome other poſition in which it will permanently float. If a cylinder, for example, having its ſpecific gravity to that of the fluid on which it floats as 3 to 4, and its axis to the diameter of its baſe as 2 to 1, be placed on the fluid with its axis vertical, it will ſink to a depth equal to a diameter and a half of the baſe; and while its axis is preſerved in a vertical poſition by external force, the centres of gravity of the whole ſolid and of the immerſed part will remain in the ſame vertical line: but when the external force that ſuſtained it is removed it will decline from its upright poſition, and will permanently float with its axis horizontal. If the axis be ſuppoſed to be half of the diameter of the baſe, and be placed vertically, the ſolid will ſink to the depth of three-eighths of its diameter; and in that poſition it will float permanently. If the axis be made to incline to the vertical line, the ſolid will change its poſition until it ſettles permanently with the axis perpendicular to the horizon.

Def. Whether a ſolid floats permanently, or overſets when placed on the ſurface of a fluid, ſo that the centre of gravity of the ſolid and that of the part immerſed ſhall be in the ſame vertical line, it is ſaid to be in a poſition of equilibrium; and of this equilibrium there are three ſpecies, viz. the equilibrium of *ſtability*, in which the ſolid floats permanently in a given poſition; the equilibrium of *inſtability*, in which the ſolid, though the two centres of gravity already mentioned are in the ſame vertical line, ſpontaneouſly overſets, unleſs ſupported by external force; and the equilibrium of *indifference*, or the inſenſible equilibrium, in which the ſolid reſts on the fluid indifferent to motion, without tendency to right itſelf when inclined, or to incline itſelf further.

Theſe ſtates of a floating body are alſo thoſe in which the ſtability may be ſaid to be *poſitive*, *negative*, and *nothing*, reſpectively.

413. From the properties inveſtigated in the chapter on the centre of gravity, it follows that the right line which joins the centres of gravity of a body and of a ſegment of that body formed by any plane whatever, muſt paſs through the centre of gravity of the other ſegment: in the caſe which we propoſe to treat here, the plane cutting the body is the upper ſurface of the fluid, or the plane of floatation; and we conclude that, in order that a body may be in equilibrium upon a fluid which is ſpecifically heavier than itſelf, the right line which paſſes through the centres of gravity of the body and either of the ſegments formed by the plane of floatation muſt be perpendicular to that plane. Thus the general problem of the determination of the poſitions of equilibrium with regard to a homogeneous body in a fluid is reduced to the following.

Prop. *To cut by a plane a body of given figure in ſuch manner that the capacity of either of the ſegments ſhall be to that of the* *whole body in a given ratio; and that the right line which paſſes through the centre of gravity of the body and thoſe of its ſegments ſhall be perpendicular to the interſecting plane.*

The ratio of the ſegments into which the body muſt be cut, or of either of them to the whole body, will depend upon the ratio of the ſpecific gravities of the body and the fluid: and the poſitions of equilibrium of any body will be given by the roots deduced from the equations of equilibrium, and the equation defining the nature of the body. We cannot here enter much into the detail: but ſhall merely preſent a few obvious examples.

414. It is manifeſt then, that right priſms and cylinders of any baſe whatever, if homogeneous, will have two poſitions of equilibrium; viz. when the generating axis is placed vertically, and when it is placed horizontally. The equilibrium obtains alſo, with regard to ſolids of revolution, and ſymmetrical bodies relative to any line, when that line is placed vertically. Hence, a floating ſphere will continue at reſt in any poſition; becauſe it is, in every poſition, ſymmetrical with reſpect to the line of ſupport paſſing through its centre of gravity.

Moreover, priſmatic or cylindric bodies, and in general thoſe which are ſymmetrical with regard to a plane (as are all which need be conſidered in the theory), have their poſitions of equilibrium, when that plane, relatively to which they are ſymmetrical, is vertical: in which caſe we have only to aſcertain the poſitions of equilibrium for an area or vertical ſection of the body, perpendicular to that with reſpect to which the whole is ſymmetrical.

415. Let us propoſe an inſtance for more particular examination, in the calculus of which it will appear in what manner different poſitions of equilibrium of a body will be furniſhed by the ſame equation.

Suppoſe that the body is a triangular priſm, which floats with its ends vertical, and its edges horizontal.

Here we muſt conſider two caſes: 1ſt. That where the baſes of the priſm have an angle F immerſed in the fluid, and the other two E, H, out of it (fig. 10. pl. XVI.). 2d. The reverſe of this, where the two angles E, H, are immerſed (fig. 11.). The two caſes may be treated at the ſame time; for the method in each is the ſame.

Let XX′ be the plane of floatation. To determine the line MN, in which this plane ſhould interſect the triangle, we muſt find FM, and FN. Call the firſt of theſe x, the ſecond y: and taking P in the middle of EG we may repreſent the data thus: $FE=a$, $FH=b$, $FP=k$, angle $PFE=m$, $HFP=n$; and let the ratio of the ſpecific gravities of the ſolid and fluid $=r$ to 1.

Then, ſhat the weight of the ſolid may be equal to that of the fluid diſplaced, we muſt have,

In the firſt caſe (fig. 10.) . . $r \,.\, \mathrm{FEH} = \mathrm{FMN}$;

In the ſecond caſe (fig. 11.) . . $r \,.\, \mathrm{FEH} = \mathrm{HEMN}$.

Now the triangles FMN, FEH, having the angle F common, have their areas proportional to the rectangles of the ſides about that angle: that is,

$$\mathrm{FEH} : \mathrm{FMN} :: \mathrm{FH} \times \mathrm{FE} : \mathrm{FM} \times \mathrm{FN} :: ab : xy;$$

$$\text{whence } \mathrm{FEH} - \mathrm{FMN}, \text{ or } \mathrm{HNME} : \mathrm{FEH} :: ab - xy : ab;$$

$$\text{conſequently } xy = rab, \text{ and } xy = ab(1 - r) \ldots (1).$$

Theſe two equations include the condition of the equality of the vertical preſſures, upwards and downwards.

Let us next find equations to ſatisfy the ſecond condition of the centres of gravity reſiding in the ſame vertical line. Now if PR be taken $= \frac{1}{3}$ PF, R (art. 114.) will be the centre of gravity of the triangle FE; in like manner, if Q is the middle point of MN, and $\mathrm{QG} = \frac{1}{3} \mathrm{QF}$, G will be the centre of gravity of the triangle FMN: the right line RG, or its parallel PQ, will therefore be vertical in the caſe of the equilibrium; and this condition will evidently be expreſſed by the equation $\mathrm{PM} = \mathrm{PN}$, in both figures. From the point P demit upon the ſides FE, FH, the perpendiculars PA, PD: then will $\mathrm{PA} = k$ ſin. m, $\mathrm{PD} = k$ ſin. n, $\mathrm{FA} = k$ coſ. m, $\mathrm{FD} = k$ coſ. n. Therefore $\mathrm{ND} = k$ coſ. $n - y$, and $\mathrm{AM} = k$ coſ. $m - x$. But $\mathrm{PM} = \mathrm{PN}$ gives $\mathrm{AP}^2 + \mathrm{AM}^2 = \mathrm{PD}^2 + \mathrm{DN}^2$: that is, k^2 ſin. ${}^2 m = (k \text{ coſ. } m - x)^2 = k^2$ ſin. ${}^2 n + (k \text{ coſ. } n - y)^2$; from which, by reduction, we find

$$y^2 - 2ky \text{ coſ. } n = x^2 - 2kx \text{ coſ. } m.$$

If for y in this equation we ſubſtitute ſeparately each of its values deduced from the two equations No. 1. there will reſult two equations containing only x and known quantities, and which will expreſs the conditions of equilibrium for the reſpective caſes of figs. 10. and 11. They are as follows:

$$\left.\begin{array}{l} x^4 - 2kx^3 \cos. m + 2abkrx \,.\, \cos. n - r^2 a^2 b^2 \ldots = 0 \\ x^4 - 2kx^3 \cos. m + 2(1-r)abkx \,.\, \cos. n - (1-r)^2 a^2 b^2 = 0 \end{array}\right\} \ldots \text{(II.)}$$

Theſe equations being of the fourth degree, and having the laſt term negative, have at leaſt *two* real roots (Lacroix's Alg. No. 219.): but the roots may be all four real, and then the diſpoſition of the ſigns indicate that three of the roots are poſitive, and the fourth negative. This latter root is obviouſly uſeleſs in the preſent caſe, becauſe the ſolicitation of gravity being conſtantly downwards, the right-line FM can only be placed on one ſide with regard to the point F. There muſt, conſequently, be either *one* or *three* poſitions of equilibrium, which will be determined by the poſitive roots of the equations marked (II.). The correſponding values of y will be furniſhed by the equations (I.). We muſt always, however, have $x \angle a$, and $y \angle b$.

416. The preceding reaſonings may now be applied to the iſoſceles triangle, as the moſt likely to occur in practice: and to prevent the calculus from being very complicated, let us merely conſider the caſe where one angle only of the triangle is immerſed in the fluid: the other caſe being eaſily deduced by transforming in the reſults r into $1-r$, as may be ſhewn by comparing together either the equations No. II. or thoſe of No. I. Here, therefore, we have $m=n$, and $a=b$, from which there ariſes

$$x\,y=a^2 r, \text{ and } x^4-2\,k\,x^3 \text{ cof. } m+2\,a^2 r\,k\,x \text{ cof. } m-r^2 a^4=0.$$

The factors of the ſecond degree of this latter equation being $x^2-a^2 r=0$, and $x^2-2\,k\,x \text{ cof. } m+a^2 r=0$, we thence conclude, taking only the poſitive roots, that,

$$x=a\sqrt{r}, \text{ and } x=k \text{ cof. } m \pm \sqrt{k^2 \cos.^2 m-a^2 r}$$

$$\text{whence, } y=x, \text{ and } y=k \text{ cof. } m \mp \sqrt{k^2 \cos.^2 m-a^2 r}.$$

The firſt root indicates that there will be only one poſition of equilibrium when EH is horizontal: and the ſame applies to the caſe of fig. 11. The other poſitions are given by the other two roots; but they muſt be ſuch as to correſpond with the conditions of the remarkable limits: for we ought to have $x \angle a$, and every root real; and from this we find

$$r \;7\; \frac{2\,a\,k \cos.\, m-a^2}{a^2}, \text{ and } r \angle \frac{k^2 \cos.^2 m}{a^2}$$

In like manner we ſhall find the limits for the ſecond caſe, by changing r into $1-r$; whence there reſults

$$r \angle \frac{2\,a-2\,k \cos.\, m}{a}, \text{ and } r \;7\; \frac{a^2-k^2 \cos.^2 m}{a^2}$$

If we ſuppoſe the triangle equilateral we ſhall have $k=a\sqrt{\tfrac{3}{4}}$, and cof. $m=\sqrt{\tfrac{3}{4}}$; therefore cof. $m=\tfrac{3}{4}a$: and the preceding limits will become—

Firſt caſe, $r \;7\; \tfrac{1}{2}$, and $r \angle \tfrac{9}{16}$: ſecond caſe, $r \angle \tfrac{1}{2}$, and $r \;7\; \tfrac{7}{16}$.

417. PROP. *When a floating body is in equilibrio in any fluid, and an extraneous cauſe as an impulſion move the body, it is propoſed to determine whether this perturbation will permit the body to return to its firſt poſition, or conſtrain it, on the contrary, to move ſtill further from that of equilibrium.*

As the inveſtigation of this problem in its utmoſt extent would lead into very tedious and complex diſcuſſions, we ſhall premiſe three hypotheſes, in order that we may obtain comparatively ſimple reſults.

1. We ſuppoſe the floating body to be ſymmetrical with reſpect to a vertical plane paſſing through its centre of gravity when the whole is at reſt; and of ſuch a kind that we need

only reſolve the problem for the area of a plane ſection of the body.

2. We conceive the derangement to be indefinitely ſmall. Thus, in fig. 12. pl. XVI. where ab is the plane of floatation, G the centre of gravity of the floating body DFE, and O the centre of gravity of AFB, or of the fluid diſplaced when the line AB coincides with the plane of floatation; the inclination of the body will then be the angle GOV formed by GO and the vertical line OV paſſing through the centre of aFb; and we put this evaneſcent angle, or the arc which meaſures it, $=A$.

3. We diſregard the vertical motion of the centre of gravity of the body as indefinitely ſmall; and ſuppoſe that the new poſition is conſiſtent with the equality of the weights of the body and of the fluid diſplaced. Thus will the portion ACa, which is moved out of the fluid, be equal to the part BCb, which has entered it, in conſequence of the motion.

Granting theſe ſuppoſitions, the equal areas ACa and BCb may be regarded as iſoſceles triangles, ſince we may ſuppoſe $AC=aC$ and $CB=Cb$: hence it will follow that $aC=Cb$; that is, the interſection C of the two lines of floatation is in the middle of AB.

The upward preſſure of the fluid on aFb is equal to the weight of a lamina of that fluid of an equal magnitude; and this vertical force acts at the centre of gravity of aFb. So that as it is neceſſary to know this centre, we ſhall proceed to aſſign its poſition. In order to this, it may be obſerved, that $aFb=AFB+CBb-ACa$; and if we conceive the laminæ of the fluid equal to theſe areas, it is evident that the moment of the upward preſſure of the fluid, taken with relation to any vertical whatever, is equal to the ſum of the moments of the fluid laminæ of which the volumes are AFB and CBb, minus the moment of the lamina ACa. Let us then eſtimate theſe moments with reſpect to the vertical Gi. So ſhall we have

I. The moment of the weight of the fluid laminæ AFB is $W \cdot GV$; W repreſenting the weight of the body, or that of the fluid diſplaced.

II. Repreſenting by $w\,[CBb]$ the weight of a volume of fluid equal to CBb, we have for the moment of this weight its product by the diſtance qi: the point q being the foot of the perpendicular demitted upon Cb from the centre of gravity of CBb, that is, taking $Cq=\frac{2}{3}Cb$. Thus the moment of CBb is $w\,[CBb] \times qi$.

III. Laſtly, the moment of ACa is, in like manner, $w\,[ACa] \times pi$; making $pC=\frac{2}{3}aC$. Where, as this latter force tends to produce a motion in the contrary direction to the former, it muſt be taken negatively; and ſince, moreover, it ought to be ſubtracted from the two others it becomes poſitive, and the moment M of the preſſure of the fluid upon aFb is (becauſe CB b = AC a) expreſſed thus; $M = (W \times GV) + w[CBb] \times pq$.

Now, ſince the fluid is ſuppoſed homogeneous, the areas AFB and BC b are proportional to the weights of the quantities of fluid to which they are reſpectively equal; that is to ſay, if we put the ſurface AFB = S, ſince CB = C b, and moreover $Be = CB \times$ ſin. $A = A \,.\, CB$, we have the triangle $CBb = \frac{1}{2} A \,.\, CB \,.\, CB$; therefore

$$S : W :: \tfrac{1}{2} A (BC)^2 : w[CBb] = \frac{W}{2S} A (BC)^2.$$

But $pq = \frac{2}{3} ab$, and $GV = A \,.\, GO$. Putting therefore GO $= a$, and $AB = b = 2BC$, there reſults

$$M = A \,.\, W \left(a + \frac{b^3}{12S}\right) \ldots\ldots\ldots \text{(i.)}$$

418. Since we conſider the body as if it were retained by a fixed axe paſſing through G, the preceding value of M will meaſure the preſſure of the fluid. So that, if we would compare together theſe preſſures upon different bodies, we muſt conſider them in reference to the ſame angle of inclination A, and the relative ſtabilities will be meaſured by

$$W \left(\frac{b^3}{12S} \pm a\right) \ldots\ldots\ldots \text{(ii.)}$$

COR. If the ſolid be of an irregular form the ſtability will be as the ſum of all the $W \left(\frac{b^3}{12S} \pm a\right)$.

By *ſtability* we would here be underſtood to mean the reſiſtance which a body oppoſes to its change of poſition; that is, the effort made by the fluid to retain the body in its poſition of equilibrium, or to carry it further from it. In fig. 12. we have placed the centre of gravity of the body *below* that of the fluid diſplaced, whence it follows that the body is not homogeneous: and this is what happens moſt frequently (ſhips of war being excepted), becauſe it is cuſtomary to diſpoſe, in the inferior parts about F, ſome ſubſtances ſpecifically heavier than the materials of the floating body. If the centre of gravity of the body were, on the contrary, higher than that of the fluid diſplaced, it would be neceſſary to take a as negative: on this account we have put in the formulæ (ii.) the double ſign ± before the quantity a.

419. The value ii. may (in conformity with art. 412.) be either poſitive, nothing, or negative. 1ſt. When the centre of gravity of the body is lower than that of the fluid diſplaced,

the ſtability will be poſitive ſo long as a retains the upper ſign: and it will be poſitive in the contrary caſe, when O is below G, whenever $a < \frac{b^3}{12\,S}$. In theſe caſes the preſſure of the fluid tends obviouſly to carry back the body to its firſt poſition. This tendency is, beſides, the ſtronger, as the value (ii) is the greater. We ſee alſo the advantage of having the floating body narrower towards the bottom than towards the plane of floatation; and the correſponding advantage of making the upper parts of ſuch materials as have the leaſt ſpecific gravity.

2dly. In the two other caſes the centre of gravity of the body is *higher* than that of the fluid diſplaced: if we have $a > \frac{b^3}{12\,S}$, the ſtability is *negative*, and the aggregate preſſure tends to move the body in the oppoſite direction, or ſo as to increaſe the angle A; for then the centre of gravity of a F b being on the other ſide of Gi muſt cauſe the body to proceed further from its firſt poſition of equilibrium. If $a = \frac{b^3}{12\,S}$, the ſtability is *nothing* or *indifferent*; and the total preſſure does not exert any ſuch effort calculated to turn the body, becauſe the centre of gravity of a F b is then upon Gi, and conſequently in the ſame vertical line as G.

420. If the quantity (ii) be divided by W, the weight of the fluid diſplaced, the quotient will expreſs the diſtance from the vertical Gi to the direction of the reſultant of the upward preſſure of the fluid: that diſtance being taken $=$ Gn, we have

$$G\,n = \frac{A\,.\,b^3}{12\,S} \pm GV = A\,.\left(\frac{b^3}{12\,S} \pm a\right) \;.\;.\;.\;.\;.\; \text{(iii)}.$$

If b is poſitive, the vertical Gi falls to the left of the centre of gravity of the fluid diſplaced; in this caſe Gn muſt be taken to the right of Gi: the ſame thing obtains in the contrary caſe, provided that we have $GV < \frac{A\,.\,b^3}{12\,S}$, the vertical $n\,g$ interſecting the right line GO in a point g more elevated than G. But if, in the latter caſe, $GV = \frac{A\,.\,b^3}{12\,S}$, we have then $G\,n = 0$, and g falls upon Gi. Finally, if we have $GV > \frac{A\,.\,b^3}{12\,S}$, and G conſtantly higher than O, the line Gn muſt be taken to the left of Gi, and the point g will fall below G. To find the diſtance Gg it may be obſerved that $G\,g\,.\,\text{ſin.}\;A = G\,n$, or, becauſe an evaneſcent arc may be ſubſtituted for its ſine, $A\,.\,G\,g = G\,n$;

$$\text{wherefore} \ldots Gg = \frac{b^3}{12\,S} \pm a \ldots\ldots (\text{iv.})$$

The point g correſponds with our definition of *metacentre* (art. 411.), we may therefore announce the reſult of this inveſtigation thus: *the ſtability of a body or veſſel is poſitive, nothing, or negative, according as the metacentre is more elevated, the ſame, or lower than the centre of gravity of the body.*

421. PROP. *If the floating body be a homogeneous rectangular parallelopiped, whoſe altitude is perpendicular to the ſurface of the fluid, its ſtability will be proportional to the difference between the ſixth part of the ſquare of the breadth of the baſe and the product of the ſquare of the altitude, into the difference between the number expreſſing the ſpecific gravity of the ſolid and its ſquare, that of the fluid being unity.*

Let h be the height of the ſolid, b the breadth of the baſe, and s the ſpecific gravity of the ſolid; then will the cube of the line coinciding with the plane of floatation $= b^3$ as before, the height of the part immerſed $= s\,h$, and S the ſpace proportional to the part immerſed $= s\,b\,h$; GO the diſtance between the centres of gravity of the whole body and of the part immerſed $= \frac{1}{2}b - \frac{1}{2}s\,h = \frac{h - s\,h}{2}$. Conſequently the ſtability which varies as $\frac{b^3}{12\,S} \pm \text{GO}$, varies as $\frac{b^3}{12\,s\,b\,h} - \frac{h - s\,h}{2} = \frac{b^2}{6} - \left(h^2 . \overline{s - s^2}\right)$.

As an example to illuſtrate this propoſition, ſuppoſe the height of the parallelopiped equal to the breadth of the baſe, or $h = b$, and let it be required to aſcertain the ſpecific gravity of the ſolid, when it will float in the equilibrium of indifference. Here $\frac{b^2}{6} = h^2 . (s - s^2)$, or, ſince $h = b$, we have $s^2 - s = -\frac{1}{6}$; whence $s = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{1}{6}} = \frac{1}{2} \pm \frac{1}{6}\sqrt{3} = \cdot 78868$ or $\cdot 21132$ nearly, either of which may be taken for the ſpecific gravity of the ſolid, that of the fluid being unity. If the fluid were rain-water the bodies would have nearly the ſame ſpecific gravity as apple-tree and cork.

422. PROP. *If the floating body be an homogeneous cylinder, whoſe axis is vertical, its ſtability will be as the quotient of the ſquare of the radius of the baſe divided by four times the height of the part immerſed, diminiſhed by the diſtance between the centres of gravity of the whole ſolid and the part immerſed.*

Let DEFG (fig. 2, pl. XVII.) be the ſection of the cylinder coinciding with the plane of floatation, AB any line in that ſection, or the breadth of any variable vertical ſection of the cylinder; $CG = r$, $CV = x$; then $AB = 2\sqrt{r^2 - x^2}$, and $AB^3 = 8 . (r^2 - x^2)^{\frac{3}{2}}$. But $(r^2 - x^2)^{\frac{3}{2}} = (r^2 - x^2)^{\frac{1}{2}} \times r^2 - (r^2 - x^2)^{\frac{1}{2}} \times x^2$. Alſo $r^2 (r^2 - x^2)^{\frac{1}{2}} = r^2 \times \frac{1}{4}$ of the circle whoſe radius is r, when x increaſes from 0 to r; and $x^2 (r^2 - x^2)^{\frac{1}{2}} = r x^2 - \frac{x^4}{2r} - \frac{x^6}{8r^3}$, &c. and the quantity generated $= \frac{r x^3}{3} - \frac{x^5}{10r} - \frac{x^7}{56r^3}$, &c. which, when x increaſes from 0 to r, becomes $r^4 \times (\frac{1}{3} - \frac{1}{10} - \frac{1}{56}$, &c.$) = r^4 \times \frac{1}{4}$ of a circle whoſe diameter is unity (vide *Anal. per Equationes*, pa. 74.) $= r^2 \times \frac{1}{16}$ of a circle whoſe radius is r. Conſequently, the whole quantity generated by $(r^2 - x^2)^{\frac{3}{2}}$ will be $r^2 \times (\frac{1}{4} - \frac{1}{16})$ of a circle whoſe radius is $r = r^2 \times \frac{3}{16}$ of that circle $= r^2 \times \frac{3}{16} \pi r^2 = \frac{3}{16} \pi r^4$. Hence the ſum of all the AB^3 in one ſemicircle $= 8 r^2 \times \frac{3}{16} \pi r^2$, $= \frac{3}{2} \pi r^4$, and the ſum of all the AB^3 in both ſemicircles $= 3 \pi r^4$; while the volume of the part immerſed is equal to its depth $(d) \times$ circle $DEFG = \pi d r^2$. Therefore $\int \frac{AB^3}{12 S} = \frac{3 \pi r^4}{12 \pi d r^2} = \frac{r^2}{4d}$; and the general expreſſion for the ſtability (ii) is proportional to $\frac{CG^2}{4d} - GO$, as in the propoſition.

Cor. If s be the ſpecific gravity of the ſolid, that of the fluid being unity, and h the height of the cylinder, then will the depth of the part immerſed $= s h$, and $GO = \frac{h - s h}{2}$: ſo that the ſtability of the cylinder will be proportional to $\frac{r^2}{4 s h} - \frac{h - s h}{2}$, and will vaniſh or become indifferent when $\frac{r^2}{4 s h} = \frac{h - s h}{2}$, or when $s^2 - s = - \frac{r^2}{2 h^2}$, or $s = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{r^2}{2 h^2}}$: where the expreſſion under the radical vaniſhes if $r = h \sqrt{\frac{1}{2}}$, or $2 r^2 = h^2$. So that, *if the ſpecific gravity of the cylinder be half that of the fluid, and the radius of the baſe be to the height of the cylinder as* 1 *to* $\sqrt{2}$, *the cylinder will float in a ſtate of inſenſible equilibrium.*

Cor. 2. *If* $s = \frac{3}{4}$, *then in the caſe of inſenſible equilibrium* r *will be to* h *as* $\sqrt{3}$ *to* $2\sqrt{2}$. And if $s = \frac{1}{4}$ *we ſhall have the ſame ratio.*

423. Prop. *Conſidering the fluid as non-reſiſting, the floating body will oſcillate inceſſantly about a horizontal axis paſſing through*

its centre of gravity, according to laws analogous to thoſe of bodies in a vacuum: it is propoſed to examine the particular nature of theſe oſcillations, regarding them as indefinitely ſmall.

Suppoſing that the vertical line paſſing through the centre of gravity of the body, in its equilibrated poſition, has been inclined by the quantity f, then denoting by A the inclination of the body at the end of the time t, and by α the arc deſcribed by the point placed at the unit of diſtance from G (fig. 12. pl. XVI.), we ſhall have $A = f - \alpha$. Now the angular accelerating force, as has been ſeen in the Dynamics, is the quotient of the ſum of the momenta of the moving forces divided by the momentum of inertia: the dividend, being the total vertical preſſure of the fluid, is given by the value art. 418. i.; the diviſor is $M k^2$, M being the maſs of the body, and k^2 the quotient of the momentum of inertia (with reſpect to an axe paſſing through the centre of gravity) divided by the maſs. Hence, as the weight W of the body is equivalent to $g M$ (art. 108.) we have, for the angular accelerating force,

$$\frac{\ddot{\alpha}}{\dot{t}^2} = \frac{g}{k^2} \cdot A \left(a + \frac{b^2}{12\,S}\right) = \frac{g}{k^2} \left(a + \frac{b^3}{12\,S}\right) \cdot (f - \alpha).$$

Or, if we put l for $\left(a + \frac{b^3}{12\,S}\right) \div k^2$, there will ariſe $\frac{\ddot{\alpha}}{\dot{t}^2} = \frac{g}{l}(f - \alpha)$. Multiplying this by $\dot{\alpha}$, we obtain for the fluent $\left(\frac{\dot{\alpha}}{\dot{t}}\right)^2 = \frac{g}{l}(2f\alpha - \alpha^2) + C$, a conſtant quantity. Here $\frac{\dot{\alpha}}{\dot{t}}$ is the angular velocity; ſo that at the ſame time that $\frac{\dot{\alpha}}{\dot{t}} = 0$, $\alpha = 0$, and $C = 0$, which gives

$$\dot{t} = \sqrt{\frac{l}{g}} \cdot \frac{\dot{\alpha}}{\sqrt{2f\alpha - \alpha^2}}.$$

Taking the fluents again, we have

$$t = \sqrt{\frac{l}{g}} \cdot \text{arc}\left(\cos. = \frac{f-\alpha}{f}\right) + C'.$$

To correct this we muſt conſider that when $t = 0$, $\alpha = 0$, and $C' = 0$; whence we find

$$\frac{f-\alpha}{f} = \cos.\left(t\sqrt{\frac{g}{l}}\right), \text{ or } \alpha = f\left(1 \cos.\left(t\sqrt{\frac{g}{l}}\right)\right)$$

The variables compriſed in this equation are the arcs α and $t\sqrt{\frac{g}{l}}$: where, if we attribute ſucceſſively to

$t\sqrt{\frac{g}{l}}$ the values . . $0,\ \frac{1}{2}\pi,\ \ \pi,\ \frac{3}{2}\pi,\ 2\pi$, &c.
we find . . . $0,\ f,\ 2f,\ f,\ \ 0$, &c. for the correſponding values of α; which ſhews that the moving body, after deſcribing the deſcending arc f, will deſcribe an aſcending arc f in the ſame time, then return and riſe the contrary way, and ſo on. It is obvious alſo, that the time of making a complete oſcillation is given by taking $\pi = t\sqrt{\frac{g}{l}}$, ſince that hypotheſis gives $\alpha = 2f$, as it ought to do. Hence there reſults

$$t = \pi\sqrt{\frac{l}{g}}.$$

This expreſſion for the time does not compriſe the quantity f; whence it follows, that the vibrations are iſochronal: and, by comparing the above with the value of t for the ſimple pendulum vibrating in indefinitely ſmall arcs (art. 269.), it will appear that they exactly agree: conſequently, the chief circumſtances traced there apply immediately to the preſent caſe. Now if we reſtore the value of l we ſhall have

$$l = \frac{12\,k^2\,\mathrm{S}}{12\,a\,\mathrm{S} + b^3}$$

An expreſſion which ſhews the length of the ſynchronal ſimple pendulum.

And, for the time of oſcillation,

$$t = \pi k\sqrt{\frac{12\,\mathrm{S}}{g\,(12\,a\,\mathrm{S} + b^3)}}.$$

424. By way of applying theſe principles, take the inſtance of a priſmatic body, ſuch that the tranſverſe ſection of the immerged part in the poſition of equilibrium is an iſoſceles triangle, of which the vertices of the equal angles are in the plane of floatation: the upper part of the ſection may have almoſt any variety of form. Let b repreſent the ſemi-baſe of the iſoſceles triangle, which is the tranſverſe ſection of the part immerſed, h the height of this triangle, and d the diſtance of the upper ſurface of the fluid from the centre of gravity of the body: the diſtance of that ſurface from the centre of gravity of the triangle immerſed will be $\frac{1}{3}h$: conſequently, the diſtance between the centres of gravity of the body and of the fluid diſplaced will be $a = d - \frac{1}{3}h$; the area immerſed will be $\mathrm{S} = bh$, and the value (418. ii.) of the ſtability will be

$$\frac{\mathrm{W}}{3h}\left(3\,dh - h^2 + 2\,b^2\right)$$

which may be either poſitive, nothing, or negative, as in the various caſes before ſpecified. The length of the ſynchronal pendulum will be $= \frac{3hk^2}{3dh-h^2+2b^2}$.

We may here remark, that the oſcillations of a floating body in *finite* angles are not analogous to thoſe of a cycloidal pendulum: for the force of ſtability varies in a proportion very different from that of the diſtance from quieſcence, unleſs the arcs of vibration be of evaneſcent magnitude.

SCHOLIUM.

425. The determination of the ſtability and the time of oſcillation of floating bodies, a ſketch of which is given in this chapter, is by no means a matter of mere curioſity: but is, when conſidered more at large, of conſiderable practical importance. Thoſe ſtudents who wiſh to purſue this ſubject may read with much advantage the admirable treatiſe by Leonard Euler, entitled *Théorie Complette de la Conſtruction et de la Manœuvre des Vaiſſeaux*, or the Engliſh tranſlation by colonel Henry Watſon; and an ingenious paper in the *Philoſophical Tranſactions* for 1796, by Mr. Atwood. In ſhips of war and merchandiſe the calculations are very complex and operoſe, and not always ſo accurate as is deſirable; but in river and canal boats the regularity and ſimplicity of the form of the veſſel itſelf, together with the compact diſpoſition and homogeneal quality of the burden, render the computations far more eaſy.

In that valuable miſcellany *Tilloch's Philoſophical Magazine* there is a paper on this ſubject by Mr. John Engliſh; from which we ſhall make an extract, ſhewing the application of the principles to river and canal boats.

"Veſſels of this kind," ſays Mr. E. "are generally of the ſame tranſverſe ſection throughout their whole length, except a ſmall part in prow and ſtern, formed by ſegments of circles or other ſimple curves; therefore a length may eaſily be aſſigned ſuch, that any of the tranſverſe ſections being multiplied thereby, the product will be equal to the whole ſolidity of the veſſel. The form of the ſection ABCD is for the moſt part either rectangular, as in fig. 13. pl. XVI. trapezoidal, as in fig. 1. pl. XVII. or mixtilineal, as in fig. 14. pl. XVI., in all which MM repreſents the line of floatation when upright, and EF that when inclined at any angle MXE; alſo G repreſents the centre of gravity of the whole veſſel, and R that of the part immerſed.

"If the veſſel be loaded quite up to the line AB, and the ſpecific gravity of the boat and burden be the ſame, then the point G is ſimply the centre of gravity of the ſection ABCD; but if not, the centres of gravity of the boat and burden muſt be found ſeparately, and reduced to one by the common method, namely, by dividing the ſum of the momenta by the ſum of weights, or areas, which in this caſe are as the weights. The point R is always the centre of gravity of the ſection MMCD, which, if conſiſting of different figures, muſt alſo be found by dividing the ſum of the momenta by the ſum of the weights as common. Theſe two points being found, the next thing neceſſary is to determine the area of the two equal triangles MXE, MXF, their centres of gravity *o*, *o*, and the perpendicular projected diſtance *n n* of theſe points on the water line EF. This being done, through R and parallel to EF draw RT = a fourth proportional to the whole area MMCD, either triangle MXE or MXF, and the diſtance *n n*; through T, and at right angles to RT or EF, draw TS meeting the vertical axis of the veſſel in S the metacentre; alſo through the points G, B, and parallel to ST, draw NGW and BV; moreover through S, and parallel to EF, draw WSV, meeting the two former in V and W; then SW is as the ſtability of the veſſel, which will be poſitive, nothing, or negative, according as the point S is above, coincident with, or below, the point G. If now we ſuppoſe W to repreſent the weight of the whole veſſel and burden (which will be equal to the ſection MMCD multiplied by the length of the veſſel), and P to repreſent the required weight applied at the gunwale B to ſuſtain the veſſel at the given angle of inclination; we ſhall always have this proportion: as VS : SW : : W : P : which proportion is general, whether SW be poſitive or negative; it muſt only in the latter caſe be ſuppoſed to act upward, to prevent an overturn.

" In the rectangular veſſel, of given weight and dimenſions, the whole proceſs is ſo evident, that any further explanation would be unneceſſary. In the trapezoidal veſſel, after having found the points G and R, let AD, BC, be produced until they meet in K. Then, ſince the two ſections MMCD, EFDC are equal, the two triangles MMK, EFK are alſo equal; and therefore the rectangle EK × KF = KM × KM = $\overline{\text{KM}}^2$; and ſince the angle of inclination is ſuppoſed to be known, the angles at E and F are given. Conſequently, if a mean proportional be found between the ſines of the angles at E and F, we ſhall have the following proportions :

" As the mean proportional thus found : ſine ∠ E : : KM : KF, and as the ſaid mean proportional : ſine ∠ F : : KM : KE; therefore ME, MF become known: from whence the area of either triangle MXE or MXF, the diſtance *n n*, and all the other requiſites, may be found.

" In the mixtilineal ſection, let AB = 9 feet = 108 inches, the whole depth = 6 feet = 72 inches, and the altitude of MM the line of floatation 4 feet or 48 inches; alſo let the two curvilinear parts be circular quadrants of two feet, or 24 inches radius each. Then the area of the two quadrants = 904·7808 ſquare inches, and the diſtance of their centres of gravity from the bottom = 13·8177 inches very nearly; alſo the area of the included rectangle *a b i e* = 1440 ſquare inches, and the altitude of its centre of gravity 12 inches; in like manner, the area of the rectangle AB *c d* will be found = 5184 ſquare inches, and the altitude of its centre of gravity 48 inches: therefore we ſhall have

Momentum of the two quad.	= 904·7808	× 13.817	= 12501·98966016
Momentum of the rectan. *a b i e*	= 1440·	× 12	= 17280·
Momentum of the rectan. AB *c d*	= 5184·	× 48	= 248832·
	7528 7808		278613·98966016

" Now the ſum of the momenta, divided by the ſum of the areas, will give $\frac{278613\cdot98966016}{7528\cdot7808}$ = 37·006 inches, the altitude of G, the centre of gravity of the ſection ABCD above the bottom. In like manner, the altitude of R, the centre of gravity of the ſection MMCD, will be found to be equal $\frac{123093\cdot98966016}{4936\cdot7808}$ = 24·934 inches; and conſequently their difference, or the value of GR = 12·072 inches, will be found.

" Suppoſe the veſſel to heel 15°, and we ſhall have the following proportion; namely, As radius : tangent of 15° : : MX = 54 inches : 14·469 inches = ME or MF; and conſequently the area of either triangle MXE or MXF = 390·663 ſquare inches. Therefore, as 4936·7808 : 390·663 : : 72 = *n n* = $\frac{2}{3}$ AB : 5·6975 inches = RT; and, again, as radius : ſine of 15° : : 12·072 = GR : 3·1245 inches = RN; conſequently RT − RN = 5·6975 − 3·1245 = 2·573 inches = SW, the ſtability required.

" Moreover, as the ſine of 15° : radius : : 5·6975 = RT : 22·013 = RS, to which, if we add 24·934, the altitude of the point R, we ſhall have 46·947 for the height of the metacentre, which taken from 72, the whole altitude, there remains 25·053; from which, and the half width = 54 inches, the diſtance BS is found = 59·529 inches very nearly, and the angle SBV = 80° 06′ 42″; from whence SV = 58·645 inches.

" Again: Let us ſuppoſe the mean length of the veſſel to be 40 feet, or 480 inches, and we ſhall have the weight of the whole veſſel equal to the area of the ſection MMCD = 4936·7808 multiplied by 480 = 2369654·784 cubic inches of water, which weighs exactly 85708 pounds avoirdupoiſe, allowing the cubic foot to weigh 62·5 pounds.

"And, finally, as SV : SW (*i. e.*) as 58·645 : 2·573 : : 85708 : 3760 + the weight on the gunwale which will ſuſtain the veſſel at the given inclination. Therefore a veſſel of the above dimenſions, and weighing 38 tons, 5 cwts. 28 lbs. will require a weight of 1 ton, 13 cwts. 64 lbs. to make her incline 15°.

"In this example, the deflecting power has been ſuppoſed to act perpendicularly on the gunwale at B; but if the veſſel is navigated by ſails, the centre velique * muſt be found; with which, and the angle of deflection, the projected diſtance thereof on the line SV may be obtained; and then the power, calculated as above, neceſſary to be applied at the projected point, will be that part of the wind's force which cauſes the veſſel to heel, and converſely, if the weight and dimenſions of the veſſel, the area and altitude of the ſails, the direction and velocity of the wind, be given, the angle of direction may be found."

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* The *centre velique* (a term firſt uſed by M. *Bouguer*) is the centre of gravity of the *equivalent* ſail; that is, of the ſail whoſe poſition and magnitude are ſuch as cauſe it to be acted upon by the wind when the veſſel is ſailing, in a manner equivalent to the action of the wind upon all the ſails together which the veſſel actually carries. See alſo Euler on Veſſels, Book III, Chap. ii. § 10, 11.

CHAPTER IV.

On the Phenomena of Attraction in Capillary Tubes.

426. THE appellation *capillary*, in a general ſenſe, is given to any thing on account of its extreme fineneſs, becauſe it in that reſpect reſembles hair. In phyſics, *capillary tubes* are ſmall pipes of glaſs whoſe canals are extremely narrow. The internal diameter of theſe tubes may vary from $\frac{1}{20}$ to $\frac{1}{50}$ of an inch: indeed Dr. Hook affirms that he drew tubes in the flame of a lamp much ſmaller, and reſembling a ſpider's thread.

If ſeveral capillary tubes of different diameters, and open at both ends, be immerſed a little way into water, the fluid will be ſeen to ſtand higher in the tubes than the ſurface of the water without, and this is in a greater degree as the tube is ſmaller; the height of the ſurface of the fluid in the tube above that of the fluid in the reſervoir, being nearly in the reciprocal ratio of the diameter of the tube: ſo that the diameter of the tube * multiplied into the altitude of the fluid in it (above that in the reſervoir) is nearly a conſtant quantity for the ſame fluid. This conſtant quantity is found by experiment to be about ·048 of an inch when the fluid is water; ·036 of an inch for vinegar or *ol. tar. per deliquium*; and ·024 of an inch for ſweet oil.

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*Among the various methods of determining the diameter of a uniform capillary tube, the following ſeems the beſt and moſt accurate. Pour into the tube a certain quantity of mercury whoſe weight in troy grains is w, and meaſure carefully the length l of the tube which it occupies; then is the diameter $d = {}^{\cdot}0192523\sqrt{\frac{w}{l}}$. For, the ſpecific gravity of mercury being 13568, a cubic inch weighs 3435·16 grains: hence $1 : d^2 l \times .785398 :: 3435{}^{\cdot}16 : w$. Whence, by multiplying means and extremes, &c. we find $d = \sqrt{\frac{w}{3435{}^{\cdot}16 \times {}^{\cdot}785398 \times l}} = {}^{\cdot}0192523\sqrt{\frac{w}{l}}$. If l be the whole length of the tube in inches, and w the difference in troy grains between its weight when empty and when full of mercury, the ſame theorem will obviouſly aſcertain the diameter.

When quickſilver is put into the tube, the contrary to all this takes place; for that fluid ſtands lower within the tube than its ſurface in the veſſel, and the *lower* as the tube is ſmaller.

Another phenomenon of theſe tubes is, that ſuch of them as would naturally only diſcharge water by drops, when electrified yield a continued and accelerated ſtream; and the acceleration is proportional to the minuteneſs of the tube. Nay, the effect of electricity is ſo conſiderable that it produces a continued ſtream from a very ſmall tube, out of which the water would not drop at all previous to the excitation by electricity. But we ſhall not attempt here any explanation of this phenomenon of the continued ſtream: our preſent object being merely to ſtate the moſt approved method of accounting for the aſcent and ſuſpenſion of fluids in theſe tubes, according to the principle of attraction or of adheſion.

In accounting for theſe phenomena of capillary tubes we muſt deviate from the method we have generally followed hitherto, of demonſtrating a ſeries of connected propoſitions: for the mode of elucidation we adopt has, at moſt, only probability on its ſide; and we would not willingly delude the ſtudent with an appearance of ſtrict demonſtration, when that kind of proof is incompatible with our preſent knowledge of the ſubject.

427. It will be neceſſary firſt to premiſe that the attraction between the particles of glaſs and water is greater than the coheſive attraction between the particles of water itſelf. For if this were not the caſe, the leaſt quantity or drop of water applied to the underſide of a glaſs tube placed horizontally would not adhere to it, but fall down immediately in the direction of gravity: whereas this does not happen till the bulk and weight of the drop be ſo far increaſed as to exceed the attractive power of the glaſs, and then it falls off.

Since, then, we find ſuch a ſtrong attractive power exerted at the ſurface of glaſs, it will be natural to conceive that ſuch power muſt act ſenſibly on the ſurface of a fluid that is not viſcid (water, for inſtance) contained within the ſmall cavity of a glaſs tube, and that it will be proportionally ſtronger as the internal diameter of the tube is the ſmaller: for that the efficacy of the power to hold up the fluid in the tube will follow the inverſe proportion of the diameter will be highly probable, if we conſider that only ſuch particles as are in contact with the fluid, and thoſe immediately above the ſurface, can affect it.

428. Hence moſt philoſophers aſſert that *the ſuſpenſion of the fluid in capillary tubes is owing to the attraction of the narrow ring of glaſs contiguous to the upper ſurface of the fluid.* The reaſoning adduced is of this kind: Every ring of glaſs below the ſurface

attracts the water above it as much downwards as it attracts the water below it upwards, and conſequently can contribute nothing towards the ſupport of the column: and the action of the loweſt ring upon all the fluid of the tube within its ſurface of attraction muſt either concur with the force of gravity to bring the fluid downwards, or acting upon it at right angles can have no effect in ſuſpending it within the tube. The fluid, therefore, can only be ſupported by the ring of glaſs contiguous to its upper ſurface, which, attracting upwards, oppoſes the action of gravitation by which the fluid is ſolicited downwards. And the ſame kind of reaſoning may be applied to the fluid raiſed between parallel planes of glaſs.

429. The preceding reaſoning being admitted, it will follow, in conformity with the experiments, that *in capillary tubes the heights to which the fluid riſes by virtue of the attraction are inverſely as the internal diameters.* For the fluid being ſuſpended by the action of the annulus of glaſs contiguous to the upper ſurface, and the diſtance to which the attraction of glaſs upon any one fluid reaches being unvaried, the force which ſuſtains the fluid will be as the number of attracting particles, that is, as the circumference, or as the diameter of the upper ring, or of the tube. Let Q, q, then, repreſent the quantities of fluid to be raiſed in two tubes of different bores; D, d, the reſpective internal diameters; and H, h, the heights to which fluids riſe in the tubes: then, becauſe Q, q, repreſent two cylinders of the fluid, we have $Q : q :: D^2H : d^2h$; and, from the nature of this attraction, which is as the diameters of the tubes, $D : d :: Q : q$; conſequently $D^2H : d^2h :: D : d$, or $DH : dh :: 1 : 1$, or, finally, $D : d :: h : H$.

Dr. *Jurin*, who firſt offered this ſolution of the phenomenon, ſays, the effect is the ſame *in vacuo* as in the air: but in his time the air-pumps would not exhauſt ſufficiently to determine this point: the pumps which are now made may perhaps ſhew that the water will not be ſupported after a *very great* degree of exhauſtion.

Mr. *Martin* ſays, the power by which the fluid is raiſed will keep it in the tube for any time without exhaling or evaporation; as he tried by hanging ſeveral capillary tubes thus charged with their fluids for months together in the ſummer ſun, whoſe heat did not appear to diminiſh the fluids in the leaſt ſenſible degree.

430. Another curious circumſtance aſcribed to the ſame cauſe is the following: *Between two glaſs plates, meeting on one ſide, and kept open at a ſmall diſtance on the other, water will riſe unequally; and its upper ſurface will form a hyperbolic curve, in*

which the altitudes of the ſeveral points above the ſurface of the fluid in the reſervoir will be to one another reciprocally as their perpendicular diſtances from the line in which the plates meet. This was firſt obſerved by Dr. *Brook Taylor*, and is thus accounted for on the ſame hypotheſis as before. Let AE (fig. 3. pl. XVII.) be the horizontal ſurface of the fluid in the reſervoir, AF the line in which the plates meet, HGIKL the curve formed by the ſurface of the ſuſpended fluid; GB, IC, KD, LE, perpendiculars to AE, ſhewing the heights of the reſpective points G, I, K, L, above the horizontal ſurface AE, and AB, AC, AD, AE, perpendiculars to AF, meaſuring the diſtances of the ſame points from the line in which the planes meet: theſe heights and diſtances are reciprocally proportional. For let the lines GB, IC, KD, LE, repreſent pillars of fluid of an equal evaneſcent breadth: thoſe portions of the glaſs plates which by their attraction ſupport theſe pillars, being of equal breadth, will ſuſtain equal quantities of fluid; that is, the pillars will be equal. But the pillars may be conſidered as parallelopipeds, which are equal when their baſes and altitudes are reciprocally proportional. And the baſes, being of equal breadth, are as their lengths; that is, as the intervals between the plates: and ſince theſe intervals continually increaſe as the diſtance from the line AF increaſes, theſe intervals at the points B, C, D, E, are as their diſtances AB, AC, AD, AE, from the line AF. Since, then, the heights of the pillars are reciprocally as the intervals, the heights GB, IC, &c. are reciprocally as the diſtances AB, AC, &c. and HGIKL is a hyperbola whoſe aſymptotes are AF and AE.

431. If two long glaſs planes are firſt ſmeared over with oil and then ſet together at their ends, and inclined to each other under a very ſmall angle, and a drop of the oil ſo placed between them as to touch both planes, it will immediately begin to move towards the touching ends, or angles of the planes; and that motion will be continued with an accelerated velocity, by reaſon of the increaſing action of the planes, on account of the decreaſing diſtance between them, and the larger portion of touching ſurface on each ſide of the drop.

432. If glaſs be applied to any other fluid whoſe particles attract each other more ſtrongly than glaſs attracts them, all the phenomena of ſuch a fluid in capillary tubes, and between glaſs planes, will be juſt the reverſe of thoſe which we have ſtated as taking place with regard to water. Now quickſilver is ſuch a fluid, and therefore it will ſtand lower within a capillary tube than without (art. 426.); the ſurface will be convex, and not concave, as in water; and between the planes it will

move the contrary way. But if a baſin or diſh be made of copper or braſs, and poliſhed well within, and then tinned all over, mercury put into ſuch a veſſel will every-where unite with the tin, and may perhaps be ſaid to *wet* it, as water does glaſs; and the mercury put into this mercurial baſin will, if clean, be attracted and riſe all around the ſides, and have the ſame phenomena with water put into a wet glaſs, tranſparency alone excepted.

433. If water riſe in any capillary tube T to a certain height, and another veſſel be put into the water, having the upper end capillary and of the ſame diameter as the tube T, but the lower part of any greater ſize; then, if the air be drawn out of this veſſel by ſuction until the water enters into the capillary part, it will ſtand at the ſame altitude as in the tube T after the ſuction ceaſes and the air is admitted into the capillary part. In this caſe the cylindrical part of the irregular veſſel whoſe diameter and height equal thoſe of the capillary tube ſeems to be ſupported by the ſame power as the water in the latter, that is, by the attraction of the glaſs annulus contiguous to the upper ſurface of the fluid: and the other part ſurrounding this capillary cylinder is ſupported by the preſſure of the air upon the ſurface of the water in the reſervoir; which is proved hence, that if the whole be placed under the receiver of an air-pump, and the air exhauſted from the ſurface of the veſſel, the water will not be ſupported as before.

434. The opinion that the ſuſpenſion of the fluid in capillary tubes is occaſioned by the attraction of the glaſs annulus contiguous to the upper ſurface has been pretty generally acquieſced in: but the ingenious Dr. *Hamilton* is of opinion, on the contrary (ſee his Eſſays), that the fluid is ſupported by the *lower* annulus contiguous to the bottom of the tube: this he imagines will firſt draw up a plate of water immediately under it, and then a ſucceſſion of plates, till the weight of the whole is equivalent to the attraction of that annulus. A ſimilar opinion is embraced by Dr. *Matthew Young*. But Dr. *Parkinſon* and Mr. *Vince* object to Dr. Hamilton's ſolution. Mr. Vince ſays, "If this were the caſe, the quantity ſupported, and conſequently the altitude of the fluid, would depend upon the orifice at the bottom; whereas experiments ſhow that the altitude at which the fluid is ſupported depends upon the diameter of the tube at the upper ſurface of the fluid, without any regard to the form of the tube below it. Again, if in a capillary tube water will ſtand at the altitude of an inch above the ſurface of the fluid in the veſſel, and you depreſs the tube till there be only an inch of it above the ſurface, the water will then not riſe to the top of

the tube, and if you depreſs the tube ſtill lower the water will not riſe to the top. Thus there will always be an annular ſurface of the tube *above* the fluid, which is a ſtrong argument in favour of the fluid being ſupported by the attraction of ſuch a ſurface. If the fluid were raiſed by the attraction of the annulus at the bottom, when the length of the tube above the ſurface was leſs than an inch, the fluid ought to run over, and thus a perpetual motion would be formed." *Hydroſtatics*, p. 68.

435. Still, though the attraction of the glaſs, and perhaps of the annulus immediately above the upper ſurface of the fluid in the tube, be the chief cauſe of the aſcent of any fluids, yet it muſt be allowed that the nature and quality of each particular fluid is to be regarded in moſt of the phenomena, which are not proportioned to the attracting power of the glaſs ſolely, but to that conjointly with the various diſpoſition of fluids to yield to it: nor is the denſity of the fluid of primary conſideration; it being manifeſt from experiment, that ſome lighter fluids will riſe to a leſs height than others which are much heavier. Nor, again, does it depend on the viſcidity or tenacity of parts; for hard white varniſh (very thick and viſcid) and ſpirit of wine aſcend nearly to the ſame height. From experiments it likewiſe appears that heat and cold are not concerned in this aſcent of fluids, very hot water ſtanding at the ſame height as cold. And a ſolution of any ſalt in water makes but very little difference in the heights.

Mr. *Martin* made and repeated frequently a ſeries of experiments on ſeveral fluids, to aſcertain the altitudes at which they would be ſuſtained in a tube whoſe internal diameter was about $\frac{1}{30}$, or more accurately $\frac{1}{25}$ of an inch: the reſults are given below:

	Alt.		Alt.
Common ſpring water	1·2 *inch.*	Red wine . . .	0·75 *inch.*
Spirit of urine . .	1·1	White wine . .	·75
Tincture of galls .	1·1	Ale	·75
Recent urine . .	1·1	Ol. ſulp. per campanum	·65
Spirit of ſalt . .	·9	Oil of vitriol . .	·65
Ol. tar. per deliq. .	·9	Sweet oil . . .	·6
Vinegar	·95	Oil of turpentine .	·55
Small-beer	·9	Geneva	·55
Strong ſpirit of nitre	·85	Rum	·5
Spirit of hartſhorn .	·85	Brandy	·5
Cream	·8	White hard varniſh	·5
Skimmed milk . .	·8	Spirit of wine . .	·45
Aqua fortis	·75	Tincture of Mars .	·45

M. *Clairaut*, in his excellent work on the figure of the earth, ſays, "The truth is, that when we compare the elevation of the ſame fluid in two different tubes, the attraction of each ſurface is the reſult of all the particular attractions exerted by the different moleculæ of the glaſs upon all thoſe of the liquid which are at diſtances ſmall enough to be ſubjected to the effect of theſe attractions."

The Abbe *Haüy*, in his *Traité Elémentaire de Phyſique*, adopts the ſame hypotheſis, and ingeniouſly applies it to the elucidation of various phenomena. As we are not willing to employ more time upon a topic which is almoſt entirely hypothetical, we refer the reader to M. Haüy's work, tome I. p. 154, &c.

HYDRODYNAMICS.

BOOK IV.

Introductory Remarks.

436. *Hydrodynamics* is that part of mechanical ſcience which relates to the motion of non-elaſtic fluids, and the forces with which they act upon bodies.

This branch of mechanics is the moſt difficult, and the leaſt advanced: whatever we know of it is almoſt entirely due to the reſearches of the moderns; for the only work on the mechaniſm of fluids which has reached us from the ancients is the piece of *Archimedes* in two books, *De Inſidentibus humido*, in which the only inquiries reſpect the ſinking and floating of bodies in fluids, their relative gravities, levities, ſituations, and poſitions, while in equilibrio. We find, it is true, ſome hints and rules upon the motion of fluids in a treatiſe attributed to *Sextus Julius Frontinus*, inſpector of public fountains at Rome, under the emperors Nerva, Cocceius, and Trajan, entitled, *De Aquæ ductibus urbis Romæ commentarius*; but they are not of ſufficient importance to deſerve much attention by a ſtudent of this ſcience. *Benedict Caſtelli* was the firſt who opened the way to a true meaſure of the flux of waters, in his treatiſe *Della meſura dell' Acque Currenti*; which meaſure he found to depend upon the area of the ſection, and the velocity of the water, conjointly. The moſt valuable and important diſcoveries and theorems in this department of ſcience are given in Sir *Iſaac Newton's Principia*, book II.; the *Hydrodynamique* of *Daniel Bernoulli*; the *Traite des Fluides* by M. *D'Alembert*; the *Examen maritimo Theorico Practico* of *D. George Juan*; the *Hydrodynamique* of M. *Boſſut*; *Principes d'Hydraulique* by M. *Buat*; and the *Handbuch der Mechanik und der Hydraulik* by Mr. *Eytelwein*. To theſe may be added an ingenious paper on the Motion and Reſiſtance of Fluids by Mr. *Vince*, in the Philoſophical Tranſactions for 1795; and thoſe by the late Dr. *Matthew Young* in the Iriſh Tranſactions.

437. Could we know with certainty the maſs, the figure, and the number of particles of a fluid in motion, the laws of its motion might be determined by the reſolution of this problem, viz. to find the motion of a propoſed ſyſtem of ſmall free bodies acting one upon the other in obedience to ſome given exterior force. We are, however, very far from being in poſſeſſion of the data requiſite for the ſolution of this problem: and even if we were in poſſeſſion of them it is doubtful whether we ſhould be much further advanced; as it might be extremely difficult to deduce any convenient practical reſults from the intricate and complex expreſſions which might ſtand at the foot of the inveſtigation. The wiſeſt philoſophers have had their doubts with regard to every abſtract theory concerning the motion of fluids; and the greateſt geometers and analyſts affirm that thoſe methods, which have directed them to ſuch curious and uſeful concluſions in the mechanics of ſolid bodies, do not furniſh any concluſions with reſpect to fluids but ſuch as are too general and uncertain for the greater number of particular caſes. On theſe accounts a detailed exhibition of the theory alone would ſcarcely be of any utility: we ſhall, therefore, enter but little into the theoretical part of Hydrodynamics; but preſent merely a few propoſitions that are leaſt dubious in their nature, and endeavour to ſupply the deficiency by ſtating the reſults of ſome of the moſt ingenious, careful, and ſatisfactory experiments with which we are acquainted.

The motion of the water in pumps, depending in part upon the principles of Pneumatics, will be treated of in our fifth book.

CHAPTER I.

On the Discharge of Fluids, through Apertures in the Bottom and Sides of Vessels, and on Spouting Fluids.

438. **Prop.** *If a fluid run through any tube, which is kept continually full, and the velocity of the fluid in every part of one and the ſame ſection be the ſame, the velocities in different ſections will be inverſely as the areas of the ſections.*

For as the tube is always equally full, the ſame quantity of fluid will run through every ſection in the ſame time: but the quantity paſſing through any ſection S with the velocity V in any given time manifeſtly varies as S and V conjointly, or as $S \, . \, V$; and, in like manner, the quantity paſſing through any other ſection s with velocity v muſt vary as $s \, . \, v$ in a given time: conſequently we muſt have $S \, . \, V = s \, . \, v$, and $S : s :: v : V$.

It is ſuppoſed in this propoſition that the changes in the diameters of the tube are *continual*, and no-where abrupt ſo as to break the law of continuity in the ſides of the tube: for if there be any angles, or conſiderable ſinuoſities in the tube, they will produce eddies in the motion of the fluid, and the propoſition will not obtain.

439. **Prop.** *If a fluid flowing through a very ſmall orifice in the bottom of a veſſel be kept conſtantly at the ſame height in the veſſel, by being ſupplied as faſt above as it runs out below, the velocity of the effluent fluid will be equal to that which a heavy body would acquire in falling freely through the height of the fluid above the orifice.*

Let MNOP (fig. 4. pl. XVII.) repreſent a veſſel filled with a fluid up to the level GH; MP the bottom in which is the aperture CD (*very* ſmall compared with MP); CIKD the column of the fluid ſtanding directly above the aperture, and CABD the loweſt plate of the fluid immediately contiguous to the aperture. Alſo let v denote the velocity which a heavy body would acquire in falling freely through BD the height of the plate, and V the velocity acquired by the ſame plate during its deſcent through the ſame ſpace until it is diſcharged by the preſſure of the column CIKD. If we ſuppoſe the loweſt plate of fluid ACBD to fall as a heavy body through the height BD, its moving force will be its own weight. Again, ſuppoſe it to be accelerated by its own weight, together with the preſſure of the ambient fluid, about the column CIKD, that is, (art. 386.) by the weight of the column CIKD, through the ſame ſpace, that is, while it is accelerated from quieſcence until it is actually diſcharged: then (by what has been ſhewn in Dynamics, art. 228.), the velocity in the former caſe will be to that in the latter as the moving forces and the times in which they act directly, and the quantities of matter moved inverſely. But the moving forces are to each other as the heights BD and KD; the times in which they act are inverſely as the velocities, the ſpace through which the body is accelerated being given; and the quantities of matter moved are equal: therefore, $v : V :: \frac{BD}{v} : \frac{KD}{V}$, whence $v^2 : V^2 :: BD : KD$, or $v : V :: \sqrt{BD} : \sqrt{KD}$. Now v is the velocity which a heavy body would actually acquire in falling through the ſpace BD; conſequently V the velocity of the effluent fluid is that which a heavy body would acquire in falling through KD, the whole altitude of the fluid above the orifice.

Cor. 1. In the ſame manner it may be ſhewn, that if a pipe be inſerted horizontally in the veſſel MNOP (fig. 5.), the plate of fluid ACBD will be diſcharged with the ſame velocity as before (if its centre of preſſure be of the ſame depth) whatever be the thickneſs of the plate; this velocity not depending upon a continual acceleration through the length of the tube, otherwiſe the effluent fluid could not attain its full velocity until a column had been diſcharged whoſe baſe is equal to the orifice and height equal to the length of the tube: whereas we find by experience that this full velocity can be attained by the thinneſt plate which can be let eſcape from the aperture.

Cor. 2. The velocities and quantities diſcharged at different depths are as the ſquare roots of the depths.

Cor. 3. The quantity run out in any time is equal to a cylinder, or priſm, whoſe baſe is the area of the orifice, and its altitude the ſpace deſcribed in that time by the velocity acquired by falling through the height of the fluid.

So that if h denote the height of the fluid,
a the area of the aperture,
g, $32\frac{1}{6}$ feet, or 386 inches,
and t the time of efflux,

we ſhall have for the quantity diſcharged $Q = a\,t\,\sqrt{2gh}$;
Or, when a and h are in feet . . . $Q = 8{\cdot}0208\,at\sqrt{h}$, feet;
When a and h are in inches, . . . $Q = 27{\cdot}7387\,at\sqrt{h}$, inch.
If the orifice is a circle whoſe diameter is d, then $\cdot785398\,d^2$ muſt be ſubſtituted for a;
And, when d and h are in feet . . . $Q = 6{\cdot}29952\,d^2t\sqrt{h}$, feet;
When d and h are in inches . . . $Q = 21{\cdot}78592\,d^2t\sqrt{h}$, inc.

And from either of theſe it will be eaſy to find either a, t, or h, when the other three quantities are given.

COR. 4. The force with which the effluent water impinges againſt any quieſcent body is proportional to the altitude of the fluid above the orifice.

For the force is as velocity × quantity of matter; but the quantity diſcharged in a given time is as the velocity: therefore the force is as the ſquare of the velocity, that is, by the demonſtration of the propoſition, as the height of the fluid.

COR. 5. The water ſpouts out with the ſame velocity whether it be downwards, or upwards, or ſideways; becauſe the preſſure of fluids is the ſame in all directions at the ſame depth.

COR. 6. Hence, if the adjutage be turned upwards, the jet will aſcend to the height of the ſurface of the water in the veſſel. And this is confirmed by experience, from which it appears that jets really aſcend *nearly* to the height of the reſervoir; the ſmall quantity abated ariſing from the friction againſt the ſides, the reſiſtance occaſioned by the oblique motion of the fluid in the bended pipe, and the reſiſtance of the air.

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440. What is ſaid in this propoſition and corollaries of the velocity of the effluent water, is true only of the middle filament of particles which iſſue through the centre of the aperture, which are ſuppoſed in theory to experience no retardation, and which, in fact, ſuffer no other retardation than what ariſes from the reſiſtance of the air, and their mutual adheſion and attrition againſt each other. But thoſe which iſſue near the edges of the aperture undergo a greater attrition, and therefore ſuffer a greater retardation. Hence it follows that the mean velocity of the whole column of effluent fluid will be conſiderably leſs than according to theory.

Sir Iſaac Newton, who examined every ſubject that came before him with peculiar accuracy, firſt diſcovered a contraction in the vein of effluent water; and found, that at the diſtance of about a diameter of the orifice the ſection of the vein contracted nearly in the ſubduplicate ratio of 2 to 1. Hence he concluded that the velocity of the water, after its exit from the aperture, was increaſed in this proportion, the ſame quantity paſſing in the ſame time through a narrower ſpace. From the quantity of water diſcharged in a given time through that narrow ſection, he found that its velocity there was that which a heavy body would acquire in falling through the height of the water above the orifice; and ſince the velocity there was greater than immediately in the orifice in the ſubduplicate ratio of 2 to 1, he concluded that the velocity of the effluent water in the orifice was equal to that which a heavy body would acquire in falling through *half* the altitude. But all this is true only of the *mean* velocity; for there is no cauſe which can actually accelerate the water after its exit from the orifice, whatever cauſes may contribute to its retardation. The manner in which the mean velocity of the water is increaſed after its diſcharge, though the actual velocity of the ſeveral particles continues unvaried, might be eaſily explained: but it need not be dwelt upon here. A circumſtance of conſiderable importance in the eſcape of fluids through a horizontal orifice ſeems to have been entirely neglected by moſt writers; we allude to a *whirling* motion: the fluid will revolve about the aperture, and at ſome diſtance from it, and eſcape with a revolving motion; the fluid ruſhes from all ſides in ſpiral ſtreams to ſupply the continual waſte. This circumſtance will be explained more diſtinctly in a ſubſequent propoſition: but, excepting that, the remainder of this chapter will be treated in conformity with the common hypotheſis; which, when corrected by the experiments in the enſuing chapter, will furniſh tolerably correct reſults.

441. Prop. *When a veſſel is left gradually to diſcharge itſelf by an orifice in the bottom, if the area of the ſection parallel to the bottom be every-where the ſame, the velocity of the ſurface of the fluid, and conſequently the velocity of the efflux, will be uniformly retarded.*

For (art. 438.) the velocity of the deſcending ſurface is to the velocity at the orifice as the area of the orifice to the area of the ſurface, which is a conſtant ratio; conſequently, the velocity of the deſcending ſurface varies as the velocity at the orifice, or as $\sqrt{h}$, by cor. 2. of the laſt article; that is, the velocity of the deſcending ſurface varies as the ſquare root of the ſpace which it has to deſcribe: ſo that this exactly correſponds with the caſe of a body projected perpendicularly upwards (art. 245.), where the velocity is as the ſquare root of the ſpace to be deſcribed: whence, as the retarding force is conſtant in the inſtance referred to, it muſt alſo be conſtant in the caſe before us, and the retardation uniform.

Comparing this with the inſtance juſt pointed out, we deduce theſe obvious corollaries:

Cor. 1. The quantities of water in a priſmatic veſſel diſcharged through an aperture in the bottom decreaſe in equal times as the ſeries of odd numbers 1, 3, 5, 7, 9, &c. taken in an inverted order.

Cor. 2. The quantity of water contained in an upright priſmatic veſſel is *half* that which would be diſcharged in the time of the entire gradual evacuation of the veſſel, if the water be kept always at the ſame altitude.

442. Prop. *To determine the time of emptying a veſſel of water by an orifice in the bottom of it, or in the ſide contiguous to the bottom, the height of the orifice being very ſmall compared with the altitude of the fluid.*

Let a = the area of the aperture;
h = the whole height of the fluid above the aperture;
x = the vertical ſpace deſcended by the upper ſurface in any time t;
A = the area of the upper ſurface;
$g = 32\frac{1}{6}$ as before, the meaſure of the force of gravity.

Then will the velocity of the effluent fluent at any time be repreſented, not by $\sqrt{2gh}$ as in art. 439. but by $\sqrt{2g(h-x)}$. This velocity will vary continually, becauſe x increaſes, and the difference $h-x$ diminiſhes continually, but it may be regarded as conſtant during the indefinitely ſmall time $\dot{t}$: ſo that in the time $\dot{t}$ there will eſcape through the orifice a priſm of the fluid which has that orifice a for its baſe, and $\sqrt{2g(h-x)}$ for its altitude. Thus the quantity of fluid diſcharged during the inſtant $\dot{t}$ is $= a\dot{t}\sqrt{2g(h-x)}$. But during the ſame time the upper ſurface has deſcended through the ſpace $\dot{x}$, and the veſſel has loſt a priſm or cylinder of the fluid whoſe height is $\dot{x}$ and baſe A, that is, a priſm whoſe capacity is $A\dot{x}$. Hence we have $A\dot{x} = a\dot{t}\sqrt{2g(h-x)}$;

$$\text{and} \quad \ldots\ldots \quad \dot{t} = \frac{A\dot{x}}{a\sqrt{2g(h-x)}} \quad \ldots\ldots \quad \text{(I.)}$$

As the area A will be given in functions of x, by the form of the veſſel, the ſecond member of this equation may be conſidered as containing only the variable quantity x; and it will be very eaſy in moſt caſes, by ſimply finding the fluents, to diſcover the ſucceſſive depreſſions and diſcharges of the fluid in any veſſel of known form.

443. By way of application, take the following examples:

I. *Let the veſſel be an upright priſm or cylinder.* Here the area A will be conſtant, becauſe every horizontal ſection of the priſm will be equal to its baſe. Hence we have

$$t = \frac{A}{a\sqrt{2g}} \cdot \int \frac{\dot{x}}{\sqrt{h-x}} = -\frac{2A}{a\sqrt{2g}} \sqrt{h-x} + C.$$

Now when the time t is nothing, the depreſſion of the upper ſurface A of the fluid is nothing alſo: thus we have at the ſame time $x=0$, and $t=0$; this condition determines the conſtant quantity $C = \frac{2A}{a\sqrt{2g}} \sqrt{h}$; and gives for the time of depreſſing the upper ſurface through the ſpace x,

$$t = \frac{2A}{a\sqrt{2g}} (\sqrt{h} - \sqrt{h-x}) \ldots\ldots (\text{II.})$$

To find the time of completely emptying the veſſel, we have only to make $x=h$, in which caſe the preceding expreſſion will become .... $t = \frac{A}{a}\sqrt{\frac{2h}{g}}$.

Cor. The time juſt found is double that in which an equal quantity would be diſcharged, if the veſſel were kept conſtantly full. For, in art. 439. cor. 3. we have $Q = at\sqrt{2gh}$, where, if $Q = Ah$, we have $t = \frac{Ah}{a\sqrt{2gh}} = \frac{A}{a}\sqrt{\frac{h}{2g}} = \frac{A}{2a}\sqrt{\frac{2h}{g}}$, which is half the preceding value of t.

Hence the reſult of this example correſponds with cor. 2. art. 441. as it manifeſtly ought to do.

II. *Let the veſſel be any ſolid of rotation, of which the axis is vertical.* Here A will be the area of a circle which has for its radius the ordinate y of the generating curve; that is, if $\pi = 3{\cdot}141593$, we ſhall have $A = \pi y^2$. Introducing this value into the equation marked I, we have

$$t = \frac{\pi}{a\sqrt{2g}} \cdot \int \frac{\dot{x} y^2}{\sqrt{h-x}} \ldots\ldots\ldots (\text{III.})$$

In any particular examples it will be neceſſary to put for y its value deduced in terms of x from the equation of the generating curve, and to find the fluents, which will be corrected by making at the ſame time $t=0$, and $x=0$.

III. *Let the ſolid of rotation be a paraboloid with its vertex downwards.* If p be the parameter of the generating parabola, the equation of the curve will be $y^2 = pX$, when the origin of the curve is ſuppoſed at V (fig. 2. pl. XIII.), the point where we conceive the aperture to be; or if we transfer the origin to A, the equation will be $y^2=p\ (h-x)$: this ſubſtituted for y^2 in equation II. gives

$$t = -\frac{2\pi p}{3a\sqrt{2g}} \cdot (h-x)^{\frac{3}{2}}+\mathrm{C}.$$

Determining the conſtant quantity as above directed, we obtain for the correct fluent

$$t = \frac{2\pi p}{3a\sqrt{2g}} \cdot (h^{\frac{3}{2}}-\overline{h-x}^{\frac{3}{2}}) \ldots\ldots\ (\mathrm{IV.})$$

If we make $h=x$, we have $t=\frac{2\pi p}{3a}\sqrt{\frac{h^3}{2g}}=\frac{2\pi y^2}{3a}\sqrt{\frac{h}{2g}}$ $=\frac{\mathrm{S}}{3a}\sqrt{\frac{2h}{g}}$, where S is the area of the upper ſurface of the fluid at the beginning of the exhauſtion, and h the whole height.

The above may ſuffice as a ſpecimen of the method, on the ſuppoſition that the velocity of efflux is that due to the *whole* height of the fluid. A very ingenious and complete paper on this ſubject, according to the hypotheſis that the velocity is that due to *half* the height of the fluid above the aperture, is given by Dr. *Hutton* in his Mathematical Miſcellany.

444. Prop. *If the water of a reſervoir which flows through a horizontal aperture be influenced by any foreign motion, it will form a hollow whirl above the orifice itſelf.*

Let DQ (fig. 6. pl. XVII.) repreſent a horizontal plane near the orifice EF, through which the fluid of the reſervoir MN flows. A fluid particle, D, ſituated in this plane, has a motion DB inclined to the axis AB; which may be decompoſed into two DC, CB. Let us ſuppoſe the plane DQ to deſcend parallel to itſelf along the axis with the motion, and then enquire into the motion of the particle D on the plane DQ; a motion which impreſſes upon all the particles ſituated in the plane DQ a centripetal force towards the centre C. Let any other horizontal motion whatever, not coincident in direction with DC, be impreſſed upon the ſame particles: under the joint influence of the two moving forces the particles will deſcribe round the centre C areas proportional to the times, and by the equilibrium of theſe motions they may aſſume a horizontal circular rotation.

Imagine that, during this horizontal circulation, the particle D in its approach towards the centre C, as in a ſpiral, ſhall deſcribe circular orbits, of which the diameters are ſucceſſively diminiſhed: put the rotatory velocity of the particle D$=v$; its diſtance CD from the centre $=r$; the time of one revolution $=t$: then, ſince the areas muſt be as the times, we ſhall have nearly $v \propto \frac{1}{r}$, $t \propto r^2$; and the centrifugal force of the particle D will $\propto \frac{1}{r^3}$. Indeed, when the particles which revolve at the ſurface of the funnel at MN are obſerved, it is ſeen that the effect which really takes place in nature is conformable to $t \propto r^2$ very nearly. Since, therefore, the centrifugal force in approaching the centre C increaſes as $\frac{1}{r^3}$, it will become ſufficient to form an equilibrium againſt the upper preſſure SD, which produces the centripetal force DC : a cavity KRTHPV will therefore be formed, round which the whirling fluid will ſupport itſelf by the centrifugal force of its rotation.

Let DQPR repreſent a circular fluid zone, the particles of which revolve about the cavity RP according to the law here indicated. Let the gravity of a fluid particle be $= w$, $CR = a$, $RD = b$, $DX = z$, $XZ = \dot{z}$, and the velocity of the particle $D = v$. If the centrifugal force of the particle D were equal to its gravity, its velocity (art. 282. cor. 3.) would be equal to that of a body falling by gravity alone through the ſpace $\frac{a+b}{2}$. And as a heavy body falls in one ſecond through the ſpace $16\frac{1}{12}$ feet $= \frac{1}{2} g$, the velocity of the particle D will be repreſented by $\sqrt{2g \cdot \frac{a+b}{2}} = \sqrt{g(a+b)}$. The centrifugal force in the circle is as v^2; therefore the centrifugal force of D will actually be $= \frac{-v^2 w}{g(a+b)}$. And ſince the centrifugal force $\propto \frac{1}{r^3}$; taking $\frac{1}{(a+b)^3} : \frac{1}{(a+b-z)^3} :: \frac{-v^2 w}{g(a+b)}$: a fourth term, we ſhall have the centrifugal force of the element of DX in $X = \frac{v^2 w (a+b)^2 \dot{z}}{g(a+b-z)^3}$; and that of the filament DX itſelf $= \frac{v^2 w (a+b)^2}{2g(a+b-z)^2} + C$. When $z = 0$, the fluent $= 0$, whence $C = \frac{-v^2 w}{2g}$: and taking $z = b$, the centrifugal force of the filament DR will be $= \frac{b v^2 w}{2 a^2 g} \cdot (2a+b)$. Now the quantity $b w$ is the gravity of the filament DR : hence the gravity of this filament is to its centrifugal force as $v^2 (2a+b)$, to $2a^2 g$.

445. When the fluid zone DRPQ is nearer the orifice EF, the preſſure SD increaſes; whence the centrifugal force of the zone muſt alſo be increaſed, in conſequence of the radius RC of the cavity being diminiſhed: hence we may determine the nature of the curve which forms the vertical ſection of the cavity KRT. For greater ſimplicity, let it be ſuppoſed that the ſides of the veſſel MD, NQ, have the ſame ſhape as thoſe of the cavity itſelf; ſo that DR $= b$ may be conſtant. Let AC $= s$, and CR $= y$. In the preceding formula ſubſtitute y for a; and, ſince the gravity of the filament DR is to that of the filament SD as b to x, we ſhall have, by compoſition of ratios, the centrifugal force of the filament DR to the preſſure SD, as $b\,v^2\,(2\,y + b)$ to $2\,g\,x\,y^2$. In order that there may be an equilibrium, this laſt muſt be a ratio of equality, that is, $b\,v^2\,(2\,y + b) = 2\,g\,x\,y^2$; whence we deduce $x\,y^2 - \frac{b\,v^2}{g}\,y - \frac{b^2\,v^2}{2\,g} = 0$, for the equation of the curve KRT. This is the 64th ſpecies in Sir Iſaac Newton's Enumeration of Lines of the Third Order: its convexity is turned toward the axis; and it has two aſymptotes, one of which is the axis AB, and the other is in MN, ſuppoſing the two points MN to be infinitely diſtant.

If the aſſumed poſitions in this and the preceding articles do not abſolutely coincide with nature, they approximate to its effects very nearly. It is not only poſſible, but there does actually exiſt in nature a whirling ſtream, of which the cavity turns its convex part to the axis, and in which $t \propto \frac{1}{r^2}$ very nearly, as is ſhewn by numerous experiments.

If any body, which floats at the ſurface of the liquor after the funnel has been formed, be of ſufficient ſize to cover the whole cavity, it will deſtroy the funnel in the upper part, and ſometimes alſo in the lower. For the body itſelf can only turn round its centre according to the law $v \propto r$; it therefore deſtroys by friction the law $v \propto \frac{1}{r}$ in the parts of the fluid contiguous to it, and conſequently deſtroys the upper part of the funnel, if not the funnel itſelf.

446. Prop. *If a notch or ſluice in form of a rectangle be cut in the vertical ſide of a veſſel full of water, or any other fluid, the quantity of water flowing through it will be $\frac{2}{3}$ of the quantity which would flow through an equal orifice placed horizontally at the whole depth, in the ſame time; the veſſel being conſtantly kept full.*

Let ABCD (fig. 7. pl. XVII.) be the vertical ſide of the reſervoir, EFGH the rectangular notch in it, and let IL $i\,l$ be a parallelogram of evaneſcent breadth. Then the velocity with which the water eſcapes at GH is to the velocity with which it eſcapes through IL il as $\surd$EG to $\surd$EI (art. 439. cor. 2.); and by the ſame corollary the quantities diſcharged in a given time through an evaneſcent parallelogram at the depth EG, and the evaneſcent parallelogram IL il, are in the ſame ratio; that is, as GH to IK, the ordinates in a parabola EKH, whoſe axis is EG, and greateſt ordinate GH. Therefore, the ſum of all the quantities diſcharged through all the parallelograms IL il, of which the rectangle EFGH is compoſed, is to the ſum of all the quantities diſcharged through as many equal parallelograms IL il at the depth EG as the ſum of all the elements IK ki of the parabola, to the ſum of all the correſponding elements IL li of the rectangle; that is, as the area of the parabola EKHG to that of the rectangle EFGH: or, the quantity running through the notch EFGH is to the quantity running through an equal horizontal area placed at GH, as EKHG to EGHF; that is, as 2 to 3, by the nature of the parabola.

Cor. 1. The mean velocity of the fluid in the notch is equal to two-thirds of that at GH.

Cor. 2. The quantity flowing through the rectangle ILHG is to the quantity diſcharged through an equal rectangle placed horizontally at GH, as the parabolic zone IKHG is to the rectangle ILHG. As is evident from the demonſtration of the propoſition.

447. Prop. *To determine the relation between the time and the quantity of water, or other non-elaſtic fluid, diſcharged from a veſſel through any vertical orifice; the velocity of the effluent fluid at any point being (as heretofore) ſuppoſed that due to the altitude of the upper ſurface of the fluid in the veſſel above that point.*

Let XYVZ (fig. 8. pl. XVII.) be the vertical ſide of the veſſel, and AM st M' a vertical orifice therein, of which the contour is a plane curve; SB a vertical line paſſing through A, the higheſt point of the orifice; and let MM', mm', be two horizontal lines indefinitely near the one to the other. Then, the upper ſurface of the fluid being ſuppoſed to paſs through S, put $SA=h'$, $SB=h$, $AP=x$, $MM'=y$, the velocity $32\frac{1}{6}$ f. which gravity communicates at the end of a ſecond $=g$, and the time $=t$: ſo ſhall $AB=h-h'$, and $Pp=\dot{x}$.

I. We propoſe firſt to aſcertain what will be the diſcharge (q) at this orifice during a determinate time T, *ſuppoſing the veſſel is all that time kept full up to the level of* S.

Now the velocity of the fluid diſcharged through the elementary trapezoid MM' $m'm$, whoſe ſurface is $y\dot{x}$, may be conſidered as that due to the height $SP=h'+x$: conſequently, if for a we ſubſtitute its value $y\dot{x}$, and for h its value in the firſt formula in art. 439. cor. 3. we ſhall have for the quantity diſcharged through that trapezoid in the time t

$$t\,y\,\dot{x}\sqrt{2g\,(h'+x)},\ \text{or}\ t\times y\,\dot{x}\sqrt{h'+x}\times\sqrt{2g}.$$

Therefore the quantity diſcharged during the time t through the portion of the orifice AMM′ is equal to

$$t\left(\int y\,\dot{x}\sqrt{h'+x}+\mathrm{C}\right)\sqrt{2g}.$$

Thus taking the fluent contained between $x=0$, and $x=\mathrm{AB}=h-h'$, we have for the quantity of efflux ſought

$$q=t\sqrt{2g}\left(\int y\,\dot{x}\sqrt{h'+x}+\mathrm{C}\right)\ \ldots\ldots\ldots\ \text{(i.)}$$

Here it may be obſerved, that as the nature of the line which bounds the orifice is conſidered as known, we may always ſubſtitute for y a function of x, and in finding the fluent there will in fact be no other variable quantity than x; of courſe the conſtant quantity C will be determined by conſidering that when $x=0$, the quantity diſcharged is alſo $=0$. And when C is known, the whole quantity diſcharged is readily found by introducing into the value of the diſcharge through AMM′, for x its value $h-h'$.

448. Let s be ſuch a height as if the fluid iſſued from all points of the orifice with the velocity due to that height, the total diſcharge through the orifice would be the ſame as has place naturally, conformably to equation i; then will the velocity of the iſſuing fluid be $=\sqrt{2gs}$, and the diſcharge through the elementary trapezoid $\mathrm{MM}'m'm=y\dot{x}$, in the time t, will be repreſented by $t\sqrt{2gs}\int y\dot{x}$: and ſince this ought to be equal to the value of q in equation i, we ſhall by making that equality, and reducing, find

$$s=\frac{\left(\int y\,\dot{x}\sqrt{h'+x}+\mathrm{C}\right)^2}{\left(\int y\,\dot{x}+\mathrm{C}'\right)^2}\ \ldots\ldots\ \text{(ii.)}$$

The fluent of the denominator muſt manifeſtly be taken between the ſame values of x as that of the numerator; and the conſtant quantity C′ is determined by conſidering that when $x=0$ the portion AMM′$=0$ alſo.

The quantity s, which we have juſt ſhewn how to determine, is generally called *the mean height* of the fluid above the orifice: the rule for finding this height is obviouſly nothing more than to *divide the value of* q *found equa.* i. *art.* 447. *by the product of* t $\sqrt{2}$ g *into the ſurface of the orifice, and to ſquare the quotient.*

449. II. Let us enquire, ſecondly, into the relation between the time and the quantity diſcharged, *ſuppoſing the veſſel empties*

itſelf; that is, let us aſcertain, according to this hypotheſis, the quantity of fluid expelled during a given time.

This determination will evidently furniſh, at the ſame time, the depreſſion of the upper ſurface of the fluid in the veſſel, of which the interior form is known; becauſe the portion of the capacity of the veſſel which is exhauſted is equal to the volume of fluid expelled at the orifice. Let us, in order to ſolve this branch of our general propoſition, ſuppoſe that at the commencement of the motion the ſurface of the fluid is at S, and that at the end of the given time t it has deſcended to K: making $SK=z$, we ſhall have to obtain an equation between z, t, g, x, h, and h'. This may be accompliſhed thus: we conceive, at firſt, that the veſſel is kept conſtantly full to the height K during the time t, and thence find, from what has juſt been done, for the diſcharge in that time the value $t\sqrt{2g}\,(\int y\dot{x}\sqrt{h'+x-z}+C)$. Since z is here ſuppoſed conſtant, it will be very eaſy to find the fluent of this expreſſion; and it may be corrected as the former fluent, by the condition that when $x=0$ the diſcharge is likewiſe nothing; the whole fluent being found by taking $x=h-h'$.

The value of $t\sqrt{2g}\,(\int y\dot{x}\sqrt{h'+x-z}+C)$ found in the manner juſt explained compriſes only z and conſtant quantities: we next ſuppoſe that the veſſel empties during the element of time $\dot{t}$, and that the ſurface Qq is depreſſed to Rr, through a diſtance $Kk=\dot{z}$; which, if we put the ſection Qq of the veſſel $=S$, gives for the quantity diſcharged $S\dot{z}$. Now, on this hypotheſis, the deſcent through Kk may be imagined to take place with a uniform velocity, and the velocity at each point of the orifice to continue the ſame as when the fluid had its ſurface at K: therefore, ſince when the velocities are equal the quantities diſcharged vary as the times, we have this analogy,

$$t\sqrt{2g}\,(\int y\dot{x}\sqrt{h'+x-z}+C) : S\dot{z} :: t : \dot{t},$$

$$\text{whence } \dot{t} = \frac{S\dot{z}}{(\int y\dot{x}\sqrt{h'+x-z}+C)\sqrt{2g}}$$

$$\text{and the time } t = \frac{1}{\sqrt{2g}}\int\frac{S\dot{z}}{\int y\dot{x}\sqrt{h'+x-z}+C}+C' \text{ . . . . . (iii.)}$$

When the form of the veſſel is known, S is given in functions of z, and of conſtant qualities; on the other hand, when the integration indicated by the denominator is effected, it contains likewiſe only the quantity z and invariable quantities: ſo that the complete value of t may be found by the integration of an expreſſion which contains only the variable quantity z. The conſtant magnitude C' will be determined from the conſideration that when $z=0$, $t=0$.

Now it is known that fluents, ſuch as thoſe in equations (i.) and (iii.) of this propoſition, which compriſe, under the ſign of integration, only one variable quantity and its fluxion, may always be referred to the rules for the quadrature of curves. In effect, let X be an expreſſion compoſed of the variable quantity x and a conſtant quantity, the fluxion $X\dot{x}$ will be the element of the ſurface of a plane, terminated by a curve of which X will be the ordinate, x the correſponding abſciſſa, and $\int X\dot{x}$ the ſurface itſelf. Thus, granting the equation to be $y'=X$, and ſquaring the curve, which is deſigned by this equation, between a certain value of x and another certain value of x, determined by the conditions of the caſe in hand, the reſulting ſurface will be equivalent to the value of $\int X\dot{x}$ taken within ſuitable limits.

450. Let us now proceed to give a few examples of the application of the preceding formula to particular caſes.

Ex. I. *Suppoſe the veſſel conſtantly kept full, and the orifice a rectangle whoſe ſides are horizontal and vertical reſpectively.*

Let the horizontal ſides of the rectangle be each equal to b: putting this value inſtead of y in the equation (i.) of art. 447. and taking the fluents, we have

$q = tb\sqrt{2g} \times \frac{2}{3}(\overline{h'+x})^{\frac{3}{2}} + C$. The conſtant quantity, determined according to the method previouſly explained, will be $C = -h'^{\frac{3}{2}}$. Subſtituting this value of C for it, in the equation juſt given, it becomes $q = \frac{2}{3}bt\sqrt{2g}\,(\overline{h'+x}^{\frac{3}{2}} - h'^{\frac{3}{2}})$. Then making, conformably to the directions in art. 447. $x = h-h'$, we obtain for the total diſcharge through the orifice

$$Q = \tfrac{2}{3}bt\sqrt{2g}\,(h^{\frac{3}{2}} - h'^{\frac{3}{2}}) \quad \ldots\ldots\ldots \quad \text{(iv.)}$$

If the theorem above be compared with the known rule for finding the area of a parabolic zone, it will thence appear that this value of Q furniſhes the ſame reſult as cor. 2. art. 446. A corroboration, if any were needed, of the theoretic truth of both.

The *mean height* of the fluid, determined by the rule in art. 44. will be found

$$s = \frac{4\,(h^{\frac{3}{2}} - h'^{\frac{3}{2}})^2}{9\,(h-h')^2}.$$

451. Ex. II. *Suppoſe the orifice a triangle whoſe vertex is uppermoſt, and the baſe horizontal, the veſſel being conſtantly kept full.*

Let the altitude of the triangle be to the baſe as m to 1, then is $y=m\,x$, and the equation art. 447. (i.) will become

$$q=t\sqrt{2g}\,\left(\int m\,x\,\dot{x}\sqrt{h'+x}+\mathrm{C}\right).$$

Making $h'+x=z$, we have $\dot{x}=\dot{z}$, and $x=z-h'$: whence ariſes $\int x\,\dot{x}\sqrt{h'+x}=\int(z-h')\,z^{\frac{1}{2}}\,\dot{z}=\int\left(z^{\frac{3}{2}}-h'z^{\frac{1}{2}}\right)\dot{z}=\frac{2}{5}z^{\frac{5}{2}}-\frac{2}{3}h'\,z^{\frac{3}{2}}+\mathrm{C}=\frac{2}{5}(h'+x)^{\frac{5}{2}}-\frac{2\,h'}{3}(h'+x)^{\frac{3}{2}}+\mathrm{C}$. The fluent vaniſhing when $x=0$, we have $\mathrm{C}=-\frac{2}{5}h'^{\frac{5}{2}}+\frac{2}{3}h'^{\frac{5}{2}}=\frac{4}{15}h'^{\frac{5}{2}}$, and the value of q becomes . . . $q=m\,t\sqrt{2g}\left[\frac{2}{5}(h'+x)^{\frac{5}{2}}-\frac{2}{3}h'(h'+x)^{\frac{3}{2}}+\frac{4}{15}h'^{\frac{5}{2}}\right]$.

When $x=h-h'$, we have for the total diſcharge at the aperture

$$\mathrm{Q}=\tfrac{2}{15}\,m\,t\sqrt{2g}\,\left(3h^{\frac{5}{2}}+2h'^{\frac{5}{2}}-5\,h'h^{\frac{3}{2}}\right)\ .\ .\ .\ .\ .\ (\mathrm{v.})$$

The area of the triangular orifice is $\frac{1}{2}m(h-h')^2$, which is the value of $\int y\,\dot{x}$ taken with reference to the preſent inſtance.

$$\text{Thus } t\sqrt{2g}\cdot\int y\,\dot{x}=\tfrac{1}{2}m\,t\sqrt{2g}\,(h-h')^2$$

whence, by the method pointed out in art. 448. we find the *mean height* of the fluid by the equation

$$s=\frac{16\left(3\,h^{\frac{5}{2}}+2\,h'^{\frac{5}{2}}-5\,h'.h^{\frac{3}{2}}\right)^2}{225\,(h-h')^4}\ .\ .\ .\ .\ .\ .\ (\mathrm{vi.})$$

452. Ex. III. *Let the orifice be a triangle, as in the preceding example, but having its vertex downwards, and its baſe horizontal.*

In this caſe $y=m\,(h-h'-x)$, and, by a calculus little more difficult than the above, we ſhall find

$$\mathrm{Q}=\tfrac{2}{15}\,m\sqrt{2g}\,\left(2h^{\frac{5}{2}}+3\,h'^{\frac{5}{2}}-5\,h\,h'^{\frac{3}{2}}\right)\ .\ .\ .\ .\ .\ (\mathrm{vii.})$$

And the mean height of the fluid as below,

$$s=\frac{16\left(2\,h^{\frac{5}{2}}+3\,h'^{\frac{5}{2}}-5\,h\,h'^{\frac{3}{2}}\right)^2}{225\,(h-h')^4}\ .\ .\ .\ .\ .\ (\mathrm{viii.})$$

453. Ex. IV. *Suppoſe the orifice a circle, and the veſſel kept conſtantly full.*

The general theorem (art. 447. i.) may be moſt readily applied to the preſent example, by an approximation which will be ſufficiently accurate for practice, thus: Put $d=$ the diameter of the circle, then from the property of that figure $\frac{1}{2}\,y=\sqrt{d\,x-x^2}$, whence the general equation juſt referred to becomes

$$q=2\,t\sqrt{2g}\int(d\,x-x\,x)^{\frac{1}{2}}\,(h'+x)^{\frac{1}{2}}\,\dot{x}+\mathrm{C}).$$

Making $\int(d\,x-x\,x)^{\frac{1}{2}}\,(h'+x)^{\frac{1}{2}}\,\dot{x}=\int z\,\dot{x}$, the integration is reduced to the finding the area of a curve whoſe equation is $z=$

$(dx - xx)^{\frac{1}{2}}(h' + x)^{\frac{1}{2}}$; its origin being at the point where $x = 0$. We may approximate to the quadrature of this curve by the method of equidiſtant ordinates, adopting the firſt general propoſition in Sect. II. Part. iv. *Hutton's Menſuration.* To this end, ſuppoſe x divided into four equal parts; then ſhall we find the five ordinates correſponding to the points of diviſion, by ſubſtituting in the preceding equation for z the values 0, $\frac{1}{4}x$, $\frac{1}{2}x$, $\frac{3}{4}x$, and x: theſe values found, we add to the ſum of the firſt and fifth four times the ſum of the ſecond and fourth, and twice the third; one third of this latter ſum, multiplied into the common diſtance between the ordinates, will be the approximate value of the ſurface $\int z\dot{x}$; which, multiplied into $2t\sqrt{2g}$, produces at length

$$q = \left\{\begin{array}{l} 2\left(\frac{1}{4}dx - \frac{1}{16}xx\right)^{\frac{1}{2}}\left(h' + \frac{1}{4}x\right)^{\frac{1}{2}} \\ + \left(\frac{1}{2}dx - \frac{1}{4}xx\right)^{\frac{1}{2}}\left(h' + \frac{1}{2}x\right)^{\frac{1}{2}} \\ + 2\left(\frac{3}{4}dx - \frac{9}{16}xx\right)^{\frac{1}{2}}\left(h' + \frac{3}{4}x\right)^{\frac{1}{2}} \\ + \frac{1}{2}\left(dx - xx\right)^{\frac{1}{2}}\left(h' + x\right)^{\frac{1}{2}} \end{array}\right\} \frac{1}{3}tx\sqrt{2g}.$$

If in this equation we make $x = d = h - h'$, and perform the requiſite reductions, we ſhall have for the total diſcharge at the orifice

$$Q = td^2\sqrt{2g}\left(\frac{\sqrt{h+3h'} + \sqrt{h'+3h}}{4\sqrt{3}} + \frac{1}{6}\,\frac{\sqrt{h+h'}}{\sqrt{2}}\right) \quad \ldots\ldots \text{(ix.)}$$

This theorem may be reduced to computation in any particular caſe with tolerable facility. Had the number of equidiſtant ordinates been much increaſed, the additional accuracy of the approximation would not have compenſated for the additional labour which would attend the reſulting formula.

454. A rigorous integration would require the aid of infinite ſeries. If r be the radius of the orifice; n the quotient of the diſtance of its centre from the upper ſurface of the fluid, divided by r; and $\pi = 3{\cdot}141593$; we ſhould then have

$$Q = \pi t r^2\sqrt{2grn} \times \left(1 - \frac{1}{32n^2} - \frac{5}{1024n^4} - \&c.\right) \quad \ldots\ldots \text{(x.)}$$

$$\text{and } s = rn\left(1 - \frac{1}{16n^2} - \frac{9}{1024n^4} - \&c.\right) \quad \ldots\ldots \text{(xi.)}$$

In both theſe ſeries the firſt three terms are all which will be needed in practice.

454. Ex. V. *To determine in what time* t, *the upper ſurface of the fluid, ſhall be depreſſed through a vertical diſtance* z, *the veſſel being ſuppoſed priſmatic, and the orifice rectangular.*

The ſection S of the veſſel, and the breadth y of the orifice, are in this inſtance conſtant quantities. According to the method explained in art. 449, we firſt determine the quantity of fluid which would eſcape through the orifice during the time t, on the ſuppoſition that the veſſel were kept full to the height $h'-z$, above the upper part of the orifice; for which purpoſe we muſt find the fluent of the expreſſion $y\int \dot{x}\sqrt{h'+x-z}$, ſuppoſing z conſtant. This gives for the diſcharge ſought $\frac{2}{3}y(h'+x-z)^{\frac{3}{2}}\times C$. But this quantity being nothing when $x=0$, we have $C=-\frac{2}{3}y(h'-z)^{\frac{3}{2}}$, and the correct fluent is $\frac{2}{3}y\left[(h'+x-z)^{\frac{3}{2}}-(h'-z)^{\frac{3}{2}}\right]$. Extending this to the whole of the orifice by making $x=h-h'$, we ſhall have $\frac{2}{3}y\left[(h-z)^{\frac{3}{2}}-(h-z)^{\frac{3}{2}}\right]$: and if this value be ſubſtituted for $\int y\dot{x}\sqrt{h'+x-z}+C$, in the equa. iii. 449. it will become

$$t=\frac{3S}{2y\sqrt{2g}}\cdot\int\frac{\dot{z}}{(h-z)^{\frac{3}{2}}-(h'-z)^{\frac{3}{2}}}\;\ldots\ldots\;\text{(xii.)}$$

If we put a for the altitude of the orifice, b for its breadth, and δ for the diſtance of the upper ſurface of the fluid from the bottom of the orifice, we ſhall have $\delta=h-z$; $\dot{z}=-\dot{\delta}$; $h=h'+a$; $h'=\delta-a$; and the preceding equation will be transformed to this:

$$t=\frac{-3S}{2b\sqrt{2g}}\cdot\int\frac{\dot{\delta}}{\delta^{\frac{3}{2}}-(\delta-a)^{\frac{3}{2}}}\;\ldots\ldots\;\text{(xiii.)}$$

The quantity which is found under the ſign of integration in this equation is ſuſceptible of being made rational; but the equation would be very complicated. The moſt eaſy method of obtaining a reſult is to ſquare by approximation the curve, which for each value of δ has an ordinate equal to $\frac{1}{\delta^{\frac{3}{2}}-(\delta-a)^{\frac{3}{2}}}$, in the ſame manner as was adopted in the laſt article. Obſerving now that when $\delta=h$, $t=0$, the ſurface may be eſtimated between the limits $\delta=h$, and $\delta=a$; and as in this caſe we ſhould only have the terms to tranſcribe from the preceding article, we truſt we have no occaſion to copy here the final equation.

If the altitude of the orifice a be equal to the height h of the veſſel, the time of exhauſtion to any variable depth z, reckoned from the bottom, would be equal to $t=\frac{3S}{b\sqrt{2g}}\times\frac{h\sqrt{z}-z\sqrt{h}}{hz}$:

when $z=0$, this expreſſion is infinite; that is, the time of complete exhauſtion is infinite.

The ſtudent, who wiſhes to purſue further this part of Hydrodynamics, may conſult *l'Hydrodynamique de M. Boſſut*, and the *Select Exerciſes* at the end of *Dr. Hutton's Conics*.

455. PROP. *If upon the altitude of the fluid in a veſſel as a diameter, we deſcribe a ſemicircle, the horizontal ſpace deſcribed by the fluid ſpouting from a vertical orifice at any point in the diameter will be as the ordinate of the circle drawn from that point; the horizontal ſpace being meaſured on the plane of the bottom of the veſſel.*

When the aperture is vertical, and indefinitely ſmall (as ſuppoſed here), the fluid will ſpout out horizontally with the velocity due to the altitude of the fluid above the orifice (art. 439.); and this velocity, combined with the perpendicular velocity ariſing from the action of gravity, will cauſe every particle, and conſequently the whole jet, to deſcribe the curve of a parabola. Now the velocity with which the fluid is expelled from any hole, as G (fig. 9. pl. XVII.), is ſuch as, if uniformly preſerved, would carry a particle through a ſpace equal to 2 BG in the time of the fall through BG: but, after quitting the orifice, it deſcribes the parabolic curve, and arrives at the horizontal plane CF in the ſame time as a body would fall freely through GD; ſo that, to find the diſtance DE, ſince the times are as the roots of the ſpaces, we have this analogy $\sqrt{GB} : \sqrt{GD} :: 2\,BG : DE = \frac{2\,BG\,.\sqrt{GD}}{\sqrt{GB}} = 2\sqrt{BG\,.\,GD} = 2\,GH$, by the nature of the circle. And the ſame will hold with reſpect to any other point in BD.

COR. 1. If apertures be made at equal diſtances from the top and bottom of the veſſel (kept full of the fluid), the horizontal diſtances to which the water will ſpout from theſe apertures will be equal. For when $Dg=BG$, we ſhall have $2\sqrt{Bg\,.\,gD} = 2\sqrt{BG\,.\,GD}$, and conſequently DE the ſame in both caſes.

COR. 2. When the orifice is at the point biſecting the altitude of the fluid in the veſſel the fluid will ſpout to the greateſt diſtance on the horizontal plane; and that diſtance, if meaſured on the plane of the bottom of the veſſel, will be equal to the depth of fluid in it. For IK, the ordinate from the centre I, is the greateſt which can be drawn in the ſemicircle, and DF, which is $=2$ IK, is then $=2$ BI $=$ BD.

COR. 3. Since the diſtance to which the fluid ſpouts depends upon the height of its ſurface AB above the orifice, and not in any degree upon the depth of the fluid below the orifice, it will follow, that whether the fluid in any veſſel reaches down to any horizontal plane CF, or whether the bottom of the veſſel ſtands at ſome higher point P, the diſtances DE, DF, to which the fluid will ſpout from the apertures G, I, will be the ſame; and the maximum diſtance will be when the fluid is expelled from an orifice half way between the planes AB and CF. If the bottom of the veſſel is higher than the point I, then the nearer to that point the orifice is placed the greater will be the diſtance to which the fluid will ſpout on CF.

Cor. 4. Since the middle filament of particles iſſuing from an orifice in the ſide of a veſſel is diſcharged with the full velocity due to the entire altitude of the fluid above the orifice (art. 440.), experiments made on the diſtance or height to which fluids ſpout will be found to agree very well with theory; but it by no means follows that *all* the filaments ſhould be expelled with the ſame velocity: conſequently, the quantity of the fluid diſcharged in a given time may be leſs than that which would be diſcharged if all the filaments were expelled with the velocity due to the entire altitude; becauſe this quantity depends on the mean velocity of all the filaments. Hence, therefore, we cannot infer (as ſeveral authors have done) from theſe experiments, compared with thoſe which relate to the height or diſtance to which the fluid ſpouts, that the velocity of the water in the orifice is leſs than that which is due to the entire altitude; and that it is accelerated immediately after it gets out of it; becauſe the diſtance to which the fluid ſpouts depends upon the central filament only; but the quantity diſcharged on the mean velocity of the whole.

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456. If the fluid iſſues from the orifice not horizontally, but in any oblique direction, it will deſcribe the curve of a parabola very nearly; conſequently the theorems given in arts. 252. 253. for the motion of projectiles in a non-reſiſting medium, may be applied to the determination of the various analogous particulars reſpecting the motion of fluids expelled from ſprings or reſervoirs. The letter I, which denotes the impetus in thoſe theorems, will here repreſent the height of the fluid in the reſervoir above the orifice, H the greateſt height to which the fluid ſpouts above any plane paſſing through the aperture, and the other letters the range, the velocity, &c. As an example of the uſe of theſe formulæ, ſuppoſe that a ſhort pipe elevated in any direction from an aperture in a conduit, throws the water in a parabolic curve to the diſtance or range R on an horizontal plane paſſing through the orifice, and that the greateſt height of the ſpouting fluid above that plane is H, we may

thence *determine the height* I *of the reſervoir which ſupplies the conduit, without actually meaſuring the angle of elevation* E *of the pipe.* For from art. 253, we have $R = 2\,I \times \text{ſin.}\ 2\,E$, or $I = \frac{R}{2\ \text{sin.}\ 2\,E}$, and $I = \frac{H}{\text{sin.}^2\,E}$. Hence, $\frac{R}{2\ \text{sin.}\ 2\,E} = \frac{H}{\text{sin.}^2\,E}$, and, by reduction, we find $\frac{R}{2\,H} = \frac{\text{sin.}\ 2\,E}{\text{sin.}^2\,E} = \frac{2\,\text{cos.}\,E}{\text{sin.}\,E} = 2\,\text{cotan.}\,E$. Thus the angle E will become known; and then either the equation $I = \frac{H}{\text{sin.}^2\,E}$, or $I = \frac{R}{2\ \text{sin.}\ 2\,E}$, will determine the height of the reſervoir, as required.

CHAPTER II.

Account of Experiments made by different Philoſophers on the Diſcharge of Water through Apertures and Tubes; and the practical Deductions from those Experiments.

457. THE greater part of the propoſitions and theorems in the preceding chapter are founded upon the hypotheſis, that the *whole* of the fluid particles iſſuing from a horizontal orifice in a veſſel is expelled with the velocity due to the height of the fluid in the reſervoir above that orifice; and the whole of the particles diſcharged through an evaneſcent horizontal element of a vertical aperture, with one and the ſame velocity. But this hypotheſis, as we have hinted more than once, is not altogether conſiſtent with fact. For, when water iſſues from an orifice, the particles will flow from all ſides towards the orifice, with an accelerated motion, and in all directions. If the orifice be horizontal, that filament of particles which anſwers to its centre will, as Dr. M. Young obſerves, deſcend in a vertical line, and ſuffer no other reſiſtance than that of the friction cauſed by the exceſs of its velocity above that of the collateral filaments, or by the retardation which ariſes from the attraction ſubſiſting between them. The other filaments are ſoon compelled to turn from their vertical courſe, and to approach the orifice in ſpiral curves, (art. 444.); and, when they arrive at it, their directions become more or leſs horizontal, according as they paſs nearer to, or further from, the edge of the orifice. The motion, therefore. may be decompoſed into two directions; the one horizontal, which is deſtroyed by the equal and contrary reſiſtance of the filaments which are diametrically oppoſite; the other vertical, in proportion to which the quantity of water diſcharged is to be eſtimated. Hence it appears that the vertical velocity of the filaments decreaſes from the centre of the orifice to the circumference; and that the total diſcharge is leſs than if the filaments had iſſued vertically, as that filament does which correſponds to the centre of the orifice. It alſo follows that the

filaments which are nearer to the centre, moving faſter than thoſe which are nearer to the edges, the vein of the fluid, after it has iſſued from the aperture (if that be circular) will form a conic fruſtrum whoſe greater baſe is the aperture; that is to ſay, its diameter will diminiſh to a certain diſtance, becauſe the exterior filaments are gradually drawn on, in conſequence of their mutual attraction, by the interior filaments whoſe velocity is greater, whence follows a diminution in the diameter of the vein.

This diminution in the ſection of the vein is often called the *contraction* of the vein; and the vein itſelf, from the orifice to the greateſt diminution, is called the *vena contracta*, the *contracted vein*.

The contraction of the ſtream is found to take place not only when water is diſcharged from horizontal apertures, but when the diſcharge is from vertical apertures, or apertures inclined to the horizon in any manner whatever: in theſe latter caſes, however, the form of the contracted vein is by no means ſo regular as in the diſcharge from horizontal orifices; the ſtream often aſſumes a very curious form, having for a ſmall diſtance from the aperture the appearance of a plaited band.

When the orifice is horizontal and circular, the length of the contracted vein is very nearly equal to the ſemi-diameter of the orifice: and *Polenus* makes the ratio of the diameters of the contraction and of the aperture to be as $5\frac{1}{2}$ to $6\frac{1}{2}$; *Bernoulli*, 5 to 7; *Chev. Du Buat*, 6 to 9; *Boſſut*, 41 to 50; *Michelotti*, 4 to 5; *Venturi*, nearly 4 to 5. And the latter author has ſhewn, by many experiments, that the contraction of the ſtream takes place at a greater diſtance under ſtrong charges than in thoſe which have but little elevation. The ratio of the ſurface of the ſection of the fluid vein at the place of greateſt contraction, to the area of the orifice, does not much depend on the figure of the orifice; but this ratio is ſubject to variations to which we ſhould have regard when the ſide or the bottom of the veſſel is more or leſs thick, or when an additional tube is adapted to the orifice. The ſituation of the orifice with reſpect to the ſides of the veſſel has likewiſe a certain influence on the contraction; but the differences thus occaſioned may be generally neglected in practice. We may infer from what has now been ſaid, that to obtain formula which will furniſh reſults applicable to practice we ſhould, in the different caſes, ſubſtitute for the actual area of the orifice the area of the ſmalleſt ſection of the contracted vein: this reduced area ſhould be conſidered as the true orifice through which the diſcharge is made, and its vertical diſtance from the upper ſurface of the fluid as the height due to the velocity of the fluid iſſuing through this little orifice.

458. From the preceding remarks the neceſſity of giving

extenſion to the theory of Hydrodynamics, by combining with it the reſults of experiments, muſt be pretty obvious: we ſhall in this chapter, therefore, give a conciſe view of the moſt uſeful concluſions deduced from the experiments of different philoſophers, commencing with thoſe of M. *Boſſut*, whoſe experiments appear to have been very numerous, judicious, and exact.

We propoſe to determine from the experiments, firſt, what is *the ratio between the area of the contraction of the vein and that of the orifice;* ſecondly, *the quantity of diſcharge through thin plates;* thirdly, *the diſcharge through additional tubes, their length being ſmall compared with the depth of the water in the reſervoir.*

In M. Boſſut's firſt experiments, the apertures for the efflux of the water were all pierced perpendicularly in plates about ½ a line thick.

M. *Boſſut* has given in his *Hydrodynamique* (tome II, pa. 47.) the following table relative to the diſcharge through orifices pierced in thin plates: the meaſure is the Paris foot royal, which is to the Engliſh foot as 1535 to 1440, or 1066 to 1000: the fourth column, which expreſſes the relation between the reſults of the experiments and thoſe of the theory, is from M. *Prony*.

Constant altit. of the water in the reserv. above the apert. in Paris ft.	Theoret. discharges in 1^{m}. through a circul. apert. of tin, diameter expressed in cubic inches.	Real discharges in the same time through the same orifice, expressed also in cubic inc.	Ratio of actual to theor. discharges, the latter being denoted by unity.
1	4381	2722	0·62133
2	6196	3846	0·62073
3	7589	4710	0·62064
4	8763	5436	0·62034
5	9797	6075	0·62010
6	10732	6654	0·62000
7	11592	7183	0·61965
8	12392	7672	0·61911
9	13144	8135	0·61892
10	13855	8574	0·61883
11	14530	8990	0·61873
12	15180	9384	0·61819
13	15797	9764	0·61810
14	16393	10130	0·61795
15	16968	10472	0·61716

It appears from this table that the actual diſcharges, as well as thoſe reſulting from the theory, are ſenſibly proportional to the ſquare roots of the depths of fluid in the reſervoir. Thus, for example, if we take the depths 4 and 9, whoſe ſquare roots are as 2 to 3; the correſponding actual diſcharges taken from the third column, are 5436 and 8135: and theſe numbers are very nearly in the ratio of 2 to 3, the ratio being 2 to 2·9931.

If the numbers in the laſt column are multiplied together, and the 15th root of the laſt product taken, we ſhall have ·61932 for the true mean of the effective diſcharges, compared with the theoretic diſcharge 1; and the arithmetical mean between the numbers in the laſt column ſtanding againſt the heights 7 and 8, is ·61938: the mean ratio between the actual and theoretic diſcharges, then, is not widely diſtant from that of ·62 to 1: whence it follows, from the remarks in art. 457, that ·62 is the number by which we muſt multiply the real area of the orifice to obtain the area of the ſmalleſt ſection of the contracted vein.

459. Another ſet of experiments made by M. *Boſſut*, with *different* apertures, are the following, in which the water was kept conſtantly at the altitude of 11 feet, 8 inches, 10 lines, from the centre of each aperture.

Exp.	No. of cubic inches furniſhed in 1 min.
1. With an horizontal circular aperture, 6 lines diameter	2311
2. With a circular horizontal aperture, 1 inch diameter	9281
3. With a circular horizontal aperture, 2 inches diameter	37203
4. With a rectangular horizontal aperture, 1 inch by 3 lines	2933
5. With a ſquare horizontal aperture, the ſide 1 inch .	11817
6. By a ſquare horizontal aperture, the ſides 2 inches	47361
Conſtant height 9 feet.	
7. Lateral circular aperture, 6 lines diameter . .	2018
8. Lateral circular aperture, 1 inch diameter . .	8135
Conſtant height 4 feet.	
9. Lateral circular aperture, 6 lines diameter . .	1353
10. Lateral circular aperture, 1 inch diameter . .	5436
Conſtant height 7 lines.	
11. By a lateral and circular orifice, 1 inch diameter .	628

From the preceding experiments we may make the following deductions:

1. 'The quantities of fluid diſcharged in equal times from different ſized apertures, the altitude of the fluids being the ſame, are nearly to each other as the areas of the apertures.'

Thus in the ſecond and third experiments the areas of the apertures are as one to four, and the water diſcharged 9281 cubic inches to 37203 is nearly in the ſame ratio.

2. 'The quantities of water diſcharged, in equal times, by the ſame aperture, with different altitudes of the reſervoir, are nearly as the ſquare roots of the correſponding altitude of the water in the reſervoir above the centre of the aperture.' Comparing together the eighth and tenth experiments, in which the reſpective altitudes of the reſervoir were of 9 and 4 feet, of which the ſquare roots are 3 and 2, we find the water diſcharged by the firſt was 8135 cubic inches, the ſecond 5436 cubic inches nearly in the proportion of 3 to 2, as before obſerved.

3. 'That, in general, the quantities of water diſcharged in the ſame time, by different apertures and under unequal altitudes of the reſervoirs, are to each other in a compound ratio of the areas of the apertures and the ſquare roots of the altitudes.

4. 'That, on account of the friction, the ſmalleſt apertures diſcharge leſs water than thoſe that are larger and of a ſimilar figure, the water in the reſpective reſervoirs being at the ſame height.'

5. 'That of ſeveral apertures whoſe areas are equal, that which has the ſmalleſt circumference will diſcharge more water than the others, the water in the reſervoirs being at the ſame altitude,' and this becauſe there is leſs friction. Hence circular apertures are moſt advantageous, as they have leſs rubbing ſurface under the ſame area.

Hence, then, to make the formulæ in the theory, for inſtance, thoſe in cor. 3 art. 439, furniſh ſuch reſults as would agree with experiments, we muſt reduce the aperture *a* in thoſe theorems in the ratio of ·62 to 1; or multiply the quantities reſulting from the theorems as they now ſtand by the decimal ·62; or, laſtly, if great accuracy be required, take, inſtead of the conſtant multiplier ·62, the number ſtanding againſt the height of fluid in the reſervoir above the orifice, in the laſt column of the table in the preceding article: thus, if the altitude of the fluid be 10 feet, the multiplier will be ·61883.

460. If the water, inſtead of flowing through an aperture pierced in a thin ſubſtance, paſſes through the end of a vertical tube of the ſame diameter as the aperture, there is a much greater diſcharge of water, becauſe the contracted ſtream is greater in the firſt inſtance than in the ſecond. In the following experiments the conſtant height of the water in the reſervoir above the upper aperture of the tube was 11 feet 8 inches 10 lines, the diameter of the tube 1 inch.

Different lengths of the tube expreſſed in lines.			Number of cubic inches of water diſcharged in 1 minute.
	Lines		
Exp. 1.	48	The ſtream filling the tube.	12274
2	24		12188
3	18		12168
4	18	The water not filling the tube.	9282

It appears, on comparing the firſt three experiments, that the longer the vertical tube is, the greater is the diſcharge of the water, becauſe the contraction of the ſtream is leſs; it is, however, always ſomewhat contracted, even when it appears to fill the tube.

By comparing the quantities of water diſcharged in the third and fourth experiments, we find the two diſcharges 12168, 9282, are to each other nearly in the proportion of 13 to 10; but we have ſeen that the water diſcharged through a thin aperture without any contraction in the ſtream, would be to the ſame aperture with a contracted ſtream as 1 to ·62, or as 16 to 10. From hence we may conclude, that, the altitude in the reſervoir and the apertures being the ſame, the diſcharge through a thin aperture without any contraction in the ſtream, the diſcharge through an additional tube, and the diſcharge through a ſimilar aperture with a contracted ſtream, are to each other nearly as the numbers 16, 13, 10: theſe proportions are ſufficiently exact for practice. Hence it is plain that an additional tube only deſtroys in part the contraction of the ſtream, which contraction is greateſt when the water paſſes through a thin aperture from a large reſervoir.

If the additional tube, inſtead of being vertical, or placed at the bottom of the reſervoir, was horizontal, or placed in the ſide, it would furniſh the ſame quantity of water, provided it was of the ſame length, and that the exterior aperture was at the ſame diſtance from the ſurface of the water in the reſervoir.

If the additional tube, inſtead of being cylindrical, was conical, having its largeſt baſe neareſt the reſervoir, it would diſcharge a greater quantity of water. The moſt advantageous form that can be given, in order to obtain the greateſt quantity of water in a given time by a given aperture, is that which the ſtream aſſumes in coming out of the aperture; i. e. the tube muſt be of the form of a truncated cone, whoſe largeſt baſe ſhould be of the ſame diameter as the aperture; the area of the ſmall baſe ſhould be to that of the larger baſe as 10 to 16; and the diſtance from one baſe to the other ſhould be the ſemidiameter of the largeſt baſe; and the efflux of water will be as abundant as it would be through a thin aperture equal to the ſmalleſt baſe, and where the ſtream was not contracted. This form may be applied where it is neceſſary to obtain a certain quantity of water from a river, an aqueduct, &c. by a canal or lateral tube.

On comparing the efflux of water through additional tubes of different diameters, and with different altitudes of the water in the reſervoirs, the following reſults were obtained; the additional tubes were two inches long, and were vertical and placed at the bottom of the reſervoir.

Conſtant altitude of the water above the tubes.		Diameter of the tubes expreſſed in lines.		Number of cub. in. in 1 min.
Ex. 1	3 feet 10 inches	6	Water filling the tube	1689
2		10		4703
3		6	The water not following the ſides	1293
4		10		3598
5	2 feet	6	The water filling the tube	1222
6		10		3402
7		6	The water not filling the tube	935
8		10		2603

It reſults from theſe experiments, 1. 'That the diſcharges by different additional tubes, with the ſame altitude of the reſervoir, are nearly in proportion to the area of the apertures, or to the ſquares of the diameters. 2. That the diſcharges of water by additional tubes of the ſame diameter, with different altitudes of water in the reſervoir, are nearly proportional to the ſquare root of the altitude of the reſervoir. 3. That in general the diſcharges of water in the ſame time, through different additional tubes, with different altitudes of water in the ſame reſervoir, are to each other nearly as the product of the ſquare of the diameters of the tubes by the ſquare root of the altitude of the reſervoirs.' So that additional tubes, tranſmitting water, follow (amongſt themſelves) the ſame laws as through the thin orifice. The following table was formed from the foregoing experiments.

Constant altitude of the water in the reservoir above the aperture, expressed in feet.	Water discharged in one minute through a hole 1 inch diameter, the stream not contracted, in cubic inches.	Water discharged in one minute through an additional tube of 1 inch diameter, 2 inches long, in cubic inches.	Water discharged during 1 minute, through a hole 1 inch diameter, with a contracted stream, in cubic inches.	Ratio between the actual and the theoretic discharges, the latter being 1.
1	4381	3539	2722	0·81781
2	6169	5002	3846	0·80729
3	7589	6126	4710	0·80724
4	8763	7070	5436	0·80681
5	9797	7900	6075	0·80638
6	10732	8654	6654	0·80638
7	11592	9340	7183	0·80573
8	12392	9975	7672	0·80496
9	13144	10579	8135	0·80485
10	13855	11151	8574	0·80483
11	14530	11693	8990	0·80477
12	15180	12205	9384	0·80403
13	15797	12699	9764	0·80390
14	16393	13197	10130	0·80382
15	16968	13620	10472	0·80270

The mean of the numbers in the laſt column of this table is ſomewhat leſs than ·81; we may, however, in moſt caſes, take ·81 as a very good approximation to the truth: uſing it as a conſtant co-efficient in the formula for the value of Q given in cor. 3. art. 439. when we wiſh to know the diſcharge through a cylindric tube of the dimenſions ſpecified at the head of column the third. Thus we ſhall have $Q = \cdot 81\, a t \sqrt{2gh}$; the dimenſions being all in feet, or all in inches.

* * *

461. We now paſs to M. *Boſſut's* experiments on the exhauſtion of veſſels (which have no extraneous ſupply) by little orifices.

The experiments upon the time of complete exhauſtion of veſſels which empty freely are not eaſy to make, at leaſt in a concluſive manner: for, beſides that in ſome caſes the complete exhauſtion would, according to the theory, require an unlimited time, it is found that, when the ſurface of the water arrives within a ſmall diſtance, as two or three inches, of a horizontal orifice, it forms above that orifice a conical or rather conoidal funnel, which diminiſhes the effect, and makes the concluſion of the diſcharge uncertain. It is beſt, therefore, not to make experiments upon the time of total diſcharge, but upon the time in which the upper ſurface is depreſſed through a certain vertical diſtance x, the greater the better, provided the upper ſurface has not ſunk ſo low as to permit the formation of the funnel juſt ſpoken of.

It was ſhewn (art. 443. equa. II.) that when the primitive height of the water in a priſmatic veſſel was $= h$, the conſtant ſection of the veſſel $= A$, the time t employed by the fluid to deſcend through the ſpace x was expreſſed by this equation:

$$t = \frac{2A}{a\sqrt{2g}}(\sqrt{h} - \sqrt{h-x}).$$

Now when the orifice a is ſuppoſed pierced in a thin plate we muſt ſubſtitute the contracted orifice $\cdot 62\, a$, and the formula will become

$$t = \frac{2}{\cdot 62 \times 8\cdot 0208} \frac{A}{a}(\sqrt{h} - \sqrt{h-x}) = \cdot 40218 \frac{A}{a}(\sqrt{h} - \sqrt{h-x}).$$

When the aperture is a circle, and its diameter d, we have $\cdot 785398\, d^2 = a$; this value of a ſubſtituted for it in the preceding equation gives

$$t = \cdot 51208 \frac{A}{a}(\sqrt{h} - \sqrt{h-x}),$$

the dimenſions being given in Engliſh ſtatute feet; or if the dimenſions are in terms of the Paris foot royal, then

$$t = \cdot 52852 \frac{A}{a}(\sqrt{h} - \sqrt{h-x}).$$

This latter theorem may be applied to the experiments made by M. *Boſſut*, in order to compare the reſults which it gives with thoſe which are furniſhed by the experiments: the whole is compriſed in the following table.

The constant section A of the vessel is 9 square feet. The primitive height h of the water is 11·667 Paris feet.	Diameter of the horizontal circular orifice, or value of d.	Depression of the water, or value of x.	Time of the depression of the water.	Time of the depression, calculated by the last theorem.
		Feet.	Seconds.	Seconds.
	·083333	4	4455	443·04
		9	1224·5	1221·2
	·166666	4	112·	110·76
		9	306·	305·25

Hence we ſee that the difference between the reſults of the computation and thoſe of the experiments are extremely ſmall: much ſmaller indeed than might be expected, conſidering the many circumſtances which may contribute to vary the times given by obſervation. So that we may regard the formulæ given in this article as ſufficiently correct for practice; at leaſt within the limit ſuggeſted by the formation of the conoidal funnel. What has been here ſaid applies principally to horizontal apertures; but it may be applied without fear of material error to ſmall lateral orifices, when the fluid in the reſervoir ſtands higher than the upper ſide of the orifice, and taking for the height *h* the diſtance of the centre of gravity of the orifice from the upper ſurface of the fluid.

Experimental Enquiries of Venturi.

462. The experiments and reſearches of M. *J. B. Venturi*, profeſſor of natural philophophy at Modena, are neither ſo extenſive nor ſo important as thoſe of M. Boſſut: but as he has noticed two or three curious circumſtances relative to the motion of fluids, which ſeem to have eſcaped the obſervation of preceding philoſophers, we ſhall preſent the reader with a conciſe account of the reſult of his enquiries.

I. In any fluid thoſe parts which are in motion carry along with them the lateral parts which are at reſt.

To ſhew the truth of this propoſition M. *Venturi* introduced a horizontal current of water into a veſſel filled with the ſame fluid at reſt. This ſtream entering the veſſel with a certain velocity, paſſes through a portion of the fluid, and is then received in an inclined channel, the bottom of which gradually riſes until it paſſes over the border or rim of the veſſel itſelf. The effect is found to be, not only that the ſtream itſelf paſſes out of the veſſel through the channel, but carries along with it the fluid contained in the veſſel; ſo that after a ſhort time no more of the fluid remains than was originally below the aperture at which the ſtream enters. This fact is adopted as a principle or primitive phenomenon by the author, under the denomination of the lateral communication of motion in fluids, and to this he refers many important hydraulic facts. He does not undertake to give an explanation of this principle, but ſhews that the mutual attraction of the particles of water is far from being a ſufficient cauſe to account for it.

II. If that part of an additional cylindric tube which is neareſt the ſide of the reſervoir be contracted, according to the form of the contracted vein of fluid which iſſues through a hole of the ſame diameter in a thin plate, the expenditure will be the ſame as if the tube were not contracted at all; and the velocity of the ſtream within this tube will be greater than that of a jet through a thin plate in the ratio of 121 to 100.

III. The preſſure of the atmoſphere increaſes the expence of water through a ſimple cylindrical tube, when compared with that which iſſues through a hole in a thin plate, whatever may be the direction of the tube.

IV. In deſcending cylindrical tubes, the upper ends of which poſſeſs the form of the contracted vein, the diſcharge is ſuch as correſponds with the height of the fluid above the inferior extremity of the tube.

V. In an additional conical tube the preſſure of the atmoſphere increaſes the expenditure in the proportion of the exterior ſection of the tube to the ſection of the contracted vein, whatever may be the poſition of the tube, provided its internal figure be adapted throughout to the lateral communication of motion.

VI. In cylindrical pipes the expenditure is leſs than through conical pipes, which diverge from the place of the contracted vein, and have the ſame exterior diameter.

For in the ſpace between the inverted contracted vein and the ſides of the cylinder eddies, or circular whirls, are produced, as in a baſin which receives water by a channel; and theſe retard the efflux of the ſtream, and produce a correſponding failure in the effect.

VII. By means of proper ajutages applied to a given cylindric tube placed horizontally, it is poſſible to increaſe the expenditure of water through that tube in the proportion of 24 to 10, the charge or height of the reſervoir remaining the ſame.

For this purpoſe, the inner extremity of the tube AD (fig. 11. pl. XVII.) muſt be fitted at AB with a conical piece of the form of the contracted vein; this increaſes the expenditure as 12·1 to 10. Every other form will afford leſs. If the diameter at A be too great the contraction muſt be made beyond B, and the ſection of the vein will be ſmaller than the ſection of the tube. Secondly, at the other extremity of the pipe BC apply a truncated conical tube CD, of which let the length be nearly nine times the diameter C, and its external diameter D muſt be 1·8 C. This additional piece will increaſe the expenditure as 24 to 12·1. So that the quantity of effluent water will be increaſed by the two adjutages in the ratio of 24 to 10. All this is on the ſuppoſition that the pipe BC has no elbows or ſinuoſities.

VIII. The expenditures out of a ſtraight tube, a curved tube in a quadrantal arc, and an elbowed tube having the angle 90°, (each being poſited horizontally) are *cæt. par.* nearly as 70, 50, and 45.

IX. The internal roughneſs of a pipe diminiſhes the expenditure, though the friction of the water againſt theſe aſperities does not form any conſiderable part of the cauſe. A right-lined tube may have its internal ſurface highly poliſhed throughout its whole length; it may every-where have a diameter greater than the orifice to which it is applied; but, notwithſtanding, the expenditure will be greatly diminiſhed if the pipe ſhould have enlarged parts, or ſwellings: for, by reaſon of theſe ſudden changes in the interior dimenſions of the pipe, much of the motion will be conſumed in eddies. This, as M. *Venturi* remarks, is a very intereſting circumſtance, to which, perhaps, ſufficient attention has not been paid in the conſtruction of hydraulic machines. It is not enough that elbows and contractions are avoided; for it may happen, by an intermediate enlargement, that the whole advantage may be loſt, which may have been procured by the ingenious diſpoſitions of the other parts of the machine.

The above compriſes what to us appeared moſt important in M. *Venturi*'s reſearches, relative immediately to the ſubject of hydrodynamics. Thoſe, however, who are deſirous of ſeeing a more detailed account of this ingenious author's experiments may conſult Mr. Nicholſon's tranſlation of his work "*On the Lateral Communication of Motion in Fluids*," ſold by Taylor, Holborn.

Practical Conclusions of Mr. Eytelwein.

463. Mr. *Eytelwein* publiſhed, at Berlin, in 1801, a treatiſe entitled *Handbuch der Mechanik und der Hydraulik*; from the ſecond part of which, relative to hydrodynamics, we ſhall extract a few particulars.

I. In the chapter on the motion of water flowing out of reſervoirs, and on the contraction of the ſtream, this gentleman makes the area of a ſection at the diſtance of about half its diameter from the orifice about $\frac{16}{25}$ of that of the aperture: hence the diameter is reduced to $\frac{4}{5}$. The quantity of water diſcharged is very nearly, but not quite, ſufficient to fill this ſection with the velocity due to the height: for, finding more accurately the quantity diſcharged, the orifice muſt be ſuppoſed diminiſhed to ·619, or nearly $\frac{5}{8}$. Hence the ſquare root of the height may be multiplied by 5 inſtead of 8 (an approximate root of $64\frac{1}{6}$) for the mean velocity in a ſimple orifice.

II. If we apply the ſhorteſt pipe that will cauſe the ſtream to adhere every-where to its ſides, which will require its length to be twice its diameter; the diſcharge will be about $\frac{13}{16}$ of the full quantity, and the velocity may be found by taking $6\frac{1}{2}$ for a multiplier.

III. The greateſt diminution is produced by inſerting a pipe ſo as to project within the reſervoir, probably becauſe of the greater interference of the motions of the particles approaching its orifice in all directions: in this caſe the diſcharge is reduced nearly to a half.

IV. A conical tube approaching to the figure of the contraction of the ſtream procured a diſcharge of ·92, and when its edges were rounded off a diſcharge of ·98, calculating on its leaſt ſection.

V. Mr. Eytelwein is of opinion that the aſſertion of Venturi (art. 462. vii.) is too ſtrong, and obſerves, that where the pipe is already very long ſcarcely any effect is produced by the addition of ſuch a tube. He proceeds to deſcribe a number of experiments made with different pipes, where the ſtandard of compariſon is the time of filling a given veſſel out of a large reſervoir, which was not kept always full, as it was difficult to avoid agitation in repleniſhing it; and this circumſtance was perfectly indifferent to the reſults of the experiments. They confirm the aſſertion that a compound conical pipe may increaſe the diſcharge to twice and a half as much as through a ſimple orifice, or to more than half as much more as would fill the whole ſection with the velocity due to the height: but where a conſiderable length of pipe intervenes the additional orifice appears to have little or no effect.

VI. The firſt chapter concludes with a general table of the coefficients for finding the mean velocity of the water diſcharged by the preſſure of a given head under different circumſtances.

1. For the whole velocity due to the height, the coefficient by which its ſquare root is to be multiplied is 8·0208.

2. For an orifice of the form of the contracted ſteam, 7·8.

3. For wide openings, of which the bottom is on a level with that of the reſervoir; for ſluices with walls in a line with the orifice; for bridges with pointed piers, 7·7.

4. For narrow openings, of which the bottom is on a level with that of the reſervoir; for ſmaller openings in a ſluice with ſide walls; for abrupt projections and ſquare piers of bridges, 6·9.

5. For ſhort pipes, from two to four times as long as their diameter, 6·6.

6. For openings in ſluices without ſide walls, 5·1.

7. For orifices in a thin plate, 5.

VII. In the chapter on the diſcharge by rectangular orifices in the ſide of a reſervoir, extending to the ſurface, this author makes uſe of the ſame principles as we have adopted in art. 446. and ſhews that the quantity of water diſcharged may be found by taking two-thirds of the velocity due to the mean height, and allowing for the contraction according to the form of the opening.

On comparing the factors here deduced by Eytelwein from his reſearches, with thoſe in our account of Boſſut's experiments, it will be found that there is no very great diſagreement between them: though Boſſut's manifeſtly claim the preference in point of accuracy. A further account of Eytelwein's labours (by Dr. *T. Young*) may be ſeen in Nicholſon's Journal, Nos. 9, 10. N. S.

Researches of Dr. M. Young and Mr. Vince.

464. Dr. *Matthew Young*, the late biſhop of Clonfort, paid particular attention to the various circumſtances connected with the diſcharge of fluids from orifices of different kinds; but he appears to have been moſt ſucceſsful in his enquiries into the cauſe of the increaſed velocity of efflux through additional tubes. This chapter will, therefore, be terminated by an extract from his paper in vol. VII. of the *Tranſactions of the Royal Iriſh Academy*, which contains ſome judicious remarks relative to his own experiments, and applicable at the ſame time to ſome experiments made by Mr. *Profeſſor Vince*.

"When a tube *m n r s* (fig. 10. pl. XVII.) is inſerted into the veſſel ABCD, it is found that the velocity is increaſed nearly in the ſubduplicate ratio of the length of the pipe; and that it approaches nearer to that ſubduplicate ratio according as the length of the pipe is increaſed. To account for this increaſe of velocity has appeared a matter of ſome difficulty, ſince the water cannot iſſue at *r s* with a greater velocity than it enters at *r n*; and it does not appear how the velocity at *m n* can be increaſed by inſerting a tube beneath it. In order to explain the cauſe of this effect, we are to conſider that the whole force with which the plate *m n* is preſſed down is the weight of a column of water equal to *e m n f*, together with the weight of a column of air of the ſame baſe, reaching to the top of the atmoſphere; and the whole force with which it is preſſed up is the weight of an equal column of air, diminiſhed by the weight of a column of water equal to *m n r s*; therefore the actual force with which the plate *m n* is preſſed down is the weight of a column of water equal to *e f r s*: the velocity therefore with which the plate *m n* will iſſue through the orifice *m n* will be the ſame as through the orifice *r s* in the veſſel A *b c* D;

that is, equal to the velocity which a heavy body would acquire in falling through the altitude *er*: and all the plates of water in the tube *mnrs* will deſcend with the ſame velocity; for they cannot deſcend faſter, becauſe otherwiſe there would be a vacuum left in the tube, which is prevented by the upward preſſure of the atmoſphere. And the velocity of the effluent water will be the ſame, whatever be the preſſure of the atmoſphere, provided the weight of a column of air of the ſame baſe with *rs*, and whoſe height is equal to that of the atmoſphere, be either greater than or equal to the weight of the pillar of water *mnrs*. This might be proved experimentally by a veſſel of water with a pipe inſerted in the bottom placed under an exhauſted receiver. But as the operation of exhauſtion is obſtructed more by the evaporation of water than of mercury, it will be better to uſe mercury in theſe experiments. Now if D be the defect of the gauge from the ſtandard altitude, it will meaſure the preſſure of the air on the ſurface of the mercury in the veſſel: let A be the altitude of the mercury in the veſſel above the upper orifice of the pipe, and P the length of the pipe; then the whole force preſſing downwards the plate of mercury, which is immediately in the upper orifice of the pipe, will be $= D + A$; and the whole force preſſing the ſame plate upwards will be $D - P$; and the difference between theſe forces will be the abſolute force preſſing the ſame plate of mercury downwards: while D is greater than P, this abſolute force will conſequently be equal to $A + P$; when $D = P$, $D - P$ vaniſhes, and the force preſſing the plate downwards is $= D + A = P + A$; hence therefore no variation in the time of the efflux will be perceived, while the altitude of the mercury in the gauge is equal to or leſs than the difference between the length of the pipe and the ſtandard altitude. When D is leſs than P, the force upwards is alſo nothing; and therefore, as before, the whole force preſſing the plate downwards is $= D + A$; and A being given, it decreaſes according as D decreaſes; and when D vaniſhes, that is, when the receiver is abſolutely exhauſted, the force becomes equal to A, and the time of the efflux will be the ſame as if the pipe had not been inſerted in the bottom of the veſſel. To try the truth of theſe things by experiment, I inſerted a tube 7·8 inches long in a cylindrical veſſel, and, cloſing the orifice of the pipe, I filled the veſſel with mercury to the height of 6 inches; then placing the apparatus under the receiver of an air-pump, when the barometer was at 30 inches, and the gauge at 28·5, the time of the efflux was 26 ſeconds; when the experiment was repeated preciſely in the ſame manner, but in the open air, the time of the efflux was only 19 ſeconds. Now as the gauge ſtood at 28·5, the defect

D was $30-28{\cdot}5=1{\cdot}5$, and the preſſure on the plate of mercury was $=6+1\frac{1}{2}=7\frac{1}{2}$; in the open air the preſſure was $=6+7{\cdot}8=13{\cdot}8$: therefore the ratio of the velocity of efflux in both caſes, which is the ſame with the reciprocal ratio of the times, was $\sqrt{7\frac{1}{2}}$ to $\sqrt{13{\cdot}8}$, or as 2·73 to 3·7; but 2·73 is to 3·7 as 19 to 26, very nearly.

"No difference was obſerved in the times of the efflux, when in the open air and exhauſted receiver, unleſs the gauge ſtood higher than 22½ inches; that is, unleſs the height of the mercury in the gauge was greater than the difference between the length of the pipe and the ſtandard altitude. In another experiment, when the gauge ſtood at 27·9, the height of the barometer was 29·9; the defect therefore was $=2$, and the preſſure $=8$. But $\sqrt{8}=2{\cdot}828$, and $\sqrt{13{\cdot}8}=3{\cdot}7$; but $2{\cdot}828 : 3{\cdot}7 :: 19 : 24$, and by experiment the time of the efflux appeared to be 23 ſeconds. When the efflux is made in vacuo, it is obvious to obſerve that the pipe is not filled during the efflux, as it is while the diſcharge is made in the open air.

465. "Since the column of water in the pipe *m n r s* adds to the preſſure which the plate *m n* ſuſtains, by diminiſhing the upward preſſure of the air through the pipe, it appears that it produces this increaſe of preſſure in the plate *m n* alone, without producing any lateral preſſure in the water which is on a level with *m n*; for it is manifeſt, that if an aperture were made in *m* B or *n* C, the velocity of the water iſſuing through it would not be affected by the inſertion of the pipe; and conſequently that the plate *m n*, which is immediately in the orifice of the pipe, is the only one, on the ſame level, whoſe tendency downwards is increaſed by the inſertion of the pipe. Hence the particles of water at the edge of the aperture having their perpendicular preſſure increaſed by the weight of the column *m n r s*, without any increaſe of their lateral preſſure, they will iſſue through the orifice *m n* more perpendicularly; the ſides alſo of the tube will obſtruct the converging motion of the particles, and conſequently, on both theſe accounts, the quantity of water diſcharged through a pipe thus inſerted will exceed that diſcharged through a ſimple orifice, in a greater ratio than the ſub-duplicate of the height of the water. And according as the length of the pipe increaſes, the ratio of the quantity of water actually diſcharged by experiment, to that which ſhould be diſcharged according to theory, will increaſe; becauſe the ratio of the perpendicular to the horizontal preſſure increaſes, in the ratio of the ſum of the depth of the veſſel, and length of the pipe, to the depth of the veſſel. It follows therefore, that experiments made in this manner will approach nearer to coincidence with theory than when made with a

ſimple orifice; except either when the tube is ſo long as that the friction of the fluid againſt the ſides of the tube ſhall produce a ſenſible effect, or when it is ſo ſhort as not to be ſufficient to give the particles a vertical direction. All which agrees very well with the experiments made by the ingenious Mr. Vince, of which he has given us an account in the Phil. Tranſ. for the year 1795. Thus he tells us, that having inſerted a tube, a quarter of an inch in length, into a cylindrical veſſel 12 inches deep, he found that the velocity did not ſenſibly differ from that through the orifice; the cauſe of which he diſcovered to be this, that the ſtream did not fill the pipe, but that the fluid was contracted, as when it flowed through the ſimple orifice. When the pipe was half an inch long, inſerted into a veſſel of the ſame depth as before, the velocity of the water from the pipe and from the orifice, which ought by theory to have been as $\sqrt{12\cdot5}$ to $\sqrt{12}$, or 49 to 48, was by experiment found to be nearly in the proportion of 4 to 3. Now if the ratio of 49 to 48 be increaſed in the ratio of 7 to 6 (becauſe this is the ratio of the diminution of the velocity on account of the contraction of the vein, and this contraction either nearly or entirely vaniſhes in a pipe), we ſhall have the ratio of 3·57 to 3. When the pipe was an inch long, the velocity from the pipe and from the orifice, which, according to theory, ought to have been as $\sqrt{13}$ to $\sqrt{12}$, or as 26 to 25, appeared by experiment very nearly in the ratio of 4 to 3; now if the ratio of 26 to 25 be increaſed in the ratio of 7 to 6, we ſhall have the ratio of 3·64 to 3. When he made uſe of longer pipes, the velocity of the effluent water by experiment approached nearer to that which ought to have been diſcharged according to theory; ſo that in long pipes the difference between theory and experiment, he ſays, was not greater than what might be expected from the friction of pipes, and other cauſes which may be ſuppoſed to retard the velocity. When he inſerted a pipe of the ſame diameter with the aperture, which terminated in a truncated cone fixed in the orifice (ſee the Philoſophical Tranſactions for the year 1795), he expected that the quantity of water diſcharged in a given time would have been diminiſhed, becauſe the water, iſſuing through the orifice *m n*, would have room to form the *vena contracta* in the enlarging cone; but he found that the ſame quantity of water was diſcharged as if the pipe had continued throughout of the ſame diameter with the orifice. The reaſon of this is manifeſt from what has been ſaid, for the preſſure of the air will not ſuffer the truncated cone to remain partly empty, as it would be if the *vena contracta* were formed; it will therefore continue full, and conſequently the water will paſs through it in the ſame manner as if the water in the cone, ſurrounding the pipe *m a b n*, were congealed.

466. "Mr. Vince likewiſe inſerted into the bottom of the veſſel a perpendicular pipe, in form of a truncated cone, the narrower part being fixed in the orifice, by which he found the efflux to be increaſed more than if he had inſerted a cylindrical pipe of the ſame length, whoſe diameter was equal to that of the narroweſt part of the conical pipe. This effect may be explained on the ſame principle by which we account for the augmentation of the diameter of a vertical vein of water through a ſimple orifice, when the velocity of the efflux is conſiderable. For when a perpendicular pipe is inſerted, the velocity of the diſcharge being conſiderably increaſed, the reſiſtance from the air will be ſo likewiſe; and thus the diameter of the vein has a tendency to enlarge itſelf: now, in the widening cone, the pipe admits of this augmentation, at the ſame time that it increaſes the velocity; but the cylindrical pipe, though it equally increaſes the velocity, yet it does not permit the vein to enlarge itſelf, and, by thus confining it, the efflux is obſtructed, and the quantity diſcharged in a given time is diminiſhed. Accordingly, under the receiver of an air-pump, even in a moderate degree of exhauſtion, there is no difference perceived between the velocities with which a fluid is diſcharged through a conical or cylindrical pipe."

CHAPTER III.

On the Effect of Water upon the Motion of Water-wheels.

467. WATER-mills are diſtinguiſhed into four kinds: *breaſt-mills*, *underſhot-mills*, *overſhot-mills*, and *mills with horizontal wheels*; of which, however, the latter kind is by far the leaſt common, being very diſadvantageous and deficient in point of utility. In a breaſt-mill the water falls down upon the wheel at right-angles to the float-boards or buckets placed all round the wheel to receive it: if float-boards are uſed, the water acts only by its impulſe; but if buckets, it acts alſo by the weight of water in the buckets in the under quadrant of the wheel, which is conſiderable. In the underſhot wheel float-boards only are uſed, and the wheel is turned merely by the force of the current running under it, and ſtriking upon the boards. In the overſhot-wheel the water is poured over the top, and thus acts principally, though not altogether, by its weight; for the fall upon the upper part cannot be very conſiderable, leſt it ſhould daſh the water out of the buckets. Hence it is evident that an underſhot-wheel muſt require a much larger ſupply of water than any other; the breaſt-mill the next, unleſs the fall is very great; and an overſhot-mill the leaſt.

It was long believed that the float-boards of an underſhot-wheel ought to be ſo proportioned that when one of them was in a vertical poſition, or at the middle of its immerſion, the next board ſhould be juſt entering the water: but it is now well known that the more float-boards ſuch a wheel has, the greater and more uniform will be its effect. According to the experiments made by M. *Boſſut* on this ſubject, a wheel furniſhed with 48 float-boards produced a much greater effect than one furniſhed with 24; and the latter a greater effect than one with 12; their immerſion in the water being equal. And the ſame thing will appear from ſome of the experiments of Mr. *Smeaton*, deſcribed in the fourth chapter of this book.

468. The varying force which communicates motion to water-wheels, and the reſiſtances occaſioned by friction, tenacity, and various other cauſes, render the application of the theory to practice in theſe caſes extremely difficult. From this reaſon probably it happens that the art of conſtructing machines to be moved by the force of water has been almoſt wholly practical; the beſt improvements having generally been deduced from conſtant obſervation and reiterated experiments. Since the theory of mechanics is confeſſedly inadequate to the complete inveſtigation of every circumſtance in the motion of hydraulic machines, it has been thence ſuppoſed that the ſame laws of motion would not extend to all branches of mechanics, but that different principles were to be accommodated to the motion and mutual action of different kinds of bodies. If this were truly the caſe the ſcience of mechanics would fall far ſhort of that ſuperior excellence and extent which it is generally allowed to poſſeſs. But it is highly probable that there is no kind of motion, whether of the moſt ſimple or complicated nature, but what may be referred to the ſame principles as were aſſumed at the commencement of this work (art. 21.): and if we are not enabled to inveſtigate the effects from the data in every caſe which can ariſe, the deficiency muſt not be imputed to the firſt principles of mechanics, but to the want of methods of applying our mathematical knowledge to the peculiar circumſtances under diſcuſſion; or to our inability to reduce to computation the various cauſes of reſiſtance and obſtruction to the moving forces. For theſe reaſons, however, we ſhall not dwell long upon the theory of the motion of water-wheels: but ſhall merely exhibit a few propoſitions, the conſideration of which may enable the ſtudent to appreciate more exactly the value and importance of the experimental reſearches of Smeaton, and ſome other practical writers.

469. Prop. *If a ſtream of water impinge on the float-boards of an underſhot-wheel, and eſcape from it the very inſtant after it has made its impact, the quantity of water which actually impinges againſt the wheel will be to the whole quantity which paſſes by it in a given time, as the difference between the velocities of the water and of the wheel to the abſolute velocity of the water.*

Let WH (fig. 2. pl. XVIII.) be the wheel, BA the ſtream of water, and let the float-board F firſt receive the impact from the water at F, and quit it at C; alſo let DF be to FC as the abſolute velocity of the water to the velocity of the float-board. Then, when F arrives at C, the particle at D will have paſſed at F; and, taking DE=FC, all the water in the ſpace DE will paſs by the wheel without impinging againſt it: for it cannot impinge on the float F, becauſe that float emerges from the water at C; neither can it impinge on the ſubſequent float, becauſe it has already paſſed it. Therefore the whole quantity of water which paſſes by the wheel in a given time is to that which actually impinges againſt it as DF to EF.

Cor. 1. Hence we may correct the miſtake of Mr. *Waring*, in his New Doctrine of Mills, who lays it down as a fundamental principle, that while the ſtream is invariable, whatever be the velocity of the wheel, the ſame number of particles or quantity of the fluid muſt ſtrike it ſomewhere or other in a given time. See the 3d vol. of the Tranſactions of the American Society, page 144.

Cor. 2. *The force of the impinging water is as the ſquare of the difference between the velocities of the wheel and the water.*

For the force is as the relative velocity into the quantity of impinging matter, and the latter is manifeſtly as the relative velocity; therefore the force will be as the ſquare of the relative velocity.

470. Prop. *If* w *be a weight faſtened to a line which is wound round the horizontal axis of an underſhot water-wheel,* A *the altitude of a column of water equal to the force of the impact of the water on the wheel, when the wheel is quieſcent,* V *the velocity with which the water impinges on the float-boards,* v *the velocity of the circumference of the wheel,* R *the radius of the wheel, and* r *the radius of the axle; then will the velocity of the wheel be* $v = V - V\sqrt{\frac{wr}{AR}}$.

For the relative velocity with which the water ſtrikes the wheel is $V - v$: whence, becauſe the force of the ſtroke is as the ſquare of the relative velocity, we have $V^2 : (V-v)^2 :: A : F$, the force of the water to turn the wheel, when its velocity is v; and $F^2 = A\left(\frac{V-v}{V}\right)^2$, or $v = V - V \cdot \sqrt{\frac{F}{A}}$. But the acceleration of the wheel will ceaſe when the force of the water to turn the wheel is equal to the force of the weight which oppoſes it; that is, when $FR = wr$, or $F = \frac{wr}{R}$. Hence, ſubſtituting this value of F for it in the preceding value of v, there reſults $v = V - V\sqrt{\frac{wr}{AR}}$, for the velocity of the wheel when its acceleration ceaſes.

This concluſion, it ſhould be obſerved, is true only on the hypotheſis that the water eſcapes from the wheel as faſt as it impinges.

COR. 1. The expreſſion $F = A\left(\frac{V-v}{V}\right)^2$, found above, is the ſame in effect as the expreſſion $F = \varphi\left(\frac{W-V}{W}\right)^2$ found in art. 377. II. Only we have now made a change in the notation, to ſuit it for more eaſy recollection in the enquiry before us. Hence, purſuing ſimilar methods to thoſe adopted in the place juſt referred to, we may deduce ſimilar practical inferences; and which, without a repetition of the inveſtigation, may be expreſſed as below.

COR. 2. *If the weight* w *vary, its momentum will be the greateſt poſſible when the wheel has acquired its uniform velocity, if* $w = \frac{4AR}{9r}$.

COR. 3. *The greateſt momentum generated in the aſcending weight will be* $= \frac{4}{27}AV$.

COR. 4. *When the momentum of the aſcending weight* w *is a maximum, that weight will be* $\frac{4}{9}$ *of the weight which would, if ſuſpended from the axle, balance the force of the ſtream.*

COR. 5. *When the momentum of the aſcending weight is a maximum, the velocity of the wheel will be* $\frac{1}{3}$ *of the abſolute velocity of the ſtream.*

COR. 6. *When the uniform velocity of the aſcending weight* w *is a maximum, the radius of the wheel will be* $= \frac{9wr}{4A}$.

For the radius of the wheel being R, and that of the axle r, the uniform velocity of the aſcending weight will be found by multiplying $V - V\sqrt{\frac{wr}{AR}}$ into the fraction $\frac{r}{R}$; that is, the velocity of w will be $= \frac{r}{R}V - V\sqrt{\frac{wr^3}{AR^3}}$: which, ſuppoſing R variable, and making the fluxion $= 0$, gives $R = \frac{9wr}{4A}$.

471. PROP. *If the velocity of the ſtream be given, the greateſt effect will be as the quantity of water expended.*

For, by cor. 3. laſt prop. the greateſt effect is as $\frac{4}{27}AV$; that is, ſince $\frac{4}{27}$ is conſtant, the effect is in a ratio compounded of the force of impact and the velocity of the ſtream: but the force of the impact is as the quantity of water expended and velocity conjointly; conſequently the effect is as the quantity of water expended and the ſquare of the velocity; or, if the velocity be given, as the quantity of water expended.

COR. 1. *When the expence of water is the ſame, the greateſt effect will be as the ſquare of the velocity.*

Cor. 2. *The expence of water being the ſame, the effect will be as the height of the head of water.*

For $V=\sqrt{2gh}$, whence $h \propto V^2$.

Cor. 3. *The aperture being the ſame, the effect will be as the cube of the velocity.*

For the effect is as Q (the quantity of water) into V^2; and when the aperture is given $Q \propto V$, whence, effect $\propto V^3$.

472. *If all the water which paſſes by an underſhot wheel be ſuppoſed to impinge againſt it, the force of the ſtream will be ſimply in the direct ratio of the relative velocity.*

Becauſe the number of particles which ſtrike the wheel in a given time is given, whatever be the velocity of the wheel.

Cor. 1. *According to this hypotheſis,* v *the velocity of the wheel will be equal to* $V-V.\frac{wr}{AR}$.

For, in this caſe, we have $V : V-v :: A : F$, whence $v=V-\frac{VF}{A}$; which, by ſubſtituting for F its value $\frac{wr}{R}$, becomes $v=V-\frac{Vwr}{AR}$.

Cor. 2. *Retaining the ſame hypotheſis, if* w *vary, its momentum will be* $=\frac{AR}{2R}$, *when it is a maximum.*

For ſince the uniform velocity of the circumference of the wheel is $=V-\frac{Vwr}{AR}$, which, multiplied into $\frac{r}{R}$, gives $\frac{Vr}{R}-\frac{Vr^2w}{AR^2}$ for the uniform veloc. of the aſcending weight w: hence its momentum is $\frac{Vrw}{R}-\frac{Vr^2w^2}{AR^2}$; which being a maximum we make its fluxion $=0$, conſidering w as variable, whence reſults $w=\frac{AR}{2r}$.

Cor. 3. *On the ſame ſuppoſition, the velocity of the wheel will be half the velocity of the ſtream, when the effect is a maximum.*

For, by cor. 1. $v=V-\frac{Vwr}{AR}$; and in the caſe of the laſt cor. $w=\frac{AR}{2r}$. Putting this for v in the former equation, it is transformed to $v=V-\frac{VARr}{2ARr}=\frac{1}{2}V$.

Cor. 4. *Still retaining the ſame hypotheſis, the greateſt momentum generated in the aſcending weight will be* $=\frac{1}{4}AV$.

For $\frac{1}{2}\text{V} \times \frac{r}{\text{R}} = \frac{\text{V}r}{2\text{R}}$ is the uniform velocity of the aſcending weight; and the weight moved w is, by cor. 2. $= \frac{\text{AR}}{2r}$. Conſequently $\frac{\text{V}r}{2\text{R}} \times \frac{\text{AR}}{2r} = \frac{1}{4}\,\text{AV}$, the momentum.

SCHOLIUM.

473. In practice, the velocity of the wheel when the machine is in its greateſt perfection will be between one-third and one-half of the velocity of the ſtream. For the water does not all eſcape the inſtant after it has made the impact, but is confined by the channel for ſome time; ſo that the ſucceeding water, which would otherwiſe paſs by the wheel inefficaciouſly, drives the confined water againſt the float-boards, and therefore acts in the ſame manner as if it actually impinged againſt the wheel. Experiments ſhew that when the moſt work is done in a given time the velocity of the wheel is much nearer the half than the third of the velocity. See the following chapter, art. 483.

The diſcrepancies between the reſults of the experiments and the concluſions from the preceding theory ariſe, as before ſuggeſted, from our ignorance of the different circumſtances which affect the motion. It may not be amiſs to point out how the inveſtigation might be conducted ſuppoſing theſe things to be diſcovered. To this end let it be conſidered, that the moving force will generally conſiſt of the impact, and the weight of ſuch portion of the water as may tend to move the wheel by its gravity: this entire moving force, whether determined by theory or by experiment, may be denoted as before by A. After the moving force which impels the circumference of the wheel has been determined, the reſiſtance oppoſed to this force muſt be found; for on the relation between the moving force and the reſiſtance will depend the acceleration of the machine. The reſiſtance will ariſe, 1. from inertia. This may be readily eſtimated: for let the diſtance of the centre of gyration of the wheel from that of motion be ϱ, the wheel's weight W, and the other quantities as in the general notation of this chapter, then will the inertia of the wheel, which reſiſts the communication of motion to the circumference, be expreſſed by $\frac{\text{W}\varrho^2}{\text{R}^2}$, (art. 310, cor. 4.). And, in a nearly ſimilar manner, the inertia of any parts of the ſyſtem may be obtained, knowing the weight, figure, and poſition, by ſome of the rules in chap. IV. of book II. 2. If the machine be one that raiſes water or other weights, the weight raiſed, allowing for its mechanical effect on the point whoſe acceleration is ſought, muſt be ſubducted from the moving force before found; and this will be a conſtant quantity. 3. There are likewiſe other kinds of reſiſtance homogeneal to weight, viz. thoſe of friction, tenacity, &c. which vary in ſome ratio of the velocity of the machine: and, in order to proceed with the inveſtigation, the exact quantity of weight to which the friction is equal, when the wheel moves with a given velocity, muſt be known, as well as the variation of the reſiſtances in reſpect of the velocities; circumſtances which muſt be determined by experiment. If then the force equivalent to the friction, &c. be ſubtracted from the moving force, the remainder will ſhew the moving power by which the circumference may be conſidered as actually impelled; and this, divided by the inertia of the maſs moved, will give the force which accelerates the circumference. Thus, if, when the velocity of the wheel's circumference is V, the friction is equal to the weight Q applied at the circumference, and varies in the nth power of the velocity, we ſhall have the force which accelerates the circumference, as follows:

$$\frac{A}{w+W} \times \left(\frac{V-v}{V}\right)^2 \times \frac{R^2}{\varrho^2} - \frac{w R r}{w+W \varrho^2} - \frac{R^2 Q v^n}{V^n (w+W) \varrho^2}:$$

whence, if x be the ſpace which has been deſcribed by the circumference when the velocity is v, and $g = 32\frac{1}{6}$ feet, the principles of acceleration will give us this equation:

$$\frac{v \dot{v}}{g} = \left[\frac{A}{w+W} \times \left(\frac{V-v}{V}\right)^2 \times \frac{R^2}{\varrho^2} - \frac{w R r}{w+W \varrho^2} - \frac{R^2 Q v^n}{V^n (w+W) \varrho^2}\right] \times \dot{x}$$

And from this if v be determined in functions of x, and conſtant quantities, the velocity communicated to the wheel may be aſcertained. It is only by taking all theſe circumſtances into the account that we can produce an exact coincidence between theory and matter of fact. Further obſervations may be ſeen in Atwood's Treatiſe on Motion, &c. § vi.

474. Prop. *In any propoſed overſhot-wheel to compute the effective weight of water in the buckets.*

An overſhot unloaded is conſidered as perfectly in equilibrio upon its axis: but when it is loaded, the equilibrium is deſtroyed; becauſe the weight of the water lies all upon one moiety of the circumference, and cauſes it to preponderate.

Let ACB (fig. 1. pl. XVIII.) repreſent an overſhot-wheel, of which the buckets a, b, c, B, d, e, are loaded with water; and let there be ſuppoſed an equal quantity of fluid W in each bucket. Now the centre of gravity of the bucket a being directly over the axle C of the wheel will have no tendency to give it a rotatory motion, but will merely cauſe it to preſs more firmly upon its pivots, juſt as it would if a weight d', equal to that of the water in the bucket a, hung directly underneath the axle. But the ſecond bucket, whoſe centre of gravity is b, acts in the ſame manner upon the wheel as would an equal weight b' hanging freely at the point f; conſequently its effort will be proportional to the product $w \times Cf = w \times \text{ſin.}\ aCb$: and the ſame will hold with reſpect to the weight in the bucket e. The water in the buckets c and d act in the ſame manner as would the equal weights c' and d', hanging freely at the point g; their joint effort, therefore, will be proportional to $2\,w \times c\,\mathrm{I} = w \times 2\ \text{ſin.}\ a\,C\,c$. And the water in the bucket B, acting at the extremity of the radius, will have its effect proportional to $w \times \text{ſin.}\ 90 = w\,\mathrm{R}$, R being the radius of the wheel. Hence, 1. if there be 12 buckets about the wheel, as in the figure, the arcs $a\,b$, $b\,c$, $c\,\mathrm{B}$, &c. will each contain 30°: and we ſhall have $(0 + 2\ \text{ſin.}\ 30^\circ + 2\ \text{ſin.}\ 60^\circ + 1)\,w = (0 + 1 + \sqrt{3} + 1)\,w = 3{\cdot}7320508\,w$, for the effective weight of the water on the wheel, while its real weight is $6\,w$: ſo that the real weight of the water is to its effective weight as 6 to 3·7320508, or as 1 to ·622008.

2. If the number of buckets were 24, and all ſuppoſed equally full; then, purſuing a ſimilar courſe, we ſhould find $(2\ \text{ſin.}\ 15 + 2\ \text{ſin.}\ 30^\circ + 2\ \text{ſin.}\ 45^\circ + 2\ \text{ſin.}\ 60^\circ + 2\ \text{ſin.}\ 75^\circ + \text{ſin.}\ 90^\circ) \times w = 7{\cdot}585754\,w$, for the effective weight of water on the wheel, its real weight being $12\,w$: hence the latter will be to the former as 12 to 7·585754, or as 1 to ·632214.

3. If we ſuppoſe the number of buckets to be ſo increaſed, that the weight may be conſidered as equally diffuſed over the ſemi-circumference a BN of the wheel; then will the ſum of all the efforts to turn the wheel be equal to the ſum of all the diſtances, as $c\,\mathrm{I}$ into all the correſponding weights at c; that is, by the nature of the centre of gravity, as the ſemicir. $a\,\mathrm{BN} \times \mathrm{C}\,k$, k being the centre of gravity of the ſemicircular arc. Now by art. 118, $\mathrm{C}\,k = {\cdot}63662\ \mathrm{CB}$: conſequently, the actual weight will be to its effective weight as arc a BN to $\cdot 63662 \times \text{arc.}\ a\,\mathrm{BN}$, or as 1 to ·63662. This being the ratio to which the others approximate.

475. PROP. *In an overſhot-wheel the machine will be in its greateſt perfection when the diameter of the wheel is two-thirds of the height of the water above the loweſt point of the wheel.*

If the wheel ABN (fig. 1. pl. XVIII.) revolve with a velocity, ſuch as a body would acquire in falling through the altitude x above the upper part a of the wheel, the water will fall into the buckets without any impulſe, (becauſe then $\mathrm{V} - v = 0$), and produce its effect by its weight only. Let $h = x + a\,\mathrm{N}$, the height of the ſupply of water above the loweſt point N of the wheel. Now, as we have already obſerved in the latter part of the preceding article, the ſum of all the effective forces of the water, in all the buckets of the ſemicircle a BN, will be equal to the ſemicircle $a\,\mathrm{BN} \times \mathrm{C}\,k$, k being the centre of gravity of the ſemicircular arc; that is, it will be = the quantity of water $a\,\mathrm{BN} \times \frac{k\,\mathrm{C}}{\mathrm{CB}}$, acting at the point B. But $k\,\mathrm{C} = \frac{\mathrm{BC}\,.\,a\,\mathrm{N}}{\mathrm{arc.}\ a\,\mathrm{BN}} = \frac{\mathrm{BC}^2}{a\,c\,\mathrm{B}}$; conſequently, the force of all the water to turn the wheel is = quantity of water $a\,\mathrm{BN} \times \frac{\mathrm{BC}}{a\,c\,\mathrm{B}}$, acting at B = quantity of water 2 BC, or a N, acting at B. Hence, if BC = R, the whole force of water $\propto \mathrm{R} \times a\,\mathrm{N} = \mathrm{R} \times (b - x)$. But the velocity of the wheel is $\sqrt{2gx}$; and, conſequently, the force of the water to produce the rotatory motion of the wheel is $= \mathrm{R}\sqrt{2gx} - (b - x)$: this being made a maximum, will require $b^2 x - 2bx^2 + x^3$ a max. or its fluxion $b^2 \dot{x} - 4bx\dot{x} + 3x^2\dot{x} = 0$, whence we find $x = \frac{1}{3}b$. Conſequently the height of the ſupply of water above the upper part of the wheel ſhould be one-third of the whole height; and the diameter of the wheel two-thirds of that height, that the machine may produce its maximum effect.

Cor. The velocity of an overſhot-wheel, when the water produces its effect by its weight only, and the machine is in its ſtate of greateſt perfection, is to the velocity of an underſhot-wheel as $\sqrt{3}$ to 1, on the ſuppoſition that all the water eſcapes from the underſhot-wheel the moment after it makes its impact.

For, b being the height of the water, the velocity of the overſhot-wheel will be $= \sqrt{\frac{1}{3}gb}$, and the velocity of the underſhot $= \sqrt{\frac{1}{9}gb}$; conſequently, the former is to the latter as $\sqrt{\frac{1}{3}}$ to $\sqrt{\frac{1}{9}}$, or as $\sqrt{3}$ to 1.

476. Prop. *The efficacy of an overſhot-wheel is to that of an underſhot-wheel, the height of the water, aperture, and diameter, being given, as* 13 *to* 5 *nearly.*

If A be the altitude of the column of water, whoſe weight is equal to the force of impact on the underſhot-wheel, when quieſcent, ſince the velocity of the wheel is equal to $\frac{1}{3}$ of that of the ſtream, (art. 470. cor 5.), we ſhall have this analogy, $3^2 : (3-1)^2 :: \mathrm{A} : \frac{4}{9}\mathrm{A}$, the weight which is equivalent to the force of the ſtream when the machine is in its greateſt perfection. Whence, if V be the velocity acquired in falling down the altitude A, the momentum of the wheel will be $= \frac{4}{9}\mathrm{A} \times \frac{1}{3}\mathrm{V} = \frac{4}{27}\mathrm{AV}$ (art. 470. cor. 3), when in its beſt ſtate. In the overſhot-wheel the weight of the water acting at the circumference is equivalent to $\frac{2}{3}$ A, and the velocity is $\sqrt{\frac{1}{3}g\mathrm{A}}$ when the machine is in its beſt ſtate. But $\sqrt{\frac{1}{3} g A} = V \sqrt{\frac{1}{3}}$: conſequently the momentum of the overſhot-wheel $= \frac{2}{3} A \times V \sqrt{\frac{1}{3}} = \frac{2}{3} AV \sqrt{\frac{1}{3}}$; and is to that of the underſhot-wheel as $\frac{2}{3} \sqrt{\frac{1}{3}}$ to $\frac{4}{27}$, or as $\frac{3}{2} \sqrt{3}$ to 1, or as 2·5987 to 1, or nearly as 13 to 5.

COR. In theory there is no limit to the weight which a given ſtream of water can raiſe by means of a water-wheel.

For either the radius of the wheel may be increaſed, or that of the axle diminiſhed, without limit. But this is far from being the caſe in practice.

CHAPTER IV.

Account of Mr. Smeaton's Experiments on Water-wheels.

477. The late ingenious and experienced engineer, Mr. *John Smeaton*, paid very great attention to the conſtruction of mills and water-wheels; and as his experiments and obſervations relative to theſe ſubjects were very extenſive and judicious, we truſt a conciſe account of his enquiries will be both intereſting and beneficial.

In the 51ſt vol. of the Phil. Tranſ. Mr. Smeaton has given a valuable paper on water-wheels, of which the following is an abridgment.

Having deſcribed the machines and models uſed for making his experiments, he obſerves, that with regard to *power* it is moſt properly meaſured by the raiſing of a weight; or, in other words, if the weight raiſed be multiplied by the height to which it can be raiſed in a given time, the product is the meaſure of the power raiſing it; and, of conſequence, all thoſe powers are equal whoſe products made by ſuch multiplication are equal: for, if a power can raiſe twice the weight to the ſame height, or the ſame weight to twice the weight, in the ſame time that another can, the former power may be conſidered as double the latter; but if a power can only raiſe half the weight to double the height, or double the weight to half the height, in the ſame time that another can, the two powers are equal. This, however, as Mr. Smeaton remarks, "muſt be underſtood only of a ſlow and equable motion without acceleration or retardation." Indeed this muſt be looked upon merely as a popular meaſure, or abridged repreſentative, of eaſy comprehenſion and remembrance, and tolerably well adapted to the uſes of thoſe engaged in the conſtruction of machines, when reſtricted to thoſe caſes where the power expended and work performed are of the ſame kind as when a heavy body deſcends, and by its preponderance raiſes another body: but it is vague, inadequate, and unfit for general adoption; for *the quantity of motion extinguiſhed or produced, and not the product of the weight and height, is the true, unequivocal, meaſure of mechanical power really expended, or the mechanical effect actually produced; and theſe two are always equal and oppoſite.* Mr. Smeaton's meaſure, however, is moſtly applicable to the caſes in which he adopts it, and as ſuch it was neceſſary to explain it, that the ſtudent might the better underſtand the foundation of Mr. Smeaton's rules and inferences.

478. To compute the effects of water-wheels exactly, it is neceſſary to know in the firſt place what is the real velocity of the water which impinges on the wheel. 2. The quantity of water expended in a given time: and, 3. How much of the power is loſt by the friction of the machinery.

1. With regard to the velocity of the water, Mr. Smeaton determined by experiments with the machinery deſcribed in the volume referred to, that with a head of water 15 inches in height, the velocity of the wheel is 8·96 feet in a minute. The area of the head being 105·8 inches, this multiplied by the weight of a cubic inch of water equal to ·579 of an ounce avoirdupois, gives 61·26 ounces for the weight of as much water as is contained in the head upon one inch in depth; and by further calculations derived from the machinery made uſe of, he computes that 264·7 pounds of water deſcend in a minute through the ſpace of 15 inches. The power of the water, therefore, to produce mechanical effects in this caſe will be 264·7 × 15, or 3970. From the reſult of the experiment, however, it appeared that a vaſt quantity of the power was loſt; the effect being only to raiſe 9·375 pounds to the height of 135 inches: ſo that the power was to the effect as 3970 to 9·375 × 135 = 1266, or as 10 to 3·18.

This, according to our author, muſt be conſidered as the greateſt ſingle effect of water upon an underſhot-wheel, where the water deſcends from an height of 15 inches; but as the force of the current is not by any means exhauſted, we muſt conſider the true proportion betwixt the power and effect to be that betwixt the quantity of water already mentioned and the ſum of all the effects producible from it. This remainder of power, it is plain, muſt be equal to that of the velocity of the wheel itſelf multiplied into the weight of the water. In the preſent experiment, the circumference of the wheel moved with the velocity of 3·123 feet in a ſecond, which anſwers to a head of 1·82 inches; and this height being multiplied by 264·7, the quantity of water expended in a minute, gives 481 for the power of the water after it has paſſed the wheel; and hence the true proportion betwixt the power and the effect will be as 3849 to 1266; or as 11 to 4.

As the wheel revolved 86 times in a minute, the velocity of the water muſt be equal to 86 circumferences of the wheel; which, according to the dimenſions of the apparatus uſed by Mr. Smeaton, was as 86 to 30, or as 20 to 7.—The greateſt load with which the wheel would move was 9 lb. 6 oz.; and by 12 lb. it was entirely ſtopped. Whence our author concludes, that the impulſe of the wâter is more than double of what it ought to be according to theory: but this he accounts for by obſerving, that in his experiment the wheel was placed not in an open river, where the natural current after it has communicated its impulſe to the float has room on all ſides to eſcape, as the theory ſuppoſes, but in a conduit, to which the float being adapted, the water cannot otherwiſe eſcape than by moving along with the wheel. It is obſervable, that a wheel working in this manner, as ſoon as the water meets the float, receiving a ſudden check, it riſes up againſt the float like a wave againſt a fixed object, inſomuch that when the ſheet of water is not a quarter of an inch thick before the float, yet this ſheet will act upon the whole ſurface of a float whoſe height is three inches: and conſequently, was the float no higher than the thickneſs of the ſheet of water, as the theory alſo ſuppoſes, a great part of the force would have been loſt by the water daſhing over the float.

479. Mr. Smeaton next proceeds to give tables of the velocities of wheels with different heights of water; and from the whole deduces the following concluſions: 1. The virtual or effective head being the ſame, the effect will be nearly as the quantity of water expended. 2. The expence of water being the ſame, the effect will be nearly as the height of the virtual or effective head. 3. The quantity of water expended being the ſame, the effect is nearly as the ſquare of the velocity. 4. The aperture being the ſame, the effect will be nearly as the cube of the velocity of the water. Hence, if water paſſes out of an aperture in the ſame ſection, but with different velocities, the expence will be proportional to the velocity; and therefore, if the expence be not proportional to the velocity, the ſection of the water is not the ſame. 5. The *virtual* head, or that from which we are to calculate the power, bears no proportion to the head water; but when the aperture is larger, or the velocity of the water leſs, they approach nearer to a coincidence: and, conſequently, in the large openings of mills and ſluices, where great quantities of waters are diſcharged from moderate heads, the head of water, and virtual head determined from the velocity, will nearly agree; which is alſo confirmed by experience. 6. The moſt general proportion betwixt the power and effect is that of 10 to 3; the extremes 10 to 3·2, and

10 to 2·8. But it is obſervable, that where the power is greateſt, the ſecond term of the ratio is greateſt alſo: hence we may allow the proportion ſubſiſting in great works to be as three to one. 7. The proportion of velocity between the water and wheel is in general about 5 to 2. 8. There is no certain ratio between the load that the wheel will carry at its *maximum*, and what will totally ſtop it; though the proportions are contained within the limits of 20 to 19, and 20 to 15; but as the effect approaches neareſt to the ratio of 20 to 15, or of 4 to 3, when the power is greateſt either by increaſe of velocity or quantity of water, this ſeems to be the moſt applicable to large works; but as the load that a wheel ought to have in order to work to the beſt advantage, can be aſſigned by knowing the effect that it ought to produce, and the velocity it ought to have in producing it, the exact knowledge of the greateſt load it will bear is of the leaſt conſequence in practice.

Mr. Smeaton, after having finiſhed his experiments on the underſhot mills, reduced the number of floats, which were originally 24, to 12; which cauſed a diminution in the effect, by reaſon that a greater quantity of water eſcaped between the floats and the floor than before: but on adapting to it a circular ſweep of ſuch a length that one float entered into the curve before the other left it, the effect came ſo near that of the former, as not to give any hopes of advancing it by increaſing the number of floats beyond 24 in this particular wheel.

480. Our author next proceeds to examine the power of water when acting by its own gravity in turning an overſhot-wheel: "In reaſoning without experiment," ſays he, "one might be led to imagine, that however different the mode of application is, yet that whenever the ſame quantity of water deſcends through the ſame perpendicular ſpace, the natural effective power would be equal, ſuppoſing the machinery free from friction, equally calculated to receive the full effect of the power, and to make the moſt of it: for, if we ſuppoſe the height of a column of water to be 30 inches, and reſting upon a baſe or aperture of one inch ſquare, every cubic inch of water that departs therefrom will acquire the ſame velocity or momentum from the uniform preſſure of 30 cubic inches above it, that one cubic inch let fall from the top will acquire in falling down to the level of the aperture: one would therefore ſuppoſe that a cubic inch of water let fall through a ſpace of 30 inches, and there impinging upon another body, would be capable of producing an equal effect by colliſion, as if the ſame cubic inch had deſcended through the ſame ſpace with a ſlower motion, and produced its effects gradually. But, however concluſive this reaſoning may ſeem, it will appear in the

courſe of the following deductions, that the effect of the gravity of deſcending bodies is very different from the effect of the ſtroke of ſuch as are non-elaſtic, though generated by an equal mechanical power."

Having made ſuch alterations in his machinery as were neceſſary for overſhot-wheels, Mr. S. next gives a table of experiments with the apparatus ſo altered. In theſe the head was 6 inches, and the height of the wheel 24 inches; ſo that the whole deſcent was 30 inches; the quantity of water expended in a minute was $96\frac{2}{3}$ pounds; which, multiplied by 30 inches, gives the power = 2900: and, after making the proper calculations, the effect was computed at 1914; whence the ratio of the power to it comes to be nearly as 3 to 2. If, however, we compute the power from the height of the wheel only, the power will be to the effect nearly as 5 to 4.

481. From another ſet of experiments the following concluſions were deduced:

1. The effective power of the water muſt be reckoned upon the whole deſcent; becauſe it muſt be raiſed to that height in order to be able to produce the ſame effect a ſecond time. The ratios between the powers ſo eſtimated, and the effects at a *maximum*, differ nearly from 4 to 3, and from 4 to 2. Where the heads of water and quantities of it expended are the leaſt, the proportion is nearly from 4 to 3; but where the heads and quantities are greateſt, it comes nearer to that of 4 to 2; ſo that by a medium of the whole, the ratio is nearly as 3 to 2. Hence it appears, that the effect of overſhot-wheels is nearly double to that of underſhot ones; the conſequence of which is, that non-elaſtic bodies when acting by their impulſe or colliſion communicate only a part of their original impulſe, the remainder being ſpent in changing their figure in conſequence of the ſtroke. The ultimate concluſion is, that the effects, as well as the powers, are as the quantities of water and perpendicular heights multiplied together reſpectively.

2. By increaſing the head, it does not appear that the effects are at all augmented in proportion; for, by raiſing it from 3 to 11 inches, the effect was augmented by leſs than one-ſeventh of the increaſe of perpendicular height. Hence it follows, that the higher the wheel is in proportion to the whole deſcent, the greater will be the effect; becauſe it depends leſs upon the impulſe of the head, and more upon the gravity of the water in the buckets: and if we conſider how obliquely the water iſſuing from the head muſt ſtrike the buckets, we ſhall not be at a loſs to account for the little advantage that ariſes from the impulſe thereof, and ſhall immediately ſee of how little conſequence this is to the effect of an overſhot-wheel. This, how-

ever, as well as other things, muſt be ſubject to limitation; for it is neceſſary that the velocity of the water ſhould be ſomewhat greater than the wheel, otherwiſe the latter will not only be retarded by the ſtriking of the buckets againſt the water, but ſome of the power will be loſt by the daſhing of the water over the buckets.

3. To determine the velocity which the circumference of the wheel ought to have in order to produce the greateſt effect, Mr. Smeaton obſerves, that the more ſlowly any body deſcends by the force of gravity when acting upon any piece of machinery, the more of that force will be ſpent upon it, and conſequently the effect will be the greater. If a ſtream of water falls into the bucket of an overſhot-wheel, it will be there retained till the wheel diſcharges it by moving round; and, of conſequence, the ſlower the wheel moves, the more water it will receive; ſo that what is loſt in velocity is gained by the greater preſſure of water upon the buckets. From the experiments, however, it appears, that when the wheel made about 20 turns in a minute the effect was greateſt; when it made only 18$\frac{1}{4}$ the motion was irregular; and when loaded ſo as not to admit its turning 18 times, the wheel was overpowered with the load. When it made 30 turns, the power was diminiſhed by about $\frac{1}{20}$th, and when the number of turns were increaſed to 40, it was diminiſhed by one-fourth. Hence we ſee, that in practice the velocity of the wheel ſhould not be diminiſhed further than what will procure ſome ſolid advantage in point of power; becauſe, *cæteris paribus*, the bucket muſt be larger as the motion is ſlower; and the wheel being more loaded with water, the ſtreſs will be proportionably increaſed upon every part of the work. The beſt velocity for practice, therefore, will be that when the wheel made 30 turns in a minute, which is little more than three feet in a ſecond. This velocity is applicable to the higheſt overſhot-wheels as well as the loweſt. Experience however determines, that high wheels may deviate further from this rule before they will loſe their power, by a given aliquot part of the whole, than low ones can be permitted to do; for a wheel of 24 feet high may move at the rate of ſix feet *per* ſecond; while our author has ſeen one of 33 feet high move very ſteadily and well with a velocity of little more than two feet. The reaſon of this ſuperior velocity in the 24 feet wheel, may probably be owing to the ſmall proportion that the head, requiſite to give the proper velocity to the wheel, bears to the whole height.

4. The maximum load for an overſhot-wheel is that which reduces the circumference of the wheel to its proper velocity;

which is known by dividing the effect it ought to produce in a given time by the ſpace intended to be deſcribed by the circumference of the wheel in the ſame time: the quotient will be the reſiſtance overcome at the circumference of the wheel, and is equal to the load required, including the friction and reſiſtance of the machinery.

5. The greateſt velocity that an overſhot-wheel is capable of, depends jointly upon the diameter or height of the wheel and the velocity of falling bodies; for it is plain that the velocity of the circumference can never be greater than to deſcribe a ſemi-circumference, while a body let fall from the top deſcribes the diameter, nor even quite ſo great; as the difference in point of time muſt always be in favour of that which falls through the diameter. Thus, ſuppoſing the diameter of the wheel to be 16 feet and an inch in diameter, an heavy body would fall through this ſpace in one ſecond; but ſuch a wheel could never arrive at this velocity, or make one turn in two ſeconds, nor could an overſhot-wheel ever come near it: becauſe, after it has acquired a certain velocity, great part of the water is prevented from entering the buckets, and part is thrown out again by the centrifugal force: and as theſe circumſtances have a conſiderable dependence upon the form of the buckets, it is impoſſible to lay down any general rule for the velocity of this kind of wheels.

6. Though in theory we may ſuppoſe a wheel to be made capable of overcoming any reſiſtance whatever, yet as in practice it is neceſſary to make the wheel and buckets of ſome certain and determinate ſize, we always find that the wheel will be ſtopped by ſuch a weight as is equal to the effort of the water in all the buckets of a ſemi-circumference put together. This may be determined from the ſtructure of the buckets themſelves; but, in practice, an overſhot-wheel becomes unſerviceable long before this time; for when it meets with ſuch an obſtacle as diminiſhes its velocity to a certain degree, its motion becomes irregular; but this never happens till the velocity of the circumference is leſs than two feet *per* ſecond, when the reſiſtance is equable.

7. From the above obſervations we may eaſily deduce the force of water upon breaſt-wheels, &c. But, in general, all kinds of wheels where the water cannot deſcend through a given ſpace unleſs the wheel moves with it, are to be conſidered as overſhot-wheels; and thoſe which receive the impulſe or ſhock of the water, whether in an horizontal, oblique, or perpendicular direction, are to be conſidered as underſhots. Hence when the water ſtrikes at a certain point below the ſurface

of the head, and after that deſcends in the arch of a circle, preſſing by its gravity upon the wheel; the effect of ſuch a wheel will be equal to that of an underſhot whoſe head is equal to the difference of level between the ſurface of the water in the reſervoir and the point where it ſtrikes the wheel, added to that of an overſhot, whoſe height is equal to the difference of level between the point where it ſtrikes the wheel and the level of the tail-water.

482. In the Philoſophical Tranſactions, vol. 51, for the year 1759, there is another paper with experiments on mills turned both by water and wind, by Mr. Smeaton. From thoſe experiments it appears, page 129, that the effects obtained by the overſhot-wheel are frequently 4 or 5 times as great as thoſe with the underſhot-wheel, in the ſame time, with the ſame expence of water, deſcending from the ſame height above the bottoms of the wheel; or that the former performs the ſame effect as the latter, in the ſame time, with an expence of only one-fourth or one-fifth of the water, from the ſame head or height. And this advantage ſeems to ariſe from the water lodging in the buckets, and ſo carrying the wheel about by their weight. But, in page 130, Mr. Smeaton reckons the effect of overſhot only double to that of the underſhot-wheel. And hence he infers, in general, "that the higher the wheel is in proportion to the whole deſcent, the greater will be the effect; becauſe it depends leſs upon the impulſe of the head, and more upon the gravity of the water in the buckets. However, as every thing has its limits, ſo has this; for thus much is deſirable, that the water ſhould have ſomewhat greater velocity than the circumference of the wheel in coming thereon, otherwiſe the wheel will not only be retarded by the buckets ſtriking the water, but thereby daſhing a part of it over ſo much of the power is loſt." He is further of opinion, that the beſt velocity for an overſhot-wheel is when its circumference moves at the rate of about 3 feet in a ſecond of time.

483. Laſtly, in the Philoſophical Tranſ. for 1776, p. 457, the ſame author ſays, "The velocity of the wheel, which according to M. Parent's determination, adopted by Deſaguliers and Maclaurin, ought to be no more than $\frac{1}{3}$ of that of the water, varies at the maximum *between one-third and one half*; but in all the caſes in which the moſt work is performed in proportion to the water expended, and which approach the neareſt to the circumſtances of great works when properly executed, the maximum lying much nearer one half than one-third, one-half ſeeming to be the true maximum, if nothing

were loſt by the reſiſtance of the air, the ſcattering of the water carried up by the wheel, &c."

Many other experiments tending to improve the theory of water-wheels are deſcribed by Mr. Banks in part IV. of his Treatiſe on Mills. See alſo a Memoir on the moſt advantageous Conſtruction of Water-wheels by Mr. *Mallet* of Geneva, in Phil. Tranſ. for 1767; and two papers by *M. Lambert* in the Berlin Memoirs for 1775.

An extenſive chronological catalogue of writings on the ſubject of Mill-work in general, may be ſeen under the word MILL, in our ſecond volume.

PNEUMATICS.

BOOK V.

Preliminary Remarks on the Mechanical Properties of Atmospheric Air.

484. The term *Pneumatics*, as well as the word *Pneumatology*, is derived from a Greek word denoting *air*, or, more properly, *breath*: but in their modern acceptation the terms differ widely, the latter denoting the ſcience of the intellectual phenomena depending upon the operations or affections of the mind of man; while the former relates to that part of natural philoſophy which treats of the mechanical properties of air, and the different elaſtic fluids. Thus Pneumatics includes, as an important branch, the doctrine of *Aeroſtatics*, common air being the moſt extenſive and univerſal of the fluids which poſſeſs elaſticity.

Previous to our entering upon the theoretic part of this ſcience, it will be proper to relate ſome of the moſt obvious and natural experiments tending to prove the exiſtence of the different mechanical affections which are uſually aſcribed to the air as a body: to this object we ſhall therefore appropriate a few of the following articles.

485. I. *Air is a ponderous fluid which ſurrounds the earth.*

For, that it is a fluid is obvious, becauſe its parts are eaſily moved, and yield to the ſmalleſt inequality of preſſure: and that it is ponderous will appear from the following conſiderations:

1. It always accompanies this globe in its orbit round the ſun, ſurrounding it to a certain diſtance, under the name of the *Atmoſphere*, which indicates the being connected with the earth by its general force of gravity. It is chiefly in conſequence of this that it is continually moving round the earth from eaſt to weſt, forming what is called the trade-wind.

2. It is in like manner owing to the gravity of the air that it ſupports the clouds and vapours which we ſee conſtantly floating in it. We have ſeen bodies of no inconſiderable weight float, and even riſe, in the air. Soap-bubbles and balloons filled with inflammable gas riſe and float in the ſame manner as a cork

riſes in water. This phenomenon proves the weight of the air in the ſame manner that the ſwimming of a piece of wood indicates the weight of the water which ſupports it.

3. But we are not left to theſe refined obſervations for the proof of the air's gravity. We may obſerve familiar phenomena, which would be immediate conſequences of the ſuppoſition that air is a heavy fluid, and, like other heavy fluids, preſſes on the outſides of all bodies immerſed in or ſurrounded by it. Thus, for inſtance, if we ſhut the nozzle and valve hole of a pair of bellows, after having ſqueezed the air out of them, we ſhall find that a very great force, even ſome hundred pounds, is neceſſary for ſeparating the boards. They are kept together by the preſſure of the heavy air which ſurrounds them, in the ſame manner as if they were immerſed in water. In like manner, if we ſtop the end of a ſyringe after its piſton has been preſſed down to the bottom, and then attempt to draw up the piſton, we ſhall find a conſiderable force neceſſary, viz. about 15 or 16 pounds for every ſquare inch of the ſection of the ſyringe. Exerting this force, we can draw up the piſton to the top, and we can hold it there: but the moment we ceaſe acting, the piſton ruſhes down and ſtrikes the bottom. It is called a ſuction, as we feel ſomething as it were drawing in the piſton; but it is really the weight of the incumbent air preſſing it in. And this obtains in every poſition of the ſyringe; becauſe the air is a fluid, and preſſes in every direction. Nay, it preſſes on the ſyringe as well as on the piſton; and if the piſton be hung by its ring on a nail, the ſyringe requires force to draw it down (juſt as much as to draw the piſton up); and if it be let go, it will ſpring up, unleſs loaded with at leaſt 15 pounds for every ſquare inch of its tranſverſe ſection.

4. Let the air be exhauſted from a glaſs veſſel, and by means of a cock let the veſſel be kept exhauſted; if the veſſel be weighed while it is exhauſted, and then again when the air is re-admitted, there will be a manifeſt difference, exhibiting the weight of as much air as the veſſel contained.

5. If a glaſs tube more than 31 inches in length, one end of which is cloſed up, be filled with mercury, and be held vertically, the other extremity being immerſed in a veſſel of the ſame fluid, then the mercury in the tube will deſcend from the upper extremity, and will remain ſuſpended at ſome altitude between 28 and 31 inches from the ſurface of the external mercury: the ſuſpenſion of the mercury is occaſioned by the preſſure of the external air upon the ſurface of the mercury in the veſſel; when this preſſure is removed by placing the tube and veſſel under a receiver and exhauſting the air, the mercury

will ſink in the tube, and on re-admitting the air, will riſe. This is called the *Torricellian* experiment.

The inſtrument here uſed is called a *Barometer;* becauſe the weight of a column of mercury whoſe baſe is the orifice of the tube, and altitude equal to that of the mercury in the tube above the ſurface in the reſervoir, is equal to the weight of a column of air extending to the top of the atmoſphere, and whoſe baſe is equal to the ſame orifice. And ſince the weight of this column of quickſilver, *cæteris paribus*, is at its altitude, it follows that the weight of the air is proportional to the altitude of the mercury in the barometer.

The altitude at which the mercury is ſuſtained in the barometer above the ſurface of the mercury in the reſervoir is called the *ſtandard altitude;* and will be the ſame in any number of tubes, whatever be their bore or their poſition, provided the tube is not ſo very ſlender as to expoſe the mercury to a ſenſible alteration from the capillary attraction (art. 432.). The preſſure of the atmoſphere is equal to about 15 lbs. avoirdupois upon every ſquare inch, at the medium height of the mercury in the barometer: for a cubic foot of mercury weighs about 13568 ounces, and $\frac{13568}{1728}\times 29\frac{3}{4}=233{\cdot}6$ oz. $=14{\cdot}6$ lbs.

6. If a barometer tube, inſtead of being hermetically ſealed at the top, be cloſely covered with a piece of bladder, the mercury will riſe to the ſame height as in a common barometer; but on piercing the bladder with a needle ſo as to admit the air, the mercury will fall: for in this caſe the weight of the air preſſes upon the mercury in the tube, and the weight of theſe two together muſt obviouſly preponderate over the contrary preſſure, and deſtroy the equilibrium.

486. II. *The preſſure of the atmoſphere varies at different altitudes.*

Let a glaſs tube, open at both ends, be put through a cork into a large phial containing a ſmall quantity of coloured water; let the lower end of the tube be in the water; and let the cork and tube be cloſely cemented to the neck of the bottle: then blow through the tube till the quantity of air within the phial is ſo increaſed that the water will riſe above the neck of the phial. Let this phial be placed in a veſſel of ſand, to keep the air within of the ſame temperature: then will the water ſtand at different heights in the tube, according to the elevation of the place where it is ſet; whence it appears that the preſſure of the atmoſphere varies at different altitudes. And the ſame thing will appear more clearly in a ſubſequent article. In aſcending the mountain of Snowden, which is 3720 feet high, the barometer ſunk 3·8 inches.

Hence the proportion of the ſpecific gravity of air near the earth's ſurface to that of water may be aſcertained. Thus, if the difference in height of the two places where the above experiment is made be 52 feet, and that difference cauſe a variation of $\frac{3}{4}$ of an inch in the height of the water; it follows, that a column of water of $\frac{3}{4}$ of an inch, or $\frac{1}{16}$ of a foot, is equiponderant to a column of air of 52 feet, having the ſame baſe: conſequently the ſpecific gravity of water is to that of air as 52 to $\frac{1}{16}$, or as 832 to 1.

487. III. *Air is elaſtic, or capable of compreſſion and expanſion.* This is proved by various experiments: 1, By the great expanſion of a ſmall quantity of air in a bladder apparently nearly empty, when the air is removed from the external parts in the receiver. 2. By the extruſion of a fluid from a glaſs bubble, by the expanſion of a bubble of air contained in it. 3. By the expulſion of the white and yolk of an egg through a ſmall hole in the little end, by the expanſion of the air contained in the greater end. 4. By putting an almoſt emptied bladder into a ſmall box, and laying a proper weight on the lid, which, on exhauſting the air, will be raiſed up by the expanſion of the air in the bladder. 5. Alſo a bladder filled with air, and juſt made to ſink with a weight, will upon exhauſtion ſoon riſe by the expanſion of the contained air.

6. Glaſs bubbles and images filled with water ſo as to make them juſt ſink in that fluid, will, on exhauſting the air from the ſurface, riſe to the top of the veſſel. 7. Beer, cyder, water, and porous bodies, emit great quantities of air under the exhauſted receiver. 8. A ſhrivelled apple, when put under an exhauſted receiver, will have its coat diſtended by the internal air ſo as to look ſmooth. 9. If the open end of a tube, whoſe other end is cloſed, be immerſed perpendicularly in water, the ſpace occupied by the air will be diminiſhed, as the depth of the tube or the upward preſſure of the water is increaſed: or, if the ſhorter leg of a bent tube be cloſed, and mercury poured into the longer, the air will be compreſſed in the ſhorter leg into a ſpace continually decreaſing as the quantity of preſſing mercury is increaſed; and if ſome of the mercury be taken from the longer leg, the air in the ſhorter will expand and occupy a proportionably larger ſpace. 10. The mercury may be raiſed by the expanſion of a ſmall quantity of confined air to the ſame height in an exhauſted tube above the air-pump, as that to which it is raiſed in the mercurial gauge by the preſſure of the atmoſphere below it.

The limits of the condenſation and rarefaction of air by human powers are not aſcertained. Dr. Hales found, that when dry wood was put into a ſtrong veſſel, which it almoſt filled, and the remainder was filled with water, the ſwelling of the wood, occaſioned by its imbibition of water, condenſed the air of his gauge into the thouſandth of its original bulk. He found that peaſe treated in the ſame way generated elaſtic air, which preſſing on the air in the gauge condenſed it into the fifteen hundredth part of its bulk. This is the greateſt condenſation that has been aſcertained with preciſion, although in other experiments it has certainly been carried much further; but the preciſe degree could not be aſcertained.

The only uſe to be made of this obſervation at preſent is, that ſince we have been able to exhibit air in a denſity a thouſand times greater than the ordinary denſity of the air we breathe, it cannot, as ſome imagine, be only a different form of water; for in this ſtate it is as denſe or denſer than water, and yet retains its great expanſibility.

Another important obſervation is, that in every ſtate of denſity in which we find it, it retains its perfect fluidity, tranſmitting all preſſures which are applied to it with undiminiſhed force, as appears by the equality conſtantly obſerved between the oppoſing columns of water or other fluid by which it is compreſſed, and by the facility with which all motions are performed in it in the moſt compreſſed ſtates in which we can make obſervations of this kind. This fact is totally incompatible with the opinion of thoſe who aſcribe the elaſticity of air to the ſpringy ramified ſtructure of its particles, touching each other like ſo many pieces of ſponge or foot-balls. A collection of ſuch particles might indeed be pervaded by ſolid bodies with conſiderable eaſe, if they were merely touching each other, and not ſubjected to any external preſſure. But the moment ſuch preſſure is exerted, and the aſſemblage ſqueezed into a ſmaller ſpace, each preſſes on its adjoining particles: they are individually compreſſed, flattened in their touching ſurfaces, and *before the denſity is doubled* they are ſqueezed into the form of perfect cubes, and compoſe a maſs, which may indeed propagate preſſure from one place to another in an imperfect manner, and with great diminution of its intenſity, but will no more be fluid than a maſs of ſoft clay.

488. IV. *The elaſtic force of the air is equal to the force of compreſſion.*

For if the air be exhauſted from an open tube whoſe lower part is immerſed in a veſſel containing mercury, the air within the veſſel being prevented from eſcaping, then will this air by its elaſticity force the mercury up the tube very nearly to the ſame height as it would be raiſed by the preſſure of the atmoſphere.

This propoſition is ſometimes proved independent of experiments in the following manner: if the force with which the air endeavours to expand itſelf when it is compreſſed were *leſs* than the compreſſing force, it would yield ſtill further to that force; if it were greater, it could not have yielded ſo far. Conſequently, when any force has ſo compreſſed the air that it remains at reſt, the force of the air ariſing from its elaſticity can neither be greater nor leſs than the compreſſing force: that is, it muſt be equal to it.

But it muſt be confeſſed that, in this caſe, the experimental proof is the moſt ſatisfactory.

489. V. *The elaſticity of air not very different from its natural ſtate, is inverſely as the ſpace occupied by it.*

1. Let a cylindrical tube BC (fig. 7. pl. XI.), open at one end B, be filled with mercury to an altitude equal to BD before inverſion, and after the immerſion of B in a baſin of mercury, the air, which before occupied a ſpace equal to CD, is dilated through a larger ſpace as CA, and, if BN be the ſtandard altitude (art. 485. v.), depreſſes the mercury from N to A. Now the elaſtic force of air in its natural ſtate, or occupying a ſpace equal to CD (E), is to the elaſtic force of air occupying the ſpace BD (*e*) as the columns of mercury which they are capable of ſupporting (art. 488.); that is, as BN to AN : but, it is collected from experiments, that BN : AN : : AC : CD; conſequently AC : CD : : E : *e*.

And the ſame thing may be otherwiſe ſhewn by experiments with bent tubes.

Hence, ſince the denſity of the air is inverſely as the magnitude or ſpace occupied (art. 10.), it is, therefore, as the elaſticity, or as the compreſſing force (art. 488.): ſo that, putting D for the denſity, C the compreſſing force, E the force of elaſticity, and S the ſpace, we have $C \propto D \propto E \propto \frac{1}{S}$. Theſe relations, however, are confined within very narrow limits, for it has been aſſerted that when the air is compreſſed into a ſpace only four times leſs than the ſpace it occupies in its natural ſtate, it does not then vary inverſely as the force of compreſſion, the reſiſtance increaſing much more rapidly. *Comment. Bonon.* vol. I. p. 209, &c.

The experiments of Boyle, Mariotte, and Amontons, were not extended to very great compreſſions: ſo that they found generally that the elaſticity of the air was proportioned to its denſity; and the law was long acquieſced in, being called the *Boylean law.* But later philoſophers have carried the compreſſion much further. Thus Sulzer compreſſed air into $\frac{1}{8}$ of its former dimenſions: the reſults of his experiments are exhibited in the following table, where the column D ſhew the denſities, and thoſe marked E the correſponding elaſticities.

1ſt Set.		2d Set.		3d Set.	
D	E	D	E	D	E
1·000	1·000	1·000	1·000	1·000	1·000
1·100	1·093	1·236	1·224	1·091	1·076
1·222	1·211	1·294	1·288	1·200	1·183
1·375	1·284	1·375	1·332	1·333	1·303
1·571	1·599	1·466	1·417	1·500	1.472
1·692	1·669	1·571	1·515	1·714	1·659
1·833	1·796	1·692	1·647		
2·000	1·958	2·000	1·964	2·000	1·900
2·288	2·130				
2·444	2·375	2·444	2·392	2·400	2·241
3·143	2·936	3·143	3·078	3·000	2·793
3·666	3·391	3·666	3·575		
4·000	3·706			4·000	3·631
4·444	4·035	4·444	4·320		
4·888	4·438				
5·500	4·922	5·500	5·096		
5·882	5·522			6·000	5·297
		7·333	6·694		
				8·000	6·835

Other experiments for the ſame purpoſe were inſtituted by profeſſor Robiſon, the reſults of which may be ſeen below.

Dry Air.		Moiſt Air.		Camp. Air.	
D	E	D	E	D	E
1·000	1·000	1·000	1·000	1·000	1·000
2·000	1·957	2·000	1.920	2·000	1·909
3·000	2·848	3·000	2·839	3·000	2·845
4·000	3·737	4·000	3·726	4·000	3·718
5·500	4·930	5·500	5·000	5·500	5·104
6·300	5·342	6·000	5·452	6·000	5·463
7·620	6·490	7·620	6·775	7·620	6·812

Here it appears again in the cleareſt manner that the elaſticities do not increaſe ſo faſt as the denſities, and the differences are even greater than in Mr. Sulzer's experiments.

The ſecond table contains the reſults of experiments made on very damp air in a warm ſummer's morning. In theſe it appears that the elaſticities are almoſt preciſely proportional to the denſities + a ſmall conſtant quantity, nearly 0·11, deviating from this rule chiefly between the denſities 1 and 1·5, within which limits we have very nearly $D = E^{1.0017}$. As this air is nearer to the conſtitution of atmoſpheric air than the former, this rule may be ſafely followed in caſes where atmoſpheric air is concerned, as in meaſuring the depths of pits by the barometer.

The third table ſhows the compreſſion and elaſticity of air ſtrongly impregnated with the vapours of camphire. Here the Boylean law appears pretty exact, or rather, the elaſticity ſeems to increaſe a little faſter than the denſity.

490. VI. *Heat increaſes the elaſticity of the air, and cold diminiſhes it. Or, heat expands and cold condenſes the air.*

This property is likewiſe proved by experience.

1. Tie a bladder very cloſe with ſome air in it, and lay it before the fire: then as it warms it will more and more diſtend the bladder, and at laſt burſt it, if the heat be continued, and increaſed high enough. But if the bladder be removed from the fire, as it cools it will contract again, as before. Indeed it was upon this principle that the firſt air-balloons were made by Montgolfier: for, by heating the air within them, by a fire underneath, the hot air diſtends them to a ſize which occupies a ſpace in the atmoſphere, whoſe weight of common air exceeds that of the balloon.

2. Alſo, if a cup or glaſs, with a little air in it, be inverted into a veſſel of water; and the whole be heated over the fire, or otherwiſe; the air in the top will expand till it fill the glaſs, and expel the water out of it; and part of the air itſelf will follow, by continuing or increaſing the heat.

Many other experiments to the ſame effect might be adduced.

The expanſion of air, though expoſed to the ſame degree of heat, is not the ſame in experiments made at different times; owing to the difference of denſity, coldneſs, humidity, &c. The expanſive force of hot ſteam may exceed the force of gunpowder more than 30 times, and indeed is irreſiſtible when the force is intenſe. Hence it follows, that when air is much impregnated with water it will poſſeſs an expanſive power by heat, much greater than that of pure air. Whether the degree of expanſion in pure air be proportional to that of the heat by which it is produced, is not known; but it is manifeſt that the variation of ſpace occupied by a portion of air expoſed to different degrees of heat may be ſufficient to convey a tolerable idea of the actual quantity of heat. Upon this principle, therefore, have been conſtructed air thermometers, to exhibit ſmall variations of heat.

There is no general rule for the degree of expanſion to which different bodies are ſubject by being expoſed to the ſame degree of heat. Taking an average, however, it has been found, by experiment, that for each degree of heat meaſured by Fahrenheit's thermometer, mercury, water, and air, expand by the following parts of their own bulk, viz.

Mercury the	9600th	part of its own bulk.
Water . . .	6666th	
Air	435th	

In mercury the correſponding expanſions for 1° gradually diminiſh, being expreſſed by ·0001177 at 2° of the thermometer, and by ·0000783 at 212°: but at 12° the expanſion correſponding to a degree of variation in heat is ·0001160, and at 102° it is ·0001003, ſo that between theſe limits the variation in the meaſure of expanſion is very trifling. Taking into the eſtimate the changes in the expanſion, &c. the ſpecific gravities of theſe fluids at different temperatures have been ſtated as below.

Spec. grav. of air -	1	when the barom. is at 29·27, and the thermom. at 53°.
water -	836	
mercury	11365	
Or thus, air - - - - -	1	when the barom. is 29·27, and the thermom. at 55°.
water - - - -	832	
mercury -	11315	
Or thus, air - - - - - -	1	when the barom. is 29·5, and the thermom. is 55°, which are their mean heights in this country.
water - -	826	
mercury -	11227	
Or thus, air - - -	1·201 or $1\frac{1}{5}$	in the laſt circumſtances.
water - -	1000	
mercury	13592	
Or thus, air - - - - - -	1·222	nearly when the barom. is - - - - 30, and thermometer - - - - 55.
or - -	$1\frac{2}{9}$	
water - - -	1000	
mercury -	13600	

On this ſubject the ſtudent may advantageouſly conſult General Roy's paper in the *Phil. Tranſ.* vol. 67. Alſo Sir Geo. Shuckburgh's and M. De Luc's papers in the ſame volume.

CHAPTER I.

Equilibrium of Elaſtic Fluids.

491. **Prop.** *To inveſtigate equations of equilibrium for elaſtic fluids.*

This will be very eaſy, if we conſider that ſuch fluids muſt, from the nature of perfect elaſticity (art. 329.), occupy a ſmaller ſpace in proportion as the forces which compreſs them are greater, and reſtore themſelves to their primitive volumes, when the action of the compreſſing forces ceaſe. Let, then, P be a preſſure exerted upon a quantity M of the fluid, whoſe denſity is D; p another preſſure, m the maſs or volume the fluid takes in conſequence of this preſſure, and d the denſity of this maſs; ſo ſhall we have theſe equations:

$$PM = p\,m,\ MD = m\,d,\ \text{and } Pd = p\,D \quad . \quad . \quad . \quad . \quad (\text{I.})$$

When $P = p$, then $M = m$, as is obvious.

Theſe values only give the preſſures exerted upon a unit of ſurface; but if we drop the conſideration of gravity, or any other force which may cauſe a variation of denſity in the different parts of the fluid, we may then reaſon from the principles of hydroſtatics. This granted, the preſſure p exerted upon any ſurface denoted by a will be

$$p = \frac{a\,d}{D}\,P,\ \text{or } p = \frac{a\,M}{m}\,P \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{II.})$$

Preſſures being commonly valued, as we have often ſeen, by weights, we may repreſent that which is exerted upon a unit of ſurface by the weight of a priſm of the ſame fluid whoſe height will be given. Let H be the height correſponding to P, and h to p; then we have HD for the maſs of this priſm, and HDg its weight (g denoting the force of gravity): hence $P = HDg$; and in like manner $p = h\,d\,g$. Subſtituting theſe for P and p in the equations marked (I.) above, we have $h\,D = H\,d$, and $h\,m = HM$. Whence we learn that the property of non-elaſtic fluids, demonſtrated art. 388, obtains likewiſe with regard to elaſtic fluids.

Inſtead of taking the ſame fluid, we may employ another whoſe denſity is δ; and, by proceeding in a ſimilar manner, get the following general equation for the preſſure,

$$p = \frac{M}{m} a g \delta H = \frac{d}{D} a g \delta H \quad . \; . \; . \; . \; . \; . \; . \quad \text{(III.)}$$

Many other equations might be deduced with equal facility; but thoſe here given are among the moſt uſeful.

492. PROP. *If an elaſtic fluid be quieſcent, and compoſed of particles equally repulſive at equal diſtances, and at unequal diſtances, repelling each other according to any law of the diſtance, its denſity will be uniform.*

For if the diſtances of any two particles from an intermediate particle be unequal, their repulſive forces muſt be unequal, and, of conſequence, motion muſt enſue; which is contrary to the hypotheſis: therefore the fluid muſt have its particles at equal diſtances, or be uniformly denſe.

COR. If any portion of an elaſtic fluid be uniformly denſe, and equally compreſſed on all ſides, it muſt be quieſcent.

493. PROP. *If the component particles of a uniform cubical maſs of a fluid repel each other, with forces varying according to any inverſe or direct ratio of their diſtances (leſs than the direct duplicate) the fluid will be elaſtic.*

For the whole repulſive force of any ſurface of the fluid is as the number of particles in that ſurface, and the force of each, or as the number in L the length of that ſurface, into the number in B the breadth, into the force of each particle; or, ſubſtituting R for the whole repulſive force of the ſurface, I for the interval or diſtance between two contiguous particles, and F for the force with which they repel each other, R will vary as $\frac{F}{I^2}$. Hence, if F vary in any inverſe ratio, or any direct ratio leſs than the duplicate ratio of I, R will vary in ſome inverſe ratio of I, which is a neceſſary condition of elaſtic fluids.

494. PROP. *If the particles of an elaſtic fluid repel each other with forces varying inverſely, as the* nth *power of their diſtances, that is, as* $\frac{1}{I^n}$, *and the compreſſing force* C *upon any ſurface be equal to its whole repulſive force* R, *then will* C *vary as that power of the denſity* D *whoſe exponent is* $\frac{n+2}{3}$.

For, let a portion of the fluid be contained in a given cubic ſpace, one of whoſe faces is the rectangle of L × B, the compreſſing force being applied to that ſurface. Now, the number of particles in the given ſquare ſurface is as $\frac{1}{I^2}$; and, by hypotheſis, the force F, with which two particles repel each other, is as $\frac{1}{I^n}$; therefore the elaſtic force of the fluid, and of conſequence the compreſſive force C, is as $\frac{1}{I^n} \times \frac{1}{I^2}$ or as $\frac{1}{I^{n+2}}$. But the denſity of the fluid contained in the given cubical ſpace is inverſely as the cube of the diſtance between the centres of the particles; that is, $D \propto \frac{1}{I^3}$, and $I \propto \frac{1}{D^{\frac{1}{3}}}$: whence, by ſubſtituting $D^{-\frac{1}{3}}$ for I, in the expreſſion $C \propto \frac{1}{I^{n+2}}$, we have $C \propto D^{\frac{n+2}{3}}$.

Cor. 1. Converſely, if $D^{\frac{n+2}{3}}$ vary as C, the repulſive force of each particle, or F, muſt vary as $\frac{1}{I^n}$.

For the quantity of matter being given, $D \propto \frac{1}{L^3} \propto \frac{1}{I^3}$ and $D^{\frac{n+2}{3}}$ will vary as $\frac{1}{I^{n+2}}$: but F varies as C divided by the number of particles in L^2, or as $C \times I^2$, or $\propto D^{\frac{n+2}{3}} \times I^2$, or $\propto I^2 \div I^{n+2}$ or $\propto \frac{1}{I^n}$.

Cor. 2. Hence again we ſee, ſince $n+2$ muſt be always poſitive to make C poſitive, that n muſt be either ſome whole poſitive number, or a negative number leſs than 2, in order to conſtitute a fluid of particles which repel each other.

Cor. 3. If water be ſuppoſed compreſſible in a very ſmall degree (art. 331.), the particles muſt be kept at a diſtance by ſome repulſive force, while D remains nearly conſtant. Now, ſince $C \propto D^{\frac{n+2}{3}}$, we ſhall have $D \propto C^{\frac{3}{n+2}}$; in which, that $C^{\frac{3}{n+2}}$ may be nearly invariable, n muſt be a very great number: *hence, according to this hypotheſis, the repulſive force of the particles of water varies inverſely in a very high power of their diſtances.*

Cor. 4. When the denſity of the fluid varies as the force which compreſſes it, or $D \propto C$, the expreſſion $C \propto D^{\frac{n+2}{3}}$ becomes $C \propto D^{\frac{1+2}{3}}$ and $n=1$; whence $F \propto \frac{1}{I^n}$ becomes $F \propto \frac{1}{I}$, or the force of each particle is inverſely as the interval between two contiguous particles.

COR. 5. Hence, becauſe the denſity of the air is nearly proportional to the force which compreſſes it (art. 489.), its conſtituent particles muſt repel one another with forces varying inverſely as their diſtances.

COR. 6. The denſity of the air varying as the compreſſing force, and that perpetually decreaſing in aſcending the atmoſphere, the denſity and elaſticity of the air alſo perpetually decreaſe.

495. DEF. A *homogeneous atmoſphere* is an atmoſphere ſuppoſed to be of the ſame weight as that which actually ſurrounds the earth; its denſity being uniform, and every-where equal to the denſity of the air at the earth's ſurface.

496. PROP. *To find the altitude of a homogeneous atmoſphere.*

Let H be the height of the homogeneous atmoſphere, its uniform denſity being D, the ſame as the denſity of the air preſſing upon the mercury in the baſin of the barometer; h the height of the mercury in the barometer tube, and d the denſity of that fluid: then (art. 491.) we have $M=m$, or $HD=h\,d$,

$$\text{whence } H=\frac{h\,d}{D}.$$

Now it appears from article 490, that when the denſities of air and mercury D and d are as $1\frac{2}{9}$ and 13600, the height h of the mercury in the barometer is $2\frac{1}{2}$ feet.

$$\text{Hence } H=\frac{h\,d}{D}=\frac{13600\times 2\frac{1}{2}}{1\frac{2}{9}}=27818 \text{ feet}, =5{\cdot}268 \text{ miles.}$$

So that the height of the homogeneous atmoſphere is rather more than $5\frac{1}{4}$ miles.

COR. *If it were not for the changes of temperature, the height* H *of the homogeneous atmoſphere would be invariable, whatever might be the height of the mercury in the barometer.*

For if d be conſtant, becauſe the ſpecific gravity of air varies as D its denſity, and this again as h, the height of the mercury in the tube, it follows that $\frac{h}{D}$ is invariable, and conſequently $H=\frac{h\,d}{D}$ is conſtant likewiſe.

Another method of aſcertaining the height of the homogeneous atmoſphere is given in the ſcholium at the end of the next propoſition.

497. PROP. *Suppoſing the force of gravity to vary as the nth*

power of the diſtance from the centre of the earth, and the compreſſive force to vary as the denſity, to find the relation between the denſity of the air and the altitude above the ſurface of the earth.

Let x repreſent the variable diſtance from the ſurface of the earth, the radius of the earth being unity, d the denſity of the air at the diſtance x, and H the height of the homogeneous atmoſphere. Now, ſince by hypotheſis the compreſſing force varies as the denſity, the fluxion of the former will vary as the fluxion of the latter; while, at any diſtance, x the fluxion of the compreſſing force muſt vary as the force of gravity, the denſity, and the fluxion of the altitude, conjointly; ſo that the fluxion of the compreſſing force will be to that of the denſity in the conſtant ratio of $x^n d \dot{x}$ to $-\dot{d}$, the latter fluxion having the negative ſign, becauſe the denſity decreaſes while the altitude increaſes. Conſequently, ſince by the definition of a homogeneous atmoſphere H will repreſent the compreſſing force at the ſurface of the earth, we have $H : 1 :: x^n d \dot{x} : -\dot{d}$, whence

$$x^n \dot{x} = -H \times \frac{\dot{d}}{d}, \text{ and } \frac{x^{n+1}}{n+1} = -H \,.\, \text{hyp. log. } d + C.$$

Now to correct the fluent we muſt conſider that, when $x = 1$, $d = 1$; whence we find $C = \frac{1}{n+1}$ for the value of the conſtant quantity: and the correct fluent is $\frac{x^{n+1}}{n+1} = \frac{1}{n+1} - H$. hyp. log. d. Hence

$$\frac{1-x}{n+1} = H \,.\, \text{hyp. log. } d.$$

which is the general equation expreſſing the relation between the altitude and the denſity.

Cor. 1. When the force of gravity varies inverſely as the ſquare of the diſtance, $n = -2$, and $\frac{1-x^{n+1}}{n+1} = H$. hyp. log. d, becomes $\frac{1}{x} - 1 = H$. hyp. log. d. So that if x increaſe in harmonic progreſſion, $\frac{1}{x}$ will decreaſe in arithmetic progreſſion, and conſequently hyp. log. d will decreaſe in arithmetic progreſſion.

Cor. 2. If the force of gravity be ſuppoſed conſtant $n = 0$, and $1 - x = H$. hyp. log. d. Conſequently, if x increaſe in arithmetic progreſſion, ſince $1 - x$ will then decreaſe in arithmetic progreſſion, the hyp. log. of d will decreaſe in arithmetic progreſſion.

COR. 3. Since the hyperbolic logarithms are to the common logarithms in a conſtant ratio, viz. that of 1 to ·43429448, &c. it follows that when x increaſes in arithmetic progreſſion the common logarithms of the denſities will decreaſe in arithmetic progreſſion, and the denſities themſelves in geometrical progreſſion, on the ſuppoſition of equal gravity.

COR. 4. Hence, retaining the ſame hypotheſis, different altitudes above the earth's ſurface will vary as the negative logarithms of the denſities or weights of air at thoſe altitudes. So that if D and d denote the denſities at the heights H and h, ſince $H \propto -\log. D$, and $h \propto -\log. d$, the difference $H-h \propto -\log. D - (-\log. d) \propto \log. D - \log. d$, or $\propto \log. \frac{D}{d}$. And if $H=0$, or $D=$ the denſity at the earth's ſurface, then any altitude above the ſurface is as the log. of $\frac{D}{d}$. Or generally the log. of $\frac{D}{d}$ varies as the altitude of the one place above the other, whether the lower place be at the ſurface of the earth or not: and upon this property is founded the method of finding the heights of mountains, &c. by the barometer.

SCHOLIUM.

498. The properties compriſed in the preceding corollaries are ſometimes proved by means of the logarithmic curve; and as a deſcription of the relation which obtains between the ordinates of that curve and the denſity of the air will furniſh an agreeable illuſtration of what is done above, ſuch deſcription is ſubjoined.

Let ARQ (fig. 3. pl. XVIII.) repreſent a ſection of a terreſtrial hemiſphere by a plane through its centre O, m OAM a vertical line, AE a horizontal line through A, a point on the earth's ſurface. Let this line AE repreſent the denſity of the air at A; and let DH parallel to AE be taken to AE, as the denſity at D to the denſity at A: then it is manifeſt that if a logiſtic or logarithmic curve EHN be drawn, having AM for its axis, and paſſing through the points E and H, the denſity of the air at any other point C in the vertical AM will be repreſented by CG, the ordinate to the curve in that point: for it is a known property of this curve, that if portions AB, AC, AD, of its axis be taken in arithmetical progreſſion, the ordinates AE, BF, CG, will be in geometrical progreſſion.

Another property of this curve is, that if EK or HS touch the curve in E or H the ſubtangent AK or DS is a conſtant quantity, being the modulus of the ſyſtem of logarithms which the particular curve repreſents. And a third property is, that the infinitely extended area AENM is equal to the rectangle AELK of the ordinate at A and the ſubtangent; and, in like manner, the area MDHN above DH is equal to the rectangle SD × DH or KA × DH; ſo that the area lying above or beyond any ordinate is proportional to that ordinate. (See Emerſon on Curve Lines, book I. § 5. Hutton's Dictionary, art. Logarithmic *Curve*, or the Treatiſe on Curve Lines at the end of Robertſon's Conic Sections.)

Now theſe geometrical properties of this curve are analogous to the chief circumſtances in the varying denſity of the atmoſphere, *on the ſuppoſition of equal gravity*. The area MBFN, for inſtance, repreſents the whole quantity of the elaſtic fluid which is above B: for BF is the denſity at B, and BC is the thickneſs of the ſtratum between B and C; whence BFGC will be as the quantity of air in that ſtratum: in like manner CGHD will be as the quantity of aerial matter in the ſtratum whoſe thickneſs is CD; and the ſame of their ſums, or of MBFN, the whole area. Alſo, as each ordinate is proportional to the area above it, ſo each denſity, and the quantity of air in each ſtratum, is proportional to the quantity of air above it. And again, as the whole area AENM is equal to the rectangle AELK, ſo the whole air of variable denſity above A might be contained in a column KA of the ſame baſe, if, inſtead of being compreſſed by its own weight, it were without weight, and compreſſed by an external force equal to the preſſure of the air at the ſurface of the earth. In this caſe it would be of the uniform denſity AE which it has at the ſurface of the earth, and would conſtitute the *homogeneous atmoſphere* (art. 495.)

Hence we learn that the height of the homogeneous atmoſphere is the ſubtangent of the curve whoſe ordinates are as the denſities of the air at different heights, on the ſuppoſition of equal gravity. This curve is generally called the *atmoſpherical logarithmic;* and its ſubtangent, or the height of the homogeneous atmoſphere, is the modulus of the ſyſtem of logarithms to which the curve correſponds.

We have already (art. 496.) ſhewn how to find the height of the homogeneous atmoſphere by a ſingle obſervation upon the barometer: but the ſame thing may be effected, perhaps, more accurately, in a different manner. When the mercury and the air are of the temperature 32° of Fahrenheit's thermometer, and the barometer on the ſea-ſhore ſtands at 30 inches, if we take it to a place 936 feet higher it will fall to 29 inches. Now in all logarithmic curves having equal ordinates the portions of the axes intercepted between the correſponding pairs of ordinates are proportional to the ſubtangents—(Emerſon on Log. Curve, prop. viii. cor. 3.); and the ſubtangent of the curve belonging to our common tables is ·43429448, and the difference of the logarithms of 30 and 29 (which is the portion of the axis intercepted between the ordinates 30 and 29) is ·0147233; whence we have ·0147233 : ·43429448 : : 935 : 27600 feet, or nearly $5\frac{1}{4}$ miles. This determination of the height of the homogeneous atmoſphere does not quite correſpond with the former; but a trifling error in meaſuring the altitudes of the mercury in the barometer at the two ſtations would cauſe all the difference.

499. All that has gone before in this ſcholium proceeds on the ſuppoſition of *equal* gravity, or the hypotheſis of COR. 2.; whereas we know that the weight of a particle of air decreaſes as the ſquare of its diſtance from the earth's centre increaſes, conformably to COR. 1. Hence we ſee that in order that a ſuperior ſtratum may produce an equal preſſure at the ſurface of the earth it muſt be denſer, becauſe a particle of it gravitates leſs. The denſity, therefore, at equal elevations, muſt be greater than on the ſuppoſition of equal gravity, and the law of diminution of denſity muſt be different.

$$\text{Make } OD : OA :: OA : Od;$$
$$OC : OA :: OA : Oc;$$
$$OB : OA :: OA : Ob, \text{ \&c.};$$

ſo that Od, Oc, Ob, OA, may be reciprocals to OD, OC, OB, OA; and through the points A, b, c, d, draw the perpendiculars AE, bf, cg, dh, making them proportional to the denſities in A, B, C, D; and ſuppoſe CD to be exceedingly ſmall, ſo that the denſity may be ſuppoſed uniform through the whole ſtratum. Thus we have

$$OD \times Od = OA^2, = OC \times Oc$$
$$\text{and } Oc : Od :: OD : OC;$$
$$\text{and } Oc : Oc - OD :: OD : OD - OC,$$
$$\text{or } Oc : cd :: OD : DC;$$
$$\text{and } cd ; CD :: Oc : OD;$$

or, becauſe OC and OD are ultimately in the ratio of equality, we have

$$cd : CD :: Oc : OC :: OA^2 : OC^2,$$

and $cd = CD \times \frac{OA^2}{OC^2}$, alſo $cd \times cg = CD \times cg \times \frac{OA^2}{OC^2}$. But $CD \times cg \times \frac{OA^2}{OC^2}$ is as the preſſure at C ariſing from the abſolute weight of the ſtratum CD: for this weight is as the bulk, as the denſity, and as the gravitation of each particle jointly; and CD expreſſes the bulk, cg the denſity, and $\frac{OA^2}{OC^2}$ the gravitation of each particle. Therefore, $cd \times cg$ is as the preſſure on C ariſing from the weight of the ſtratum DC. But $cd \times cg$ is evidently the element of the curvilineal area A*mn*E, formed by the curve E*fghn* and the ordinates AE, *bf*, *cg*, &c. *mn*. Hence the ſum of all the elements, ſuch as *cdhg*, that is, the area *cmng* below *cg*, will be as the *whole* preſſure on C, ariſing from the gravitation of all the air above it; but, by the nature of air, this whole preſſure is as the denſity which it produces, that is, as *cg*. Therefore the curve E*gh* is of ſuch a nature that the area lying below or beyond any ordinate *cg* is proportional to that ordinate. This is the property of the logarithmic curve, and E*gn* is a logarithmic curve.

Moreover, this curve is equal to EGN. For, let B continually approach to A, and ultimately coincide with it. It is evident that the ultimate ratio of BA to A*b*, and of BF to *bf*, is that of equality; and if EFK, E*fk*, be drawn, they will contain equal angles with the ordinate AE, and will cut off equal ſubtangents AK, A*k*. The curves EGN, E*gn*, are therefore equal, but in oppoſite poſitions.

Laſtly, if OA, O*b*, O*c*, O*d*, &c. be taken in arithmetical progreſſion decreaſing, their reciprocals OA, OB, OC, OD, &c. will be in harmonical progreſſion increaſing, as is well known: but, from the nature of the logarithmic curve, when OA, O*b*, O*c*, O*d*, &c. are in arithmetical progreſſion, the ordinates AE, *bf*, *cg*, *dh*, &c. are in geometrical progreſſion. Therefore when OA, OB, OC, OD, &c. are in harmonical progreſſion, the denſities of the air at A, B, C, D, &c. are in geometrical progreſſion; and thus may the denſity of the air at all elevations be diſcovered. Thus to find the denſity of the air at K, the top of the homogeneous atmoſphere, make OK : OA :: OA : OL, and draw the ordinate LT, ſo ſhall LT expreſs the denſity at K.

Dr. Halley was the firſt who obſerved the relation between the denſity of the air and the ordinates of the logarithmic curve, or common logarithms. This he did on the ſuppoſition of equal gravity; and his diſcovery is acknowledged by Sir Iſaac Newton in *Princip.* ii. *prop.* 22. *ſchol.* Halley's diſſertation on the ſubject is in No. 181 of the Phil. Tranſ. Newton, with his uſual ſagacity, extended the ſame relation to the true ſtate of the caſe, where gravity is as the ſquare of the diſtance inverſely; and ſhowed that when the diſtances from the earth's centre are in harmonic progreſſion the denſities are in geometric progreſſion. He ſhews indeed, in general, what progreſſion of the diſtance, on any ſuppoſition of gravity, will produce a geometrical progreſſion of the denſities, ſo as to obtain a ſet of lines OA, O*b*, O*c*, O*d*, &c. which will be logarithms of the denſities. The ſubject was afterwards treated in a more familiar manner by Cotes in his *Hydroſt. Lect.* and in his *Harmonia Menſurarum*; alſo by Brooke Taylor, *Method. Increment.*; Wolf, in his *Aerometria*; Herman, in his *Phoronomia*, &c.; and, on account of the ſimplicity and elegance of this method, it now obtains a place in almoſt every treatiſe on pneumatics.

500. Prop. *To determine the actual denſity of the atmoſpheric air at any altitude above the earth's ſurface.*

This may be done with the aid of the equation in cor. i. art. 497. but with greater facility by means of the atmoſpherical logarithmic, thus. By the property in Emerſon's Treatiſe, referred to in art. 498. ſay, as the height of the homogeneous atmoſphere is to the modulus of Briggs's ſyſtem, ſo as the given altitude in feet to a fourth number, which in the common tables is the logarithm of the ratio of the denſity of the air at the earth's ſurface to its denſity expreſſed by unity at the propoſed height, on the ſuppoſition of equal gravity. But, if we attend to the variation of gravity, the proceſs will be this. Suppoſe C to be the point at which the denſity is required; make, as directed in the laſt article, OC : OA : : OA : O *c*, or OC : OA : : AC : A *c*, and A *c* thus obtained will be the height to which the denſity is to be calculated by the preceding analogy, on the hypotheſis of equal gravity.

Let us take for an example the height of 7 miles, and conceive the radius of the earth to be 4000 miles. Then ſhall we have OC (=4007) : OA (=4000) : : AC (=7) : A *c* = $\frac{4000 \times 7}{4007}$ =6·98777 miles =36895½ feet. Wherefore, taking 27600 feet for the height of the homogeneous atmoſphere, we ſhall have

$$27600 : \cdot 43429448 : : 36895\tfrac{1}{2} : \cdot 5806957,$$

which is the common logarithm of 3·80799, or 3·808 nearly. Conſequently the denſity of the air at the earth's ſurface is to its denſity at the altitude of 7 miles as 3⅘ to 1 nearly, allowing for the diminution of the force of gravity.

This reſult agrees nearly with experiments. Thus Mr. Cotes inferred from the French experiment at the Puy de Dome, that at the altitude of 7 miles the air was rather more than 4 times rarer than at the ſurface of the earth: but from the experiments of Mr. Caſwell, at Snowden, he concluded that at the ſame altitude of 7 miles the air was not quite 4 times rarer than at the ſurface. And Sir Iſaac Newton, in the laſt edition of his Optics, ſtates it as 4 times rarer at the height of 7½ miles: which, properly reduced, gives 3·86 for the comparative rarity at 7 miles.

501. Returning to the hypotheſis of equal gravity, we may readily find an equation for the altitude, which ſhall include the changes in temperature. Thus, let D and d repreſent the denſity of the air at any two places, meaſured by the column of mercury in the barometer: then, ſince the difference of altitude a (art. 497. cor. 4.) is always as log. $\frac{D}{d}$, aſſume h, ſo that $a = h \times \text{log.} \frac{D}{d}$, where h will be of one conſtant value for all altitudes: and to determine that value let a caſe be taken in which we know the altitude a correſponding to a known denſity d; as, for inſtance, take $a = 1$ foot, or 1 inch, or ſome ſuch ſmall altitude; then, becauſe the denſity D may be meaſured by the preſſure of the atmoſphere, or the uniform column of 27600 feet, when the temperature is 55°; therefore 27600 feet will denote the denſity D at the lower place, and 27599 the leſs denſity d at 1 foot above it; conſequently $1 = h \times \text{log.} \frac{27600}{27599}$; which, by the nature of logarithms, is nearly $= h \times \frac{.43429448}{27600} = \frac{h}{63551}$ nearly; and hence $h = 63551$ feet; which gives, for any altitude in general, this theorem, viz. $a = 63551 \times \text{log.} \frac{D}{d}$, or $= 63551 \times \text{log.} \frac{M}{m}$ feet, or $10592 \times \text{log.} \frac{M}{m}$ fathoms; where M is the column of mercury which is equal to the preſſure or weight of the atmoſphere at the bottom, and m that at the top of the altitude a; and where M and m may be taken in any meaſure, either feet or inches, &c.

This formula is adapted to the mean temperature of the air 55° of Fahrenheit's thermometer: but it has been found by the experiments of Sir Geo. Shuckburgh and Gen. Roy, that for every degree of the thermometer different from 55° the altitude a will vary by its 435th part: hence if we would change the factor h from 10592 to 10000, becauſe the difference 592 is the 18th part of the whole factor 10592, and becauſe 18 is nearly the 24th part of 435; therefore the change of temperature anſwering to the change of the factor h is 24°, which reduces the 55° to 31°. So that $a = 10000 \times \text{log.} \frac{M}{m}$ fathoms is the eaſieſt expreſſion for the altitude, and anſwers to the temperature of 31°, or very nearly the freezing point: and for every degree above that the reſult muſt be increaſed by ſo many times its 435th part, and diminiſhed when below it.

This method was firſt given by Dr. Hutton, under the article *Atmoſphere*, in his Mathematical Dictionary.

A correct and extenſive table of the comparative rarity and expanſion, denſity and compreſſion, of air at different altitudes, aſſuming the homogeneous atmoſphere at 29725 feet, and allowing for the variation of gravity, is inſerted at p. 253 of Helſham's Lectures.

CHAPTER II.

On the Admeaſurement of Altitudes by the Barometer and Thermometer.

502. M. Pascal and Deſcartes hinted that if a barometer were carried to a higher ſituation the quickſilver would ſink in the tube; and the fact was verified by M. Perier. But the admeaſurement of altitudes by means of the barometer was firſt diſtinctly ſuggeſted by Dr. Halley, in a paper in No. 181 of the Philoſophical Tranſactions, to which we have before referred (art. 499.). He there ſtates that, according to experiments ſhewn before the Royal Society, when the barometer ſtood at 30 inches, and in a mean ſtate of heat and cold, the ſpecific gravity of air was to that of water as 1 to 800; and then proceeds as follows: "By the like trials the weight of mercury to water is as $13\frac{1}{2}$ to 1, or very near it; ſo that the weight of mercury to air is as 10800 to 1, and a cylinder of air of 10800 inches or 900 feet is equal to an inch of mercury; and were the air of an equal denſity like water, the whole atmoſphere would be no more than $5\frac{1}{10}$ miles high; and in the aſcent of every 900 feet the barometer would ſink an inch. But the expanſion of the air increaſing in the ſame proportion as the incumbent weight of the atmoſphere decreaſes, that is, as the mercury in the barometer ſinks, the upper parts of the air are much more rarefied than the lower, and each ſpace anſwering to an inch of quickſilver grows ſtill larger, ſo that the atmoſphere muſt be extended to a much greater height. Now, on theſe principles, to determine the height of the mercury at any aſſigned height in the air, and *è contra*, having the height of the mercury given to find the height of the place where the baro-

meter ſtands, are problems not more difficult than curious." He then points out a method of ſolution, and gives tables of the heights of mercury at different altitudes, and *vice verſa*; which need not be copied in this place, as they depend upon erroneous eſtimates of the ſpecific gravities of mercury and air. He alſo mentions the condenſation and rarefaction by heat and cold, and the effect of aqueous and other vapours; but was of opinion that theſe would compenſate each other.

503. It appears, then, that Dr. Halley's method cannot apply to every caſe, ſince it depends upon the ſuppoſition of the relative denſity of air and mercury being only affected by *a change of preſſure*. But ſince all bodies are expanded by heat, and as they are not all *equally* expanded by it, it follows that a change of temperature may change the relative gravity of mercury and air, even although both undergo the ſame change of temperature: and ſince the air may be warmed or cooled when the mercury is not, or may change its temperature independent of it, ſtill greater variations of ſpecific gravity may be expected. Hence many corrections are required to adjuſt the barometrical method to the various attending circumſtances, and it was not till long after Dr. Halley's time that the method was turned to any real uſe. The chief improvements and adjuſtments are due to M. *de Luc*, who had favourable opportunities of making numerous obſervations among the high hills of Switzerland, and who availed himſelf of theſe opportunities in almoſt every variety of circumſtances. The reſult of his experiments and enquiries was publiſhed at Geneva in a treatiſe on the barometer and thermometer, and in a paper in the 57th volume of the Philoſophical Tranſ. Other curious and valuable papers on this ſubject by Dr. Maſkelyne, Dr. Horſley, Sir George Shuckburgh, and General Roy, are inſerted in the different volumes of theſe Tranſactions.

The moſt accurate experiments for aſcertaining the expanſion of mercury are thoſe of General Roy (vol. 67, Philoſophical Tranſactions). Theſe are contained in the following table; where the firſt column expreſſes the temperature by Fahrenheit's thermometer, the ſecond column expreſſes the bulk of the mercury, and the third column the expanſion of an inch of mercury for an increaſe of one degree in the adjoining temperatures.

TABLE I.

Temp.	Bulk of ☿.	Expans. for 1°
212°	30·5117	0·0000763
202	30·4888	0·0000787
192	30·4652	0·0000810
182	30·4409	0·0000833
172	30·4159	0·0000857
162	30·3902	0·0000880
152	30·3638	0·0000903
142	30·3367	0·0000923
132	30·3090	0·0000943
122	30·2807	0·0000963
112	30·2518	0·0000983
102	30·2223	0·0001003
92	30·1922	0·0001023
82	30·1615	0·0001043
72	30·1302	0·0001063
62	30·0984	0·0001077
52	30·0661	0·0001093
42	30·0333	0·0001110
32	30·0000	0·0001127
22	29·9662	0·0001143
12	29·9319	0·0001160
2	29·8971	0·0001177
0	29·8901	

The General alſo made experiments on the expanſion of the mercury in a real barometer: and in ſome rules for barometrical admeaſurement, it will be neceſſary to know the *logarithmic* difference to the expanſion for the mean temperature of the two barometers which are uſed. Theſe logarithmic differences are contained in the following table, which is carried as far as 112°, beyond which it is not probable that any obſervations will be made. The number for each temperature is the difference between the logarithms of 30 inches, of the temperature 32, and 30 inches expanded by that temperature.

TABLE II.

Temp.	Log. diff.	Dec. of Fath.	Ft.	In.
112°				
102	0·0000427	·427	2	7
92	0·0000436	·436	2	7
82	0·0000444	·444	2	8
72	0·0000453	·453	2	9
62	0·0000460	·460	2	9
52	0·0000468	·468	2	10
42	0·0000475	·475	2	10
32	0·0000482	·482	2	11
22	0·0000489	·489	2	11
12	0·0000497	·497	3	0
0	0·0000504	·504	3	0

General Roy compared a mercurial and an air thermometer, each of which was graduated arithmetically, that is, the units of the ſcales were equal bulks of mercury, and equal bulks (perhaps different from the former) of air. He found their progreſs as in the following table.

TABLE III.

Merc.	Diff.	Air.	Diff.
212	20	212·0	17·6
192	20	194·4	18·2
172	20	176·2	18·8
152	20	157·4	19·4
132	20	138·0	20·0
112	20	118·0	20·8
92	20	97·2	21·6
72	20	75·6	22·6
52	20	53·0	21·6
32	20	31·4	20·0
12		11·4	

The following table contains the expanſion of 1000 parts of air, nearly of the common denſity, by heating it from 0 to 212. The firſt column contains the height of the barometer; the ſecond contains this height augmented by the ſmall column of mercury in the tube of the manometer, and therefore expreſſes the denſity of the air examined; the third contains the total expanſion of 1000 parts; and the fourth contains the expanſion for 1°, ſuppoſing it uniform throughout.

Table IV.

Barom.	Density of Air examined.	Expansion of 1000 parts by 212°.	Expansion by 1°.
29·95	31·52	483·89	2·2825
30·07	30·77	482·10	2·2741
29·48	29·90	480·74	2·2676
29·90	30·73	485·86	2·2918
29·96	30·92	489·45	2·3087
29·90	30·55	476·04	2·2455
29·95	30·60	487·55	2·2998
30·07	30·60	482·80	2·2774
29·48	30·00	489·47	2·3087
Mean	30·62	484·21	2·2840

Hence it appears, that the mean expanſion of 1000 parts of air of the denſity 30·62 by one degree of Fahrenheit's thermometer is 2·284, or that 1000 becomes 1002·284.

If this expanſion be ſuppoſed to follow the ſame rate that was obſerved in the compariſon of the mercurial and air thermometer, we ſhall find that the expanſion of a thouſand parts of air for one degree of heat at the different intermediate temperatures will be as in the following table.

Table V.

Temp.	Total Expanſion.	Expanſion for 1°.
212	484·210	2·0099
192	444·011	2·0080
172	402·452	2·1475
152	359·503	2·2155
132	315·193	2·2840
112	269·513	2·3754
92	222·006	2·4211
82	197·795	2·5124
72	172·671	2·5581
62	147·090	2·6037
52	121·053	2·5124
42	95·929	2·4211
32	71·718	2·3297
22	48·421	2·2383
12	26·038	2·1698
0		

If we would have a mean expanſion for any particular range, as between 12° and 92°, which is the moſt likely to comprehend all the geodætical obſervations, we need only take the difference of the bulks 26·038 and 222·006 = 195·968, and divide this by the interval of temperature 80°, and we obtain 2·4496, or 2·45 for the mean expanſion for 1°, whence a complete table may readily be conſtructed.

504. Having thus ſtated the moſt material circumſtances which are to be taken into the account in this kind of admeaſurement, and given ſuch tables as will be uſeful in the computation, we ſhall proceed to exhibit the moſt approved precepts and rules.

In order that the obſervations may be carefully and properly made, the perſon who undertakes them ſhould be provided with two portable barometers of the beſt conſtruction (both filled with mercury of the ſame ſpecific gravity), on which, by means of a nonius properly adapted to the ſcale, he may read off the height of the mercurial columns to the 200th part of an inch: each barometer ſhould be fitted up with an attached thermometer, ſet in the wooden frame in the ſame manner as the barometer tube is. The ball of each thermometer had beſt be nearly of the ſame diameter as the barometer tube. Beſides theſe he muſt alſo be provided with two other thermometers, detached from the barometers. Of theſe barometers, one with its attached and detached thermometers is to be placed in the ſhade at the top of the eminence whoſe height is required, while the other remains below. Let them continue in their places at leaſt a ſufficient time for the detached thermometer to acquire the temperature of the air, that is to ſay, till the contained fluid is ſtationary. Then the obſerver on the eminence muſt note down the height of the mercurial column in the barometer, as well as the temperatures exhibited by the attached and detached thermometers; and at *the ſame time* the other obſerver muſt make like obſervations upon the inſtruments below. If, in this manner, two or three ſets of obſervations be taken at each ſtation after ſhort intervals of time, and the mean of the reſults furniſhed by theſe ſets reſpectively be taken as the true altitude (following one of the ſubſequent rules), the probability of error will be much diminiſhed.

In practice, however, it will often be neceſſary to deviate from the preceding directions, and ſuch deviations may be pretty well guarded againſt miſtake. In caſes where better inſtruments cannot be had, any well-made portable barometer ſo graduated as to ſhew the true change in the altitude of the mercury, may afford ſuch obſervations as ought not to be neglected. A ſmall error in the poſition of the point from whence the ſcale of inches commences, will not ſenſibly affect the reſult; for that reſult depends chiefly (art. 501.) upon $\log. \frac{M}{m}$ or $\log. M - \log. m$, which will not be much affected by a ſmall and equal change in M and *m*. Suppoſe, for example, M was 29·36 inches, and *m* 27·84 inches, then $\log. M - \log. m =$ 230869; but if M were 29·11, and *m* 27·59, each leſs by a quarter of an inch, we ſhould have $\log. M - \log. m = 232905$, which would not cauſe a difference in the reſult of more than 1 fathom in a hundred. And an error of $\frac{1}{4}$ of an inch in the poſition of the zero is far greater than may be reaſonably expected to occur. If an hour or more can be allowed for the mercury in the barometer to acquire the temperature of the ſurrounding air, which is ſhewn by the detached thermometer, then the attached thermometer may be diſpenſed with. A ſingle barometer may ſupply the place of two, if the obſervations can be made firſt at the baſe, then at the ſummit, and again at the baſe, in a moderate ſpace of time: and if the two ſets of obſervations at the baſe correſpond pretty nearly, it may be preſumed that the denſity of the air below has undergone no material change during the operations. The neceſſary obſervations being made, the altitude of the object at whoſe top and bottom the inſtruments were placed, may be aſcertained by one or other of the following rules.

I. *Dr. Robiſon's method.*

505. In this method no tables are required; it will be ſufficiently exact for moſt purpoſes, and is not difficult to remember. It was deduced from theſe conſiderations.

1. The height through which we muſt riſe in order to produce any fall of the mercury in the barometer, is inverſely proportional to the denſity of the air, that is, to the height of the mercury in the barometer.

2. When the barometer ſtands at 30 inches, and the air and quickſilver are of the temperature 32°, we muſt riſe through 87 feet to produce a depreſſion of $\frac{1}{10}$ of an inch.

3. But if the air be of a different temperature, this 87 feet muſt be increaſed or diminiſhed by about 0·21 of a foot, for every degree of difference of the temperature from 32°.

4. Every degree of difference of the temperatures of the mercury at the two ſtations makes a change of 2·833 feet, or 2 feet 10 inches in the elevation.

Hence the following rule:

I. Take the difference of the barometric heights in tenths of an inch. Call this D.

II. Multiply the difference d between 32° and the mean temperature of the air by ·21, and take the ſum or difference of this product and 87 feet. This is the height through which we muſt riſe to cauſe the barometer to fall from 30 inches to 29·9; and may be called h.

III. Let m be the mean between the two barometric heights;

Then $\frac{30\,D\,h}{m}$ is the approximated elevation very nearly.

IV. Multiply the difference δ of the mercurial temperatures by 2·833 feet, and add this product to the approximated elevation if the upper barometer has been the warmeſt, otherwiſe ſubtract it: then will the reſulting ſum or difference be the corrected elevation.

Or, this rule may be expreſſed by the following formula; where d is the difference between 32° and the mean temperature of the air, D is the difference of barometric heights in tenths of an inch, m is the mean barometric height, δ the difference between the mercurial temperatures, and E is the correct elevation. $E = \frac{30\,(87 \pm 0{\cdot}21\,d)\,D}{m} \pm \delta \times 2{\cdot}833.$

For an example, ſuppoſe that the mercury in the barometer at the lower ſtation was at 29·4 inches, its temperature 50° of Fahrenheit's thermometer, and the temperature of the air 45°: the height of the mercury at the upper ſtation 25·19 inches, its temperature 46°, and the temperature of the air 39°.

$$\begin{aligned} \text{Here } D &= 294 - 251{\cdot}9 = 42{\cdot}1 \\ h &= 87 + (10 \times {\cdot}21) = 89{\cdot}1 \\ m &= \tfrac{1}{2}\,(29{\cdot}4 + 25{\cdot}19) = 27{\cdot}295 \end{aligned}$$

$\frac{30\,D\,h}{m}$ = approximate elevation = . . .	4123·24
Correction for temp. of mercury 4 × 2·833 =	11·33
Corrected elevation in feet . . .	4111·91
Ditto in fathoms	685·32

II. *M. de Luc's method.*

506. I. Subtract the logarithm of the barometrical height at the upper ſtation from the logarithm of that at the lower,

and count the index and firſt four decimal figures of the remainder as fathoms, the reſt as a decimal fraction. This may be called the *elevation*.

II. Note the different temperatures of the mercury at the two ſtations, and the mean temperature. Multiply the logarithmic expanſion correſponding to this mean temperature (in table II. art. 503.) by the difference of the two temperatures, and ſubtract the product from the elevation if the barometer has been coldeſt at the upper ſtation, otherwiſe add it. This difference, or ſum, will be the *approximated elevation*.

III. Obſerve the difference of the temperatures of the air at the two ſtations by a detached thermometer, alſo the mean temperature, and its difference of temperature from 32°. Multiply this difference by the expanſion of air for the mean temperature, and multiply the approximated elevation by 1 ± this product, according as the air is above or below 32°. The laſt product is the correct elevation in fathoms and decimals.

Taking the ſame example as before, we have

☿al heights.	Temp. ☿.		Mean.	Temp. Air.	Mean.
29·4	50	4	48	45	42
25·19	46			39	

I.	Log. of 29.4		1·4683473
	Log. of 25·19		1·4012282
	Elevation in fathoms		671·191
II.	Expanſ. for 48° (tab. II.)	·473	
	Multiply by (50 − 46)	4	
			1·892
	Approximated elevation		669·299
III.	Expanſ. of air at 42°	00·00238	
	Multiply by (42 − 32)	10	
		0·0238	

Hence 669·299 × 1·0238 = 685·228, the correct elevation.

III. *Sir George Shuckburgh's method.*

507. I. Reduce the barometric heights to what they would be if they were of the temperature 32°.

II. The difference of the logarithms of the reduced barometrical heights will give the approximate elevation.

III. Correct the approximated elevation as before.

Same example.

I.	Mean expanſ. for 1° from tab. I. is 0·000111.	
	18° × 0·000111 × 29·4 =	0·059
	Subtract this from	29·4
	Reduced barometric height	29·341
	Expanſ. from tab. I. is 0·000111.	
	14° × 0·000111 × 25·19	0·039
	Subtract from	25·190
	Reduced barometric height	25·151
II.	Log. 29·341	1·4674749
	Log. 25·151	1·4005553
	Approximated elevation	669·196
III.	This multiplied by 1·0238 gives	685·125

Remark 1. If 0·000101 be ſuppoſed the mean expanſion of mercury for 1°, as Sir George Shuckburgh determines it, the reduction of the barometric heights will be had ſufficiently exact by multiplying the obſerved heights of the mercury by the difference of its temperatures from 32, and cutting off four more decimal places; thus $29{\cdot}4 \times \frac{18}{10000}$ gives for the reduced height 29·347, and $25{\cdot}19 \times \frac{14}{10000}$ gives 25·155, and the difference of their logarithms gives 669·4 fathoms for the approximated elevation, which differs from the one given above by no more than 15 inches.

Remark 2. If 0·0024 be taken for the expanſion of air for one degree, the correction for this expanſion will be had by multiplying the approximated elevation by 12, and this product by the ſum of the differences of the temperatures from 32°, counting that difference as negative when the temperature is below 32, and cutting off four places; thus $669{\cdot}196 \times 12 \times \overline{13+07} \times \frac{1}{10000} = 16{\cdot}061$, which added to 669.196 gives 685·257, differing from the former only 9 inches.

IV. *Dr. Hutton's method.*

508. I. Obſerve the height of the barometer at the bottom of any height or depth intended to be meaſured, with the temperature of the quickſilver by means of a thermometer attached to the barometer, and alſo the temperature of the air in the ſhade by a detached thermometer.

II. Let the ſame thing be done alſo at the top of the ſaid height or depth, and at the ſame time, or as near the ſame time as may be. And let thoſe altitudes of barometer be reduced to the ſame temperature, if it be thought neceſſary, by correcting either the one or the other; that is, augment the height of the mercury in the colder temperature, or diminiſh that in the warmer, by its $\frac{1}{9600}$ part for every degree of difference of the two. The altitudes of mercury ſo corrected being what we denoted by M and *m*, in art. 501.

III. Take the difference of the common logarithms of the two heights of the barometer, corrected as above if neceſſary, cutting off three figures next the right hand for decimals, the reſt being fathoms in whole numbers.

IV. Correct the number laſt found for the difference of temperature of the air, as follows: Take half the ſum of the two temperatures for the mean one; and for every degree which this differs from the temperature 31°, take ſo many times the $\frac{1}{435}$ part of the fathoms above found, and add them if the mean temperature be above 31°, but ſubtract them if the mean temperature be below 31°; and the ſum or difference will be the true altitude in fathoms; or, being multiplied by 6, it will be the altitude in feet. *Math. Dict.* art. *Barometer.*

Same Example.

Thermometers.		Barometers.
Detached.	Attached.	
45	50	29·4 lower.
39	46	25·19 upper.
Mean 42	Diff. 4	

As 9600 : 4 : : 29·4 : ·0123
corr. ·0123

Mean 42 $M=29{\cdot}3877$ . . . log. 4681656
Stand. 31 $m=25{\cdot}19$ 4012282

Diff. 11 As 435 : 11 : : 669·374 : 16·924
16·924

The altitude ſought is 686·298 fathoms.

509. Such are the moſt approved rules for the determination of altitudes by the barometer. By a compariſon of the four methods it will be ſeen that no two of the reſults differ more than a 600th part; ſo that in point of accuracy we know not which ſhould be preferred. In practice, perhaps, the firſt and fourth may be moſt expeditious; and, in general, an arithmetical mean between the reſults of theſe two will not be far from the truth.

As to the advantages of the barometric compared with the geometrical method of meaſuring elevations, we ſhall ſtate them chiefly in the language of Mr. Nicholſon, (Natural Philoſophy, vol. ii.). Firſt, the inſtruments are neither very expenſive, nor even difficult for an ingenious philoſopher to make in any country where he can procure quickſilver and glaſs tubes; but the geometrical method requires inſtruments of conſiderable price, which cannot at all be accurately conſtructed by the moſt ingenious perſon who is deſtitute of the tools, and unacquainted with the artifices neceſſary to render them correct. Secondly, the barometers require no other adjuſtment than to obſerve previouſly whether they agree, and, if they do not, to allow for their difference. The barometrical obſervations are likewiſe eaſily made; whereas, on the contrary, the previous adjuſtment and ſubſequent uſe of inſtruments for meaſuring angles require a degree of preciſion and ſkill not uſually obtained without practice. Thirdly, the error *of obſervation* in the barometrical method for all elevations is nearly a conſtant quantity, never amounting to ſo much as half a fathom for a miſtake of the 500th of an inch; but any error either in the meaſurement of lines or angles proportionally affects the reſult; ſo that the greater the elevation required to be meaſured, the larger the quantity of error. Fourthly, the barometrical obſervations require no particular circumſtances of advantages either in the figure or ſituation of the mountains, &c. to be meaſured, nothing more being required than that both ſtations be acceſſible. Theſe obſervations, and the computation, are performed after the ſame method in all caſes; but, in the geometrical method, if the horizontal diſtance of the two ſtations be conſiderable, or if there be not a convenient plain for meaſuring a fundamental baſe, the operation becomes very complicated, and the probability of error is multiplied.

After all, it muſt not be diſguiſed that the principles of the geometrical method are eſtabliſhed and ſure, and that an extreme degree of exactneſs may often be obtained in this way by good inſtruments in the hands of a ſkilful obſerver: whereas the modifications of the atmoſphere with regard to the effect which exhalations of various kinds, and the greater or leſs abundance of the electric matter, may have in expanding the air without changing its temperature, are not yet ſufficiently known to render the corrections altogether ſo perfect as might be wiſhed. Theſe remain to be aſcertained more accurately by future obſervations: mean while it ſhould be remembered, that the elevations determined by the barometer are moſt to be depended upon when the extreme temperatures of the column of air do not greatly differ, and when the air is cold and dry.

The various circumſtances affecting the denſity of the atmoſphere and likely to occaſion errors in barometrical admeaſurements, ſuch as the expanſion from various degrees of heat, the change in the relation between the denſity and the compreſſing force, the variation in the force of gravity, the ſolution of moiſture in the air, the diminution of the weight of mercury in the upper barometer, &c. have all been made the ſubjects of inveſtigation in an elaborate paper by Mr. Profeſſor Playfair, publiſhed in the Tranſactions of the Royal Society of Edinburgh, vol. I.

M. *Laplace* has propoſed a method of combining the obſervations of the barometer with geographical meaſures, to determine in a more deciſive manner the poſition of different places. This poſition, ſo far as it depends upon geographical admeaſurement, is ſhewn by the interſection of two rectangular co-ordinates; one of which exhibits the diſtance from the firſt meridian, or the longitude, and the other the diſtance from the equator, or the latitude. He ſuppoſes a third co-ordinate perdendicular to the two preceding ones, by which may be meaſured the vertical diſtance between the ſame point of interſection and the level of the ſea. Thus he takes, for France, the level at Breſt, where the mean height of the barometer is very nearly 76 centimetres, or about 29·92 of our ſtandard inches. Then making in each place a great number of barometrical obſervations during a year or two, the mean between all theſe obſervations gives the elevation of the place propoſed above the level of the ſea. It is recommended to chooſe in each reſpective tract of country, for the level to which all the obſervations are to be referred, the mean height of the neareſt river. Such a courſe of obſervations carefully made at different places with accurate barometers, might furniſh reſults very intereſting in the topography of different countries.

CHAPTER III.

On the Motion of Air when the Equilibrium of Preſſure is removed.

510. In the preſent chapter we ſhall conſider air (taken for a repreſentative of elaſtic fluids in general) as acted upon by gravity equally and in parallel directions; and in a ſeries of propoſitions enquire into the circumſtances of its paſſage from a veſſel into a vacuum, or from one veſſel to another, in which the air is of leſs denſity.

Prop. *To determine with what velocity air will ruſh into a void ſpace when impelled by its weight alone.*

This is manifeſtly analogous to the hydraulic problem of water flowing through an orifice in the bottom or ſide of a veſſel (art. 439.): and the manner of demonſtration there adopted will immediately apply to the preſent inſtance. For when the moving force, and the matter to be moved, vary in the ſame proportion, the velocity will continue the ſame; that is, ſince $F \propto BV$ (art. 216.) when $F \propto B$, V is conſtant. If, therefore, there be ſimilar veſſels of air, water, oil, or any other fluid, all of the height of a homogeneous atmoſphere (art. 495.), they will be diſcharged through equal and ſimilar holes with the ſame velocity: for, in whatever proportion the quantity of matter moving through the hole be varied by a change of denſity, the preſſure which forces it out, acting in circumſtances perfectly ſimilar, varies in the ſame proportion. Hence it follows, that *air ruſhes from the atmoſphere into a void with the velocity which a heavy body would acquire by falling from the top of a homogeneous atmoſphere.*

Let us take, then, for the height of the homogeneous atmoſphere $H=27818$ feet (art. 496.); and if we put $g=32\frac{1}{6}$ feet, as we have heretofore done, we ſhall have $V=\sqrt{2gH}=1339$ feet nearly $=8\sqrt{H}$ nearly, the velocity ſought: no regard being here paid to the velocity which the air acquires after its iſſuing into the previous void by its continual expanſion.

SCHOLIUM.

511. The firſt ſolution to this problem we recollect was given by Dr. Papin in the Phil. Tranſ. No. 184. See alſo the new abridgment, part x. p. 334. It is deduced from this propoſition, viz. that of different liquors under the ſame preſſure, thoſe ſpecifically lighter muſt acquire a greater celerity, and their different velocities will be to one another as the roots of the ſpecific gravities of the ſaid liquors. "If, therefore," ſays Dr. Papin, "we would know what is the velocity of air when driven by any degree of preſſure whatever, we need only find what would be the velocity of water under the ſame preſſure, and then take the ſquare roots of the ſpecific gravities of theſe two fluids, becauſe, as much as the ſquare root of the ſpecific gravity of water exceeds the ſquare root of the ſpecific gravity of the air, ſo much in proportion will the velocity of the air exceed that of the water. For example, when I would compute what would be the velocity of a bullet ſhot by the pneumatic engine, deſcribed in Phil. Tranſ. No. 179, I ſhould firſt compute what was the velocity of the air itſelf that drove the bullet: I therefore obſerve that on this occaſion the air ſuſtains a preſſure much about the ſame as that of water when its head is 32 feet high; now ſuch water would ſpout out with a ſufficient velocity to aſcend 32 feet perpendicular, and therefore it has the velocity of 45 feet in a ſecond. It remains, therefore, only to know the proportion of the gravity of air to that of water: this we have found not to be always the ſame; becauſe the height, the heat, and the moiſture of the atmoſphere, are variable; yet we may ſay in general, that the ratio of the ſpecific gravities of water and air is about 840 to 1. Taking, then, their ſquare roots, which are 29 and 1, we may conclude that the velocity of air muſt exceed that of water 29 times; and ſo multiplying 45, the velocity of water, by 29, we ſhall find that the velocity of the air, driven by the whole preſſure of the atmoſphere, is about 1305 feet in a ſecond."

This number, as Dr. Hutton remarks, is too ſmall; for the mean preſſure of the atmoſphere is now known to be about that of a column of water of $33\frac{3}{4}$ feet, and this will give for its velocity $46\frac{12}{25}$ feet, inſtead of 45; which, multiplied by the 29, gives nearly 1348 feet: a determination which differs from that in the preceding article by about its 150th part.

512. Prop. *To determine the velocity with which the air of the atmoſphere will ruſh into a ſpace containing rarer air.*

Whatever the denſity of the rarer air may be, its elaſticity, which varies as its denſity, will balance a proportional part of the preſſure of the atmoſphere; and it is the exceſs of this laſt only which is the moving force, the matter to be moved is the ſame as before. Let D, then, be the natural denſity of the air, and δ the denſity of the air contained in the veſſel into which it is ſuppoſed to run: let P be the preſſure of the atmoſphere, and therefore equal to the force which impels it into a void, and π the force with which this rarer air would run into a void. Then (art. 491.) we have $P\delta = \pi D$, and $\pi = \frac{P\delta}{D}$. Alſo the moving force in the preſent caſe is $P - \pi$, or $P - \frac{P\delta}{D}$. Laſtly, let V be the velocity of air of denſity D ruſhing into a void, and v the velocity with which it will ruſh into the rarified air of denſity δ. Now, in this as well as other fluids, the preſſures are as the ſquares of the velocities of efflux; therefore (art. 469. cor. 2.) $P : P - \frac{P\delta}{D} :: V^2 : v^2$, and by reduction we find $v = V \times \sqrt{1 - \frac{\delta}{D}}$; an expreſſion for the velocity ſought, not conſidering the reſiſtance which the air of denſity D will experience from the inertia of that in the veſſel, which it muſt diſplace in its motion.

COR. 1. Hence it appears that there will always be a current into the veſſel while δ is leſs than D.

COR. 2. Hence, alſo, we learn the gradual diminution of the velocity as the veſſel fills; for, becauſe δ gradually increaſes $1 - \frac{\delta}{D}$, and conſequently the value of v continually diminiſhes.

513. PROP. *To determine the time* t *in ſeconds, in which the air of the atmoſphere will flow into a given veſſel from its ſtate of vacuity, till the air in the veſſel has acquired any propoſed denſity* δ.

Let H be the height in feet due to the velocity V (art. 244.), s the ſolid content or capacity of the veſſel in cubic feet, and a the area or ſection of the aperture in ſquare feet, D repreſenting, as before, the natural denſity of the air. Now, ſince the quantity of air to fill the veſſel will depend upon its denſity and the capacity of the veſſel conjointly, we may expreſs it by Ds when the air is in its ordinary ſtate, and by δs when it has acquired the denſity δ. To find the rate at which the veſſel fills we muſt take the fluxion of the quantity δs, which will be $s\dot{\delta}$, becauſe s is invariable. The velocity of influx at the firſt inſtant is V or $\sqrt{2gH}$ (art. 510.), and when the air in the veſſel has acquired the denſity δ, that is, at the end of the time t, the velocity is $\sqrt{2gH} \times \sqrt{1 - \frac{\delta}{D}}$

or $\sqrt{2gH \cdot \frac{D-\delta}{D}}$. Hence the rate of influx, which may be meaſured by the indefinitely ſmall quantity of air which will enter during the time $\dot{t}$ with this velocity, will be denoted by $\sqrt{2gH} \times \sqrt{\frac{D-\delta}{D}} \times D a \dot{t} = a \dot{t} \sqrt{2gDH\,(D-\delta)}$. Making theſe two values of the rate of influx equal, we have $a \dot{t} \sqrt{2gDH\,(D-\delta)} = s\dot{\delta}$, or $\dot{t} = \frac{s}{\sqrt{2gDH}} \times \frac{\dot{\delta}}{\sqrt{D-\delta}}$. Taking the fluents, we have $t = \frac{s}{\frac{1}{2}a\sqrt{2gDH}} \times \sqrt{D-\delta} + C$.

To determine the conſtant quantity C we muſt recollect that when $t=0$, $\delta=0$, and $\sqrt{D-\delta} = \sqrt{D}$: hence $C = \sqrt{D}$; and the correct fluent is

$$t = \frac{s}{a\sqrt{\frac{1}{2}gDH}} \times (\sqrt{D} - \sqrt{D-\delta}).$$

Cor. When $\delta = D$ the motion ceaſes, in which caſe t becomes $\frac{s\sqrt{D}}{a\sqrt{\frac{1}{2}gDH}}$ or $\frac{s}{a\sqrt{\frac{1}{2}gH}}$, or $\frac{s}{4a\sqrt{H}}$ nearly, for the time of completely filling the veſſel.

514. To illuſtrate this by an example in numbers, let us ſuppoſe the capacity of the veſſel to be 8 cubic feet, or nearly a wine hogſhead, and that the orifice by which the air of the ordinary denſity (which we ſhall make = 1) enters, is an inch ſquare, or $\frac{1}{144}$ of a foot. Here $4\sqrt{H} = 4\sqrt{27818} = 668$, and $t = \frac{8}{\frac{1}{144}668} = \frac{1152}{668} = 1^{s}\cdot7245$. If the hole be only $\frac{1}{100}$ of a ſquare inch, or the ſide $\frac{1}{10}$ of an inch, the time of completely filling the veſſel will be nearly $172\frac{1}{2}$ ſeconds, or rather leſs than 3 minutes.

If we make the experiment with a hole cut in a thin plate, we ſhall find the time greater nearly in the ratio of 62 or 63 to 100. As is likewiſe the caſe with water flowing through ſmall orifices (art. 458. 517.), and for ſimilar reaſons.

In like manner we can find the time neceſſary for bringing the air in the veſſel to $\frac{3}{4}$ of the ordinary denſity. For the only variable part of the correct fluent above is $\sqrt{D-\delta}$, which in this caſe becomes $\sqrt{1-\frac{3}{4}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$, whence $\sqrt{D} - \sqrt{D-\delta} = \frac{1}{2}$. And if the hole is a ſquare, each ſide being $\frac{1}{10}$ of an inch, the time is $\frac{1}{2}$ of $172\frac{1}{2}$, that is, $86\frac{1}{4}$ ſeconds.

515. Prop. *Let the air in the veſſel* ABCD (fig. 4. pl. XVIII.) *be compreſſed by a weight acting on the cover* AD, *which is move-*

able down the veſſel, to determine the velocity of the air expelled at the aperture C.

The denſity D of the external air correſponds to its preſſure P: and, ſince the immediate effect of the external preſſure is to compreſs the air in the veſſel and give it another denſity, let the additional preſſure on the cover of the veſſel be p, and the denſity of the air in the veſſel be d; then ſhall we have $P:P+p::D:d$; whence $p=P\times\frac{d-D}{D}$. Now, becauſe the preſſure which expels the air is the difference between the force which compreſſes the air in the veſſel and that which compreſſes the external air, the expelling force is p; and becauſe the quantities of motion are as the forces which ſimilarly produce them, we ſhall have $P:P.\frac{d-D}{D}::MV:mv$, where M and m expreſs the quantities of matter expelled, V the velocity with which air ruſhes into a vacuum, and v the velocity required. But becauſe the quantities of aerial matter which iſſue from the ſame orifice in an inſtant are as the denſities and velocities jointly, we ſhall have $MV:mv::DVV:dvv$. Conſequently $P:P.\frac{d-D}{D}::DV^2:dv^2$. And hence we deduce $v=V\sqrt{\frac{d-D}{d}}$.

COR. Another expreſſion for the velocity may be obtained without conſidering the denſity: for ſince $P:P+p::D:d$, therefore $d=\frac{D(P+p)}{P}$, and $d-D=\frac{D(P+p)}{P}-D=\frac{D(P+p)-DP}{P}=\frac{Dp}{P}$; whence $\frac{d-D}{d}=\frac{DP}{P}\div\frac{D(P+p)}{P}=\frac{p}{P+p}$. Subſtituting this value of $\frac{d-D}{d}$ for it in the final value of v, it becomes $v=V.\sqrt{\frac{p}{P+p}}$, which is both a ſimple and convenient expreſſion.

516. PROP. *To enquire into the effect of the air's elaſticity, when aſcertaining the velocity with which the air will iſſue from a veſſel into a vacuum.*

Let ABCD (fig. 4. pl. XVIII.) be a veſſel containing air of any denſity D. This air is in a ſtate of compreſſion, and if the compreſſing force be removed it will expand, and its elaſticity will diminiſh with the denſity. Now its elaſticity in any ſtate is meaſured by the force which keeps it in that ſtate (art. 488.); and the force which keeps common air in its ordinary denſity is the preſſure of the atmoſphere, which (art. 485.) is at a medium the ſame with the weight of a column of mercury $29\frac{3}{4}$ inches high. If therefore we ſuppoſe that this air, inſtead of being confined by the top of the veſſel, is preſſed down by a moveable piſton carrying a cylinder of mercury of the ſame baſe, and $29\frac{3}{4}$ inches high, its elaſticity will balance this preſſure juſt as it does the preſſure of the atmoſphere: and, ſince it is a fluid, and propagates through every part and in every direction the preſſure exerted on any one part, it will preſs on any ſmall portion of the veſſel by its elaſticity, in the ſame manner as when loaded with this column. Hence, if this ſmall portion of the veſſel be removed, and a paſſage made into the void, the air will begin to flow out with the ſame velocity as it would flow out when impelled by its weight alone, or with the velocity acquired by falling from the top of the homogeneous atmoſphere; that is, a velocity of 1339 feet nearly (art. 510.). But as ſoon as any air has paſſed through the orifice the denſity of that remaining in the veſſel is diminiſhed, and its elaſticity is diminiſhed, conſequently the expelling force is diminiſhed: but the matter to be moved is diminiſhed in the very ſame proportion as the denſity, the capacity of the veſſel remaining unchanged: therefore, ſince the denſity and elaſticity follow the ſame law, the quantity of matter moved will vary as the moving force, and the velocity will continue the ſame from the beginning to the end of the efflux.

SCHOLIUM.

517. Hence, ſince the velocity with which the air iſſues out of ſuch a veſſel is *conſtant*, we may readily compare the velocity of theory in art. 510. with that found by experiment. But for this purpoſe we ſhall here deſcribe the ſimple apparatus and experiments of Mr. Banks, in which a change of denſity is avoided. (Banks on the Power of Machines, p. 10.) A (fig. 8. pl. XVIII.) is a caſk of known capacity, into the top of which is ſcrewed an aperture *a* of a known area. The tube T *d*, recurve at *d*, is ſoldered or ſcrewed into the top of the ſaid caſk. The hole *a* is ſtopped, and water poured into the tube T till it is full; at which time a quantity of water will have paſſed out of the tube at *d*, and condenſed the air in the caſk till its ſpring is equal to the weight of the water in the tube. At this time a cock placed over the tube T, ſufficiently large to ſupply water as faſt as it can deſcend into the veſſel A, muſt be opened to keep the tube conſtantly filled: for this purpoſe one perſon muſt attend it, and another muſt open the aperture *a*, which need only have been cloſed by a finger, and he muſt meaſure the ſeconds from the moment that the finger is removed till the water flies out at the jet. Hence, from know-

ing the capacity of the veſſel and the area of the jet, the velocity may be obtained. If the tube T *d* ſhould be continued nearly to the bottom of A, while A was filling with water, the length of the compreſſing column would be gradually diminiſhed, and of conſequence the preſſure would be conſtantly changing: to avoid any irregularity this might produce, the open end of the tube is as near the top of the caſk as is conſiſtent with a free paſſage for the water.

Experiments. The veſſel contained 15 lb. 6 oz. of water; from which we find its capacity 425·088 cubic inches. The area of the aperture *a*, through which the air is diſcharged, is ·0046 inches.

Exp. I. { The altitude of T above the caſk 30 inches;
Time of expelling the air 33^s, by ſeveral trials.

Exp. II. { The altitude of T 6 feet;
Time of expelling the air $21{\cdot}3^s$, by ſeveral trials.

In the firſt experiment 425·088, the capacity of the caſk, being divided by ·0046, the area of the hole, gives 92410·4 inches, for the length of the ſtream driven out in 33^s. Hence

$$\frac{92410{\cdot}4}{12 \times 33} = 233{\cdot}3 \text{ feet, the velocity per ſecond.}$$

From the ſecond experiment we deduce, by a ſimilar proceſs, 361·6 feet, for the velocity per ſecond: and, to ſhew the correſpondence of this with the firſt, ſay, as $\sqrt{2\frac{1}{2}}$ (the head) : 233·3 :: $\sqrt{6}$ (the head) : 361·8 feet; differing from the former by only a fifth of a foot.

To compare the velocity thus found by experiment with that aſſigned by theory (art. 510.), we may ſay as $\sqrt{6} : 361{\cdot}6 :: \sqrt{33}$ (the height of a column of water equivalent in preſſure to the atmoſphere) : 845·2 feet, the velocity with which the atmoſphere would begin to enter into a vacuum. Making the allowance ſpoken of in art. 514. we ſhall have $1339 \times {\cdot}63 = 843{\cdot}57$, agreeing as nearly with the experimental reſult as can reaſonably be expected.

518. PROP. *To find the quantity of aerial matter which will be expelled from the orifice* C *of the veſſel* ABCD (fig. 4.) *during any time* t, *and the denſity of the remaining air at the end of that time.*

In the element of time $\dot{t}$ there iſſues (by art. 513.) the capacity of air $a\dot{t}\sqrt{2gH}$, the velocity V being conſtant (art. 516.), and conſequently the *quantity* of air $ad\dot{t}\sqrt{2gH}$. On the other hand, the quantity of air at the beginning of the efflux was sD, s being, as before, the capacity of the veſſel; and when the air has acquired the denſity d, the quantity in the veſſel is sd, and $sD - sd$ is the quantity expelled: conſequently the quantity diſcharged in the element of time $\dot{t}$ muſt be the fluxion of $s\,\mathrm{D} - s\,d$; that is, $-\dot{s}\,d$. Hence we have the equation $a\,d\,\dot{t}\sqrt{2g\mathrm{H}} = -s\,\dot{d}$, and $\dot{t} = \frac{-s\dot{d}}{ad\sqrt{2g\mathrm{H}}} = \frac{s}{a\sqrt{2g\mathrm{H}}} \times -\frac{\dot{d}}{d}$. The fluent of this is $t = \frac{s}{a\sqrt{2g\mathrm{H}}}$. hyp. log. d. Now this fluent muſt be ſo taken that when $t=0$, $d=\mathrm{D}$, or hyp. log. $\frac{\mathrm{D}}{d}$ = hyp. log. $1=0$. So that the correct fluent is

$$t = \frac{s}{a\sqrt{2g\mathrm{H}}}\,.\,\text{hyp. log.}\,\frac{\mathrm{D}}{d} = \frac{s}{8a\sqrt{\mathrm{H}}}\,.\,\text{hyp. log.}\,\frac{\mathrm{D}}{d},\ \text{nearly.}$$

Cor. Hence it follows that the whole air of a veſſel ſituated as in the propoſition will not flow out of it into a void, in any finite time whatever.

419. Prop. *To determine the time when the veſſel, inſtead of diſcharging its air into a void, emits it into air of a leſs denſity, that denſity remaining invariable during the efflux; as may be ſuppoſed the caſe when a veſſel holding condenſed air emits it into the ſurrounding atmoſphere.*

Let the initial denſity of the air in the veſſel be δ, and that of the atmoſphere D. Then it is manifeſt that the expelling force is $\mathrm{P} - \frac{\mathrm{PD}}{\delta}$ (art. 512.); and that after the time t it is $\frac{\mathrm{P}d}{\delta} - \frac{\mathrm{PD}}{\delta}$. We have therefore $\frac{\mathrm{P}\delta - \mathrm{PD}}{\delta} : \frac{\mathrm{P}d - \mathrm{PD}}{\delta} :: \mathrm{MV} : mv :: \delta\mathrm{V}^2 : dv^2$: whence, by reduction, we have $v = \mathrm{V}\sqrt{\frac{\delta(d-\mathrm{D})}{d(\delta-\mathrm{D})}}$. From which equation we may learn, that when $d=\mathrm{D}$ the motion will be at an end; and that if $\delta=\mathrm{D}$ there can be no efflux.

Now, to find the relation between the time and the denſity, let H, as before, be the height due to the velocity V. Hence the height due to the velocity of efflux v muſt be $\mathrm{H} \times \frac{\delta(d-\mathrm{D})}{d(\delta-\mathrm{D})}$, and the ſmall parcel of air which will flow out in the element of time $\dot{t}$ will be $a\,d\,\dot{t}\sqrt{2g\mathrm{H}\,.\,\frac{\delta(d-\mathrm{D})}{d(\delta-\mathrm{D})}}$. And another expreſſion for the ſame is, as in the preceding article, $-s\,\dot{d}$. Making theſe two values equal, we ſoon deduce the fluxionary equation $\dot{t} = \frac{s\sqrt{\delta-\mathrm{D}}}{a\sqrt{2g\delta\mathrm{H}}} \times \frac{-\dot{d}}{\sqrt{d^2 - d\mathrm{D}}}$. The fluent of this, ſo corrected that when $t=0$, $d=\delta$, is

$$t = \frac{s}{a} \times \sqrt{\frac{\delta - D}{2g\delta H}} \times \text{hyp. log.} \left(\frac{\delta - \frac{1}{2}D + \sqrt{\delta^2 - \delta D}}{d - \frac{1}{2}D + \sqrt{d^2 - dD}}\right).$$

COR. When d becomes $= D$ the efflux is complete, and the expreſſion for the time will then become

$$t = \frac{s}{a} \times \sqrt{\frac{\delta - D}{2g\delta H}} \times \text{hyp. log.} \left(\frac{\delta - \frac{1}{2}D + \sqrt{\delta^2 - \delta D}}{\frac{1}{2}D}\right).$$

520. PROP. *Let the capacities of two veſſels containing airs of different denſities be* A *and* B; *if they communicate by a tube whoſe ſection is* a, *there will be a current from the veſſel containing the denſer air into that containing the rarer: it is propoſed to find an expreſſion for the time of efflux.*

Let P be the elaſtic force of the air in the veſſel A, Q its denſity, and V its velocity; D being the denſity of the air in the veſſel whoſe capacity is B. After the time t let the denſity of the air in A be q, its velocity v, the denſity of the air in B having become δ. The force expelling from A will then be $P - \frac{PD}{\mathcal{Q}}$ at the firſt inſtant, and at the end of the time t it will be $\frac{Pq - P\delta}{\mathcal{Q}}$. So that we ſhall have $\frac{P\mathcal{Q} - PD}{\mathcal{Q}} : \frac{Pq - P\delta}{\mathcal{Q}} :: QV^2 : qv^2$; whence ariſes $v = V \cdot \sqrt{\frac{\mathcal{Q}(q - \delta)}{q(\mathcal{Q} - D)}}$; and we ſee that the motion will ceaſe when $\delta = q$.

Now the capacities of the veſſels being A and B reſpectively, we have for a ſecond equation, including the denſities, $AQ + BD = Aq + B\delta$; from which we find $\delta = \frac{A(\mathcal{Q} - q) + BD}{B}$. This value of δ being ſubſtituted in the preceding value of v, tranſforms it to $v = V \cdot \sqrt{\frac{\mathcal{Q}[B(q - D) - A(\mathcal{Q} - q)]}{qB(Q - D)}}$; an equation which gives the relation between the velocity v and the denſity q.

In order to find the time when the air in A is reduced to the denſity q, it will be convenient to abridge the work by ſome ſubſtitutions: thus, make $Q(B + A) = M$, $BQ(D + Q) = N$, $BQ - BD = R$, and $\frac{N}{M} = m$. Then, proceeding as in the propoſitions immediately before this, we obtain the fluxionary equation

$$aq\dot{t}\sqrt{2gH \cdot \frac{M(q - N)}{Rq}} = \text{flux. of } (AQ - Aq) = -\dot{A}q.$$

This gives $\dot{t} = \frac{A\sqrt{R}}{a\sqrt{2gMH}} \times \frac{-\dot{q}}{\sqrt{q^2 - mq}}$; the fluent of which, ſo corrected that $t = 0$ when $q = Q$, is as below:

$$t = \frac{A}{a} \times \sqrt{\frac{R}{2gMH}} \times \text{hyp. log.} \left(\frac{Q - \frac{1}{2}m + \sqrt{Q^2 - mQ}}{q - \frac{1}{2}m + \sqrt{q^2 - mq}}\right).$$

In a manner not widely different from ſome of the propoſitions in this chapter might various other problems relative to the paſſage of air from one veſſel to another be ſolved: but ſuch problems are not of very frequent uſe in the common applications of Pneumatics to practice, and are therefore not inſerted here. Some particular caſes are conſidered in the article *Pneumatics*, Encyclo. Britan. to which the ſtudent, who wiſhes to inveſtigate the ſubject further, is referred for other information.

CHAPTER IV.

On the Theory of Air Pumps, and Pumps for raiſing Water.

521. THE *Air-pump* is a machine formed for exhauſting the air out of a proper veſſel, and ſo to make what is called a vacuum: it is one of the moſt uſeful philoſophical inſtruments whoſe operations depend upon the properties of the air. By the help of this machine the chief propoſitions relative to the weight and elaſticity of the air are proved experimentally, in a ſimple and ſatisfactory manner. This machine is conſtructed in various ways, one of the beſt of which will be deſcribed in the ſecond volume of this work. At preſent we ſhall deſcribe one of a portable and convenient form; being well adapted to moſt of the purpoſes for which air-pumps are employed.

EFGH (fig. 1. pl. I. vol. II.) is a ſquare table of wood; AA are two ſtrong barrels or tubes of braſs, firmly retained in their poſition by the croſs-piece TT, which is preſſed on them by ſcrews OO, fixed on the tops of the braſs pillars N, N. Theſe barrels communicate with a cavity in the lower part D. At the bottom within each barrel is fixed a valve, opening upwards; and in each barrel a piſton works, having a valve likewiſe opening upwards. The piſtons are moved by a cog-wheel in the piece TT, turned by the handle B, of which wheel the teeth catch in the racks of the piſtons C, C. PQ is a circular braſs plate, having near its centre the orifice K of a concealed pipe that communicates with the cavity: in the piece D at V is a ſcrew that cloſes the orifice of another pipe, for the purpoſe of admitting the external air when required. LM is a glaſs receiver, out of which the air is to be exhauſted, and which has obtained the name of *receiver*, becauſe it receives or holds the ſubjects on which the experiments are to be made. This receiver is placed on the plate PQ, which is previouſly covered with a wet ſheep-ſkin, or ſmeared with wax, to prevent the air from inſinuating under the edge of the glaſs.

When the handle B is turned one of the piſtons is raiſed, and the other depreſſed; conſequently a void ſpace is left between the raiſed piſton and the lower valve in the correſpondent barrel: the air contained in the receiver LM communicating with the barrel by the orifice K immediately raiſes the lower valve by its ſpring, and expands into the void ſpace; and thus a part of the air in the receiver is extracted. The handle then, being turned the contrary way, raiſes the other piſton, and performs the ſame act in its correſpondent barrel; while, in the mean time, the firſt-mentioned piſton being depreſſed, the air by its ſpring cloſes the lower valve, and, raiſing the valve in the piſton, makes its eſcape. The motion of the handle being again reverſed, the firſt barrel again exhauſts, while the ſecond diſcharges the air in its turn: and thus, during the time the pump is worked, one barrel exhauſts the air from the receiver, while the other diſcharges it through the valve in its piſton. Hence it is evident that the vacuum in the receiver of this air-pump (and the ſame may be ſaid of all others) can never be perfect; that is, the air can never be entirely exhauſted: for it is the elaſticity of the air in the receiver that raiſes the valve, and forces air into the barrel; and the barrel at each exſuction can only take away a certain part of the remaining air, which is in proportion to the quantity before the ſtroke, as the capacity of the barrel to the ſum of the capacities of the barrel, receiver, and communicating pipe.

522. Now, if we ſuppoſe no vapour from moiſture, &c. to riſe in the receiver, the degree of exhauſtion after any number of ſtrokes of the piſton may be determined by knowing the reſpective capacities of the barrel and of the receiver, including the pipe of communication, &c. For, as we have ſeen above that every ſtroke diminiſhes the denſity in a conſtant proportion, namely, as much as the whole content exceeds that of the cylinder or barrel; the exhauſtion will go on in a geometrical progreſſion, the ratio of which is the ſame as that which the ſum of the receiver and barrel together bears to that of the receiver: and this ratio of exhauſtion will be followed until the elaſticity of the included air is ſo far diminiſhed by its rarefaction as to render it too feeble to puſh up the valve of the piſton.

Let, then, the capacity of the barrel, receiver, and pipe of communication together be expreſſed by $b+r$, and that of the barrel alone by b, and let 1 repreſent the primitive denſity of the air in the pump: ſo ſhall we have

$b+r : r :: 1 : \frac{r}{b+r}=$ the denſity after 1 ſtroke of the piſton,
$b+r : r :: \frac{r}{b+r} : \frac{r^2}{(b+r)^2}=$ denſity after 2 ſtrokes,
$b+r : r :: \frac{r^2}{(b+r)^2} : \frac{r^3}{(b+r)^3}=$ denſity after 3 ſtrokes;
and the nth power of the ratio $\frac{r}{b+r}$, that is $\left.\right\} \frac{r^n}{(b+r)^n}=d$, denſity after n ſtrokes,

From which we may eaſily find the denſity after any number of ſtrokes, when the ratio of $b : r$, and conſequently that of $b+r$ to r, is given.

From the ſame formula, $\frac{r^n}{(b+r)^n}=d$, we may alſo derive another for finding the number of ſtrokes of the piſton neceſſary to rarefy the air a number of times, or to give it a certain denſity d, the primitive denſity being 1. For the above equation, expreſſed logarithmically, is $n \times \log. \frac{r}{b+r}=\log. d$; or $n \times (\log. r - \log. b+r)=\log. d$: conſequently, $n=\frac{\log. d}{\log. r-\log. (b+r)}$: in which expreſſion d will be a fraction. If the number of times which the air is rarefied be expreſſed by N an integer, then the logarithmic equation will be $n=\frac{\log. N}{\log. (b+r)-\log. r}$.

A further reduction of the ſame theorem will furniſh us with the proportion between the capacities of the receiver and the barrel, when the air is rarefied to the fractional denſity d by a definite number n of ſtrokes of the piſton. For ſince $\frac{r^n}{(b+r)^n}=d$, if we take the nth root of both members of the equation we ſhall have $\frac{r}{b+r}=\sqrt[n]{d}$. Thus, if d be equal $\frac{1}{177147}$, and the number of ſtrokes $n=11$; we ſhall find $\frac{\log. d}{11}=\log. \frac{1}{3}$; ſo that $r : b+r :: 1 : 3$, and $b : r :: 2 : 1$.

523. For the numerous uſes of the air-pump the reader may conſult the popular treatiſes on natural philoſophy: we ſhall merely obſerve in this place, that the ſpecific gravity of air may be accurately aſcertained by means of this machine. The method is as follows: To the neck of a glaſs bottle, made in the form of a Florence flaſk, adapt a cap and valve opening outwards; ſcrew it on the pump, and exhauſt it to a known degree, which

will be ſhewn by the gauge attached to the pump for that purpoſe: then, from the weight of the bottle before and after exhauſtion, we have the weight of the exhauſted air; and from the ratio of the height of the mercury in the guage to the ſtandard altitude we know the proportion which the exhauſted part bears to the whole air originally in the veſſel, whoſe weight is therefore known. Subtracting this weight from the weight of the veſſel when full of air, there will remain the weight of the veſſel itſelf: fill it with water and weigh it, and ſubduct the weight of the veſſel from this weight; the remainder is the weight of a bulk of the ſame magnitude with the air which fills the veſſel, and whoſe weight was alſo previouſly aſcertained.

Following this method, it has been found by a mean of ſeveral experiments, that the ſpecific gravity of air is to that of water as 1·222 to 1000, very nearly, when the barometer ſtands at 30 inches, and in the mean temperature of 55° of Fahrenheit's thermometer. This agrees with the reſult already given in art. 490.

Pumps for raising Water.

524. The term *Pump* is generally applied to a hydraulic machine for raiſing water by means of the preſſure of the atmoſphere. Of pumps there are a great many different ſorts, ſome of the beſt of which will be deſcribed in the ſecond volume of this work: at preſent we ſhall only ſpeak of three or four of the moſt common, and ſhall give merely ſuch a general deſcription of their conſtruction as will enable the ſtudent to underſtand the principles on which their operation depends.

The four kinds of pumps of which we ſhall now treat are, the *ſucking pump*, the *lifting pump*, the *forcing pump*, and the *centrifugal pump*: of theſe the firſt three have ſome parts in common, and particularly the piſtons and ſuckers; they will therefore be treated in a rather connected way: the properties of the centrifugal pump will be conſidered ſeparately.

The *piſton* is a body ABCD of circular baſe (figs. 5. 6. 13. pl. XVIII.), which may be moved through the interior part of the tube or body of the pump, filling it exactly as it moves along. The ſucker E is moveable about a joint in ſuch a manner as either to permit or to prevent the paſſage of the water, according as it preſſes upwards or downwards. In figs. 5. 6. there are likewiſe ſuckers in the piſtons. FGHK (figs. 5. 13.) is another tube joined to the body of the pump, and is generally called the *pipe* or *ſucking pipe*: its lower extremity is immerſed in the water, of which we ſuppoſe RS is the horizontal ſurface.

525. The *ſucking* pump is repreſented in fig. 5. In this

pump if we ſuppoſe a power P applied to the handle of the piſton ſo as to raiſe it from I to C, the air contained in the ſpace DVKHGFC tends by its ſpring to occupy the ſpace that the piſton leaves void: it therefore forces up the valve E, and enters into the body of the pump, its elaſticity diminiſhing in proportion as it fills a greater ſpace. Hence it will exert on the ſurface GH of the water a leſs effort than is made by the exterior air in its natural ſtate upon the ſurrounding parts of the ſame ſurface RG, HS: and the exceſs of preſſure on the part of the exterior air will cauſe the water to riſe in the pipe GK to a certain height HN, ſuch that the weight of the column GN, together with the ſpring of the ſuper-incumbent air, ſhall juſt be a counterpoiſe to the preſſure of the exterior air. At that time the ſucker E cloſes of itſelf; and if the piſton be lowered, the air contained between the piſton and the baſe IV of the body of the pump having its denſity augmented as the piſton is lowered, will at length have its denſity, and conſequently its elaſticity, greater than that of the exterior air: this difference of elaſticity will conſtitute a force which will puſh the ſucker L in the piſton upwards, and ſome air will eſcape till the exterior and interior air are reduced to the ſame denſity. The ſucker L then falls again: and if we again elevate the piſton, the water will be raiſed higher in FGHK, for the ſame reaſon as before. Thus, after a certain number of ſtrokes of the piſton, the water will reach the body of the pump; where, being once entered, it will be forced at each ſtroke of the piſton through the ſpout X: for the water above the piſton will then preſs upon the ſucker, and keep it ſhut whilſt the piſton is riſing; ſo that a cylinder of water whoſe height is equal to the ſtroke OT of the piſton (or the vertical diſtance through which it paſſes) will be raiſed by each upward motion and forced through the aperture X, provided it is of an adequate magnitude.

526. The *lifting pump* is repreſented in fig. 6. pl. XVIII. Its manner of operation is this: The piſton PCD is here placed below the horizontal ſurface RS of the water, and when it is cauſed to deſcend it produces a vacuum between the ſucker E (which is puſhed down by the external air) and the baſe CD of the piſton. The weight of the water, together with that of the exterior air about R and S, preſſes up the ſucker L, and the water paſſes into the body of the pump: and when the water ceaſes to enter, the weight of the ſucker L cloſes it. Then, if the piſton be raiſed, it raiſes all the water above it, forces up the ſucker E, and introduces the water into the part IVYX. When the piſton is raiſed to its higheſt poſition the ſucker E is made to cloſe by the ſuper-incumbent water, and retains the fluid there until by a freſh ſtroke of the piſton more water is forced up-

wards through the ſucker E; that which was before in the upper part of the pump being expelled through a proper orifice or ſpout in the neighbourhood of X, in order to make way for a new ſupply. And ſo the operation is continued, and water delivered at every ſtroke of the piſton.

527. The *forcing pump* unites in ſome meaſure the properties of the other two. The piſton ABCD (fig. 13. pl. XVIII.) which here has no ſucker, being elevated, rarefies the air in the ſpace DGHVOC, and the water riſes towards K: the ſubſequent deſcent of the piſton forces ſome of the air in this ſpace through the valve L: the next aſcent of the piſton cloſes the valve L, and raiſes the water in GK; and ſo on till the water paſſes through the ſucker E and enters the ſpace DIVOC. Then the piſton being puſhed down cloſes the ſucker E, and ſome of the condenſed air is forced through the valve L. A further ſtroke raiſes more water into the ſpace DIVOC, and expels more air through L. At length the water reaches L, and the ſubſequent ſtrokes raiſe it into the tube MO mn; from whence it is carried off by a ſpout, as in the other pumps. Or, if this pump be cloſed at mn, excepting a narrower pipe ps, then when the water is raiſed by the proceſs juſt deſcribed to or, above the bottom s of the tube, the elaſtic force of the compreſſed air in the ſpace $morn$ will compel the water to iſſue from the aperture p in a continued ſtream or jet; thus forming an artificial fountain.

528. Let us now enquire into the fundamental properties of theſe machines. By means of the *lifting* pump, water may be elevated to any height we pleaſe, provided we employ a ſufficient force. But the eſtimation of this force requires various conſiderations. We muſt have regard to the dimenſions of the piſton, the barrel of the pump, the height to which the water is to be raiſed, and the velocity with which it is elevated; beſides the effects of friction, &c. At preſent, however, we ſhall not examine theſe particulars in all their extent; but ſhall confine ourſelves to one of them. Now it is certain that the power neceſſary to raiſe the water to any propoſed height muſt at leaſt be capable of ſuſtaining in equilibrio the preſſure experienced by the baſe of the piſton when it is kept at reſt, and the fluid has attained the required height. This preſſure, then, we proceed to eſtimate.

In general the power muſt be, at leaſt, capable of ſuſtaining the weight of a column of water which has for its baſe that of the piſton, and for its altitude the diſtance between the ſurface RS of the water in the reſervoir and the upper ſurface XY of that in the pump. For when the baſe DC (fig. 6.) of the piſton is below the ſurface RS of the water in the reſervoir, it is manifeſt that the power has not to ſuſtain the preſſure of the water contained between RS and DC; becauſe that preſſure is counterbalanced by that of the water ſurrounding the lower part of the pump, and which is tranſmitted by means of the inferior orifice of the pipe. The power, therefore, has only to ſuſtain the preſſure exerted upon the ſurface DC by the fluid compriſed between RS and XY; which preſſure (art. 386.) is equal to the weight of a column of water whoſe baſe is CD and altitude the vertical diſtance between RS and XY.

When the piſton is above R′S′, the ſurface of the water in the reſervoir, then it is evident the water contained between DC and R′ S′ does not preſs the piſton downwards. But, as in that caſe it can only be ſuſtained above R′ S′ by the preſſure of the air upon the water ſurrounding the pump, and as this preſſure is only capable of ſuſtaining in equilibrium the contrary preſſure of the air upon the ſurface XY, it follows that the ſurface DC of the piſton is ſurcharged by a weight equivalent to the column which has DC for its baſe and CR′ for its altitude. And this preſſure, joined to that which is exerted upon DC by the ſuper-incumbent fluid between DC and XY, makes the whole preſſure upon the piſton as before equal to that of a column of water whoſe baſe is DC, and height the diſtance between XY and R′S′.

529. The *ſucking pump* requires in its theory the aid of other principles. To judge of its effect a mere evaluation of the power will not ſuffice: we muſt enquire if under the propoſed circumſtances the water can poſſibly be raiſed to the piſton, and made to paſs through the ſucker L; for in ſome caſes the water will never paſs a certain altitude, how many ſtrokes ſoever we give to the piſton. To underſtand this, conceive that the water has been actually raiſed to T (fig. 5. pl. XVIII.), and that the ſituation of the piſton in the figure is the loweſt which can be given to it: and for greater ſimplicity, ſuppoſe that the pump is of the ſame internal diameter throughout. It is obvious that the air compriſed in the ſpace CDTZ is of the ſame denſity and elaſticity as the exterior air (at leaſt dropping the conſideration of the weight of the ſucker L and the friction attending its motion); for if its ſpring were *leſs* the water would riſe higher than ZT, and if it were *greater* it would raiſe the ſucker L, and mix with the exterior air till both became of the ſame denſity. Suppoſe now that the play of the piſton, or the diſtance through which it is raiſed or lowered at each ſtroke, is DQ: then when the baſe CD is raiſed to QO, the air which previouſly occupied the ſpace CDTZ will tend to expand and fill the ſpace QOTZ; and if the water did not riſe would actually be ſo expanded. Its elaſtic force would then be leſs than that of the natural air,

in the ratio of CDTZ to QOTZ (art. 489.), or of DT to OT. If, therefore, this elaſtic force, together with the weight of the column of water whoſe height is ZR, conſtitute a preſſure equal to that of the atmoſphere, or equal to the weight of a column of water of equal baſe and height at a medium 33 feet, there will be an equilibrium, and the water will not riſe further: if this joint preſſure is greater than that of 33 feet of water, the water cannot be retained ſo high; but if it is leſs than the column of 33 feet, the water will continue to riſe in the pump.

530. From theſe conſiderations we may readily inveſtigate a general theorem.

Let a, the altitude or vertical diſtance from the point O to the ſurface RS of the water in the reſervoir, $p = \mathrm{OD}$ the play of the piſton, and x the diſtance OT: then we have $\mathrm{DT} = x - p$, and ST the height of the point T will be $a - x$. Since the air contained in CDTZ has the ſame denſity and elaſticity as the exterior air, its force may be meaſured by a column of water of the ſame baſe ZT and 33 feet high; and becauſe when this air is ſo expanded as to fill the ſpace QOTZ the elaſtic force will be leſs in the ratio of DT to OT, we ſhall have (rejecting the baſe of the column, as equally affecting every part of the proceſs) this latter force expreſſed by the fourth term of this proportion, $x : x - p :: 33 : \frac{33}{x}(x - p)$. But the force which the water compriſed between ZT and RS exerts in oppoſition to the exterior preſſure of the air, is meaſured by the height $a - x$: conſequently, the elaſtic force of the air in the ſpace QOTZ, together with the weight of the water between ZT and RS, will be expreſſed by $\frac{33(x-p)}{x} + a - x$. Now in order that the water may always riſe, this joint preſſure muſt be leſs than the weight of a column of water of 33 feet by ſome variable quantity, which we will call y: ſo that the following equation muſt always obtain, viz. $\frac{33(x-p)}{x} + a - x = 33 - y$. The value of x deduced from this equation is ambiguous, being thus expreſſed:

$$x = \tfrac{1}{2}a + \tfrac{1}{2}y \pm \sqrt{(\tfrac{1}{2}a + \tfrac{1}{2}y)^2 - 33p}.$$

Now, when the water ſtops and does not riſe any further, y vaniſhes, and the equation becomes $x = \frac{1}{2}a \pm \sqrt{\frac{1}{4}a^2 - 33p}$; of which the two values are real, ſo long as $\frac{1}{4}a^2$ is greater than $33p$. Hence we conclude that *when one-fourth of the ſquare of the greateſt height of the piſton above the ſurface of the water in the reſervoir is greater than 33 times the play of the piſton, there are always two points in the ſucking pump where the water may ſtop in* *its motion;* and the pump muſt be reputed *bad* when the loweſt point to which the piſton can be brought is found between theſe two points.

But if 33 p be greater than $\frac{1}{4}a^2$, the two values of x, when y is ſuppoſed $=0$, become imaginary: ſo that in a pump ſo conſtructed it is impoſſible that y ſhould vaniſh; that is, the preſſure of the exterior air always prevails, and the water is not arreſted in its paſſage. Hence we conclude, ſecondly, that *in order that the ſucking pump may infallibly produce its effect, the ſquare of half the greateſt elevation of the piſton above the water in the reſervoir muſt always be leſs than* 33 *times the play of the piſton.*

531. This general rule may alſo be eaſily deduced geometrically thus: Suppoſe the ſucker or valve E be placed at the ſurface RS of the water (fig. 5), the tube to be of uniform bore, and YS to be the height of a column of water whoſe preſſure is equal to that of the atmoſphere; that is, $YS=33$ feet. Conceive the water raiſed by working to N: then the weight of the column of water SN, together with the elaſticity of the air above it, exactly balances the preſſure of the atmoſphere YS. But the elaſticity of the air in the ſpace OM, (QO being the higheſt and CD the loweſt ſituation of the piſton,) is proportional to $YS \times \frac{DN}{ON}$; and, conſequently, in the caſe where the limit obtains, and the water riſes no further, it will be $YS = NS + (YS \cdot \frac{DN}{ON})$. Tranſpoſing NS, we have $YS - NS$ $(=NY) = YS \cdot \frac{DN}{ON}$; whence $ON : DN :: YS : YN$; or, *dividendo*, $ON - DN\ (=DO) : ON :: YS - YN\ (=NS) : YS$; conſequently $DO \times YS = ON \times NS$. Hence we ſee, that if SO, the diſtance of the piſton in its higheſt poſition from the water, and OD, the length of the ſemi-ſtroke, or the play of the piſton, be given, there is a certain determinate height, as SN, to which the water can be raiſed by the difference of the preſſures of the exterior and interior air: for YS is to be conſidered as a conſtant quantity, and, of courſe, when OD is given, $ON \cdot NS$ is given likewiſe. To enſure, therefore, the delivery of water by the pump, the ſtroke muſt be ſuch that the rectangle $OD \cdot YS$ may be greater than any rectangle that can be made of the parts of SO; that is, greater than the ſquare of $\frac{1}{2}SO$, by a well-known theorem.

Hence we deduce a practical maxim of the ſame import as the preceding, i. e. *No ſucking pump can raiſe water effectually unleſs the play of the piſton in feet be greater than the ſquare of the greateſt height of the piſton, divided by* 132.

532. Reſuming the equation $\frac{33(x-p)}{x}+a-x=33-y$, and finding thence the value of y, we have $y=\frac{x^2-ax+33p}{x}$. Now let AB (figs. 9. 10. pl. XVIII.) repreſent the greateſt height of the piſton above the ſurface of the water in the reſervoir, and AD the play of the piſton: ſuppoſe the different portions AP of the line AB to repreſent the ſucceſſive values of x, and lay down upon the perpendiculars PM the values of y which correſpond to theſe aſſumed values of x: ſo ſhall we have a curve MMC (fig. 9.) which, while $\frac{1}{4}a^2$ is greater than $33p$, will cut AB in two points I and I′, in ſuch manner that the ordinates PM will lie on different ſides of AB: the ordinates which are below AB ſhewing the poſitive values of y, and thoſe which are above AB the negative values. We ſee, therefore, that ſo long as $\frac{1}{4}a^2$ is greater than $33p$ the preſſure of the exterior air is ſtrongeſt, until the water has attained the height BI′. At this point I′ it will ſtop (abſtracting from the conſideration of the motion acquired), becauſe the value of y is $=0$. But if the water by the motion it has acquired continues to riſe till it reaches ſome point between I′ and I, it will not ſtop there, but will deſcend, if the ſucker does not oppoſe its deſcending motion; becauſe the value of y being there negative indicates that the preſſure of the exterior air is weaker than the united preſſures of the water and the internal air. If the water reach the point I it will ſtop there, for the ſame reaſon as it would at the point I′: but if once it gets above I, there is then no reaſon to fear that it will deſcend; for all the ordinates PM between I and A being poſitive, ſhew that in that portion of the pump the preſſure of the external air exceeds the combined efforts of the internal air and water.

533. When, on the contrary, the value of $\frac{1}{4}a^2$ is leſs than that of $33p$, the curve (fig. 10.) will no where interſect the axis AB; all the ordinates are poſitive, and conſequently the preſſure of the external air is always the ſtrongeſt. This confirms and illuſtrates what has been laid down in art. 530.

If the ſucking pump were to be placed ſo high above the uſual ſurface of the earth (as at the top of a high mountain), or ſo low beneath it (as in a deep mine), that the preſſure of the atmoſphere would be ſenſibly different from the aſſumed mean preſſure equivalent to 33 feet of water, we muſt then in all the preceding inveſtigation change the co-efficient 33 to that which would expreſs the height in feet of the correſponding column of water. And theſe equivalent columns may always be aſcertained by means of the height of the mercurial column in the barometer: the analogy being this;—as $29\frac{3}{4}$ inches, the mean altitude of the mercurial column, to 33 feet, the mean height of the column of water; ſo is any other mercurial column in inches to its correſponding column of water in feet.

534. In the preceding calculus the pump has been ſuppoſed of uniform bore throughout: when this is not the caſe the ſolution is rendered ſomewhat more complex, but not difficult. To calculate the effort of the internal air when the water has not reached the body of the pump, having only attained the height HN, for example (fig. 5.), we muſt uſe this proportion: as the ſpace QOVNMIQ : the ſpace CDVNMIC :: 33 feet: a fourth term, which being added to the weight of the column of water whoſe height is NH, ought again to be equal to $33-y$, as before. Beſides, when the ſucking pipe FG is of a ſmaller diameter than the body of the pump, if the conditions which we have before ſpecified obtain, the pump cannot fail to produce the proper effect; for the air is dilated with more facility in this latter caſe than when the whole is of the ſame internal diameter. We need only add on this point, that, if the length of the ſtroke in a uniform pump, which is requiſite to render the machine effectual, be greater than can conveniently be made, it may be diminiſhed *by contracting the diameter of the ſucking pipe in the ſubduplicate ratio of the diminution of the length of the ſtroke.*

535. As to the effort of which the power ought to be capable to ſuſtain the water to a determinate height YH (fig. 5.), it will be meaſured according to what we have ſaid reſpecting the lifting pump (art. 528.), by the weight of a column of water whoſe baſe is equal to CD, and height that of XY above RS. Here, too, we drop the conſideration of friction and the weight of the piſton.

536. The velocity of the water flowing from the ſucking pipe into the barrel ſhould be equal to the velocity with which the piſton moves. For if it be greater, leſs work will be done than the pump is competent to effect; and if it be leſs, a vacuum will be produced below the piſton, which will therefore be moved upwards with great difficulty. If V be the velocity of the water in the ſucking pipe, d the diameter of that pipe, D the diameter of the barrel or body of the pump, and v the velocity of the piſton; then $V \cdot \frac{d^2}{D^2}$ will be the velocity of the water in the barrel, and we muſt have $v=V \cdot \frac{d^2}{D^2}$, if the machine be perfect. If h be the height of a column of water whoſe weight is equivalent to the preſſure of the atmoſphere, a the altitude of the water in the ſucking pipe, x any other height to which it aſcends in following the piſton, $g=32\frac{1}{6}$ feet; then will the moving force $\propto h-x$, the quantity of matter moved $\propto x$, and, conſe-

quently, the accelerating force $\propto \frac{h-x}{x}$. Hence we have $v\dot{v} = g\left(\frac{h\dot{x}}{x} - \dot{x}\right)$; and, taking the fluents, there reſults v the velocity of the water $= \sqrt{2g \times (h\,.\,\text{hyp. log. } x) - x}$. But when $x=a$, $v=0$; conſequently the correct fluent will be

$$v = \sqrt{2g \times h\,.\,\text{hyp log. } \frac{x}{a} - (x-a)}.$$

It may juſt be added here, that the meaſure we have all along given of the external force is only what is neceſſary for *balancing* the preſſure of the water in the riſing pipe. But in order that the pump may perform work it muſt *ſurmount* this preſſure, and cauſe the water to iſſue at X with ſuch a velocity that the required quantity of water may be delivered in a given time. This requires force, even although there were no oppoſing preſſure; which would be the caſe if the main were horizontal. The water fills it, but it is at reſt. In order that a gallon, for inſtance, may be delivered in a ſecond, the whole water in the horizontal main muſt be put in motion with a certain velocity. This requires force. We muſt therefore always diſtinguiſh between the ſtate of equilibrium and the ſtate of actual working. It is the equilibrium only that we have conſidered, and no more is neceſſary for underſtanding the operation of the different ſpecies of pumps.

On the Centrifugal Pump.

537. The *centrifugal pump* conſiſts of a vertical tube, and a horizontal arm of equal bore, or ſometimes of two horizontal arms of leſs bore than the vertical pipe: that of which we ſhall ſpeak has only one horizontal arm. In fig. 3. pl. XV. KL may repreſent the vertical tube, the end K of which is immerſed in the water of the reſervoir, and LI the horizontal arm communicating with the former. There is a conical valve at K opening upwards, and one at the end of the horizontal arm opening outwards. The whole machine being filled with water and turned ſwiftly round upon pivots at K and L, the arm LI retaining its horizontal poſition, the water will, when the motion is properly regulated, continue to be diſcharged from the moveable extremity I of the arm LI in an uninterrupted ſtream.

In order to inveſtigate the moſt uſeful theorems relative to the operation of this machine, put a = the length of the arm LI in feet, l=the length of the leg KL in feet, t=the time of a revolution in ſeconds, $g=32\frac{1}{6}$ feet, the meaſure of the force of gravity, and $\pi=3{\cdot}141593$ the circumference of a circle whoſe diameter is unity. Then, ſince the centrifugal force is as the velocity it generates in a unit of time, we ſhall have the centrifugal force of a particle $\dot{x}$ of the fluid at the diſtance x from L equal to $\dot{x}\left(\frac{2\pi x}{t}\right)^2 \div x = \frac{4\pi^2 x\dot{x}}{t^2}$, the fluent of which is $\frac{2\pi^2 x^2}{t^2} =$ the centrifugal motive force of the column x: which when $x = a$ becomes $\frac{2\pi^2 a^2}{t^2}$, for the whole centrifugal motive force of the water in the arm LI. Now the preſſure of a column whoſe length is a will, *cæt. par.* vary as $g\,a$; hence, it will be $g\,a : a ::$ $\frac{2\pi^2 a^2}{t^2} : \frac{2\pi^2 a^2}{g\,t^2} =$ the length of a column of water whoſe preſſure is equivalent to the centrifugal force. If from this we deduct the altitude l of the vertical leg, the remainder $\frac{2\pi^2 a^2}{g\,t^2} - l$ will be the length of a column whoſe action would expel the water from the orifice I, with the ſame force as the whirling motion will occaſion; and the column moved will be $a + l$, which will alſo denote the ſpace through which the fluid is accelerated. Conſequently the circumſtances of the caſe before us are the ſame as if a conſtant head of water of the height $\frac{2\pi^2 a^2}{g\,t^2} - l$ impelled a column of water horizontally at the bottom, of the depth $a + l$. Hence the accelerative force ϕ is $\left(\frac{2\pi^2 a^2}{g\,t^2} - l\right) \div (a + l)$, and the ſpace s is $a + l$; ſo that the velocity generated will, by the rules of dynamics, be $v = \sqrt{2\phi g s} = \sqrt{2g \cdot \left(\frac{2\pi^2 a^2}{g\,t^2} - l\right)}$ $= 8{\cdot}0208\sqrt{\frac{2\pi^2 a^2}{g\,t^2} - l}$. Or, if we adapt the theorem to practical purpoſes, the co-efficient 8·0208 muſt (art. 458.) be multiplied by ·62 or ·63, and the equation will become $v = 5\sqrt{\frac{2\pi^2 a^2}{g\,t^2} - l}$, very nearly.

In this inveſtigation we have paid no regard to the rotatory motion of the fluid moleculæ in the vertical tube, but have conſidered it as though it were indefinitely narrow.

538. It is manifeſt that l muſt never exceed 33 feet, becauſe a greater column cannot be ſupported by the preſſure of the atmoſphere: ſo that this machine cannot raiſe water higher than the common ſucking pump. Beſides this, the time of a

revolution of the arm has limits, between which alone the pump is effective, ſo that the inveſtigation muſt be carried a little further.

When the centrifugal force is barely equal to the weight to be raiſed the expreſſion under the radical vaniſhes, and $v=0$, or the water is ſtationary in the pump. In this caſe we evidently have $\frac{2\pi^2 a^2}{g t^2} = l$; from which we find $t = \pi a \sqrt{\frac{2}{gl}} = \cdot 78336 \frac{a}{\sqrt{l}}$, for the time of a revolution in ſeconds, when no work is done.

If the machine be effectual, the velocity of the ſtream can never exceed the velocity which a heavy body would acquire in falling down the difference between 33 feet and the height of the vertical leg: for a greater velocity than that would cauſe a vacuity in the machine which would not be ſupplied by the preſſure of the atmoſphere. Now, by the laws of falling bodies, the velocity acquired in falling through the height $33-l$ is $\sqrt{2g\,(33-l}$, and, by what is done above, the velocity of efflux is $\sqrt{\left(\frac{2\pi^2 a^2}{g t^2} - l\right)}$. Making theſe values equal, we have $\frac{2\pi^2 a^2}{g t^2} - l = 33 - l$. From which we deduce $t = \pi a \sqrt{\frac{2}{33g}} = \cdot 78336 \frac{a}{\sqrt{33}} = \cdot 13636\, a$.

If $l=33$, then will the laſt value of t be the ſame as the preceding, and no diſcharge can take place; agreeably to the obſervations at the beginning of this article. In all other caſes, having found the time of revolution when the water is ſtationary, we have only to diminiſh that time in the ratio of $\sqrt{33}$ to $\sqrt{l}$, and we ſhall have the time of revolution when the work done is *the greateſt this pump will admit of.*

CHAPTER V.

On the Resistance of Fluids to Bodies moving in them.

539. THE force with which bodies moving in fluid mediums, as water, air, &c. are impeded and retarded in their motions, is uſually termed the RESISTANCE of fluids: and as all our machines move either in water or in air, or both, it becomes a matter of importance in the theory of mechanics to enquire into the nature of this kind of force.

We know by experience that force muſt be applied to a body in order that it may move through a fluid, ſuch as air or water; and that a body projected with any velocity is gradually retarded in its motion, and generally brought to reſt. The analogy of nature makes us imagine that there is a force acting in the oppoſite direction, or oppoſing the motion; and that this force reſides in, or is exerted by, the fluid. And the phenomena reſemble thoſe which accompany the known reſiſtance of active beings, ſuch as animals. Therefore we give to this ſuppoſed force the metaphorical name of RESISTANCE. We alſo know that a fluid in motion will hurry a ſolid body along with the ſtream, and that it requires force to maintain it in its place. A ſimilar analogy makes us ſuppoſe that the fluid exerts force, in the ſame manner as when an active being impels the body before him; therefore we call this the IMPULSION *of a fluid*. And as our knowledge of nature informs us that the mutual actions of bodies are in every caſe equal and oppoſite, and that the obſerved change of motion is the only indication, characteriſtic, and meaſure, of the changing force, the forces are the ſame (whether we call them impulſions or reſiſtances) when the relative motions are the ſame, and therefore depend entirely on theſe relative motions. The force, therefore, which is neceſſary for keeping a body immoveable in a ſtream of water, flowing with a certain velocity, is the ſame with what is required for moving this body with this velocity through ſtagnant water.

A body in motion appears to be reſiſted by a ſtagnant fluid, becauſe it is a law of mechanical nature that force muſt be employed in order to put any body in motion. Now, the body cannot move forward without putting the contiguous fluid in motion, and force muſt be employed for producing this motion. In like manner, a quieſcent body is impelled by a ſtream of fluid, becauſe the motion of the contiguous fluid is diminiſhed by this ſolid obſtacle; the reſiſtance, therefore, or impulſe, no way differs from the ordinary communications of motion among ſolid bodies, at leaſt in its nature; although it may be far more difficult to reduce the various circumſtances to accurate computation, or to obtain all the requiſite data on which to found the calculus.

540. The reſiſtance which a body ſuffers from the fluid medium through which it is impelled depends on the velocity, form, and magnitude of the body, and on the inertia and tenacity of the fluid. For fluids reſiſt the motion of bodies through them, 1. by the inertia of their particles: 2. by their tenacity, *i. e.* the adheſion of thoſe particles: 3. by the friction of the body againſt the particles of the fluid. In perfect fluids the latter cauſes of reſiſtance are very inconſiderable, and therefore are not taken into the account; but the former is always very conſiderable, and obtains equally in the moſt perfect as in the moſt imperfect fluids. And that the reſiſtance varies with the velocity, ſhape, and magnitude of the moving body is ſufficiently obvious.

We muſt carefully diſtinguiſh between *reſiſtance* and *retardation:* reſiſtance is the quantity of *motion*, retardation the quantity of *velocity*, which is loſt; therefore, the retardations are as the reſiſtances applied to the quantities of matter; and in the ſame body the reſiſtance and retardation are proportional.

541. Prop. *To determine the force of fluids in motion, or the reſiſtance of fluids againſt bodies moving in them.*

1. In fluids uniformly tenacious the reſiſtance is as the velocity with which the body moves. For, ſince the cohesion of the particles of the fluid is always the ſame in the ſame ſpace, whatever be the velocity, the reſiſtance from this coheſion will be as the ſpace deſcribed in a given time; that is, as the velocity.

2. In a fluid whoſe particles move freely without diſturbing each other's motions, and which flows in behind as faſt as a plane body moves forward, ſo that the preſſure on every part of the body is the ſame as if the body were at reſt, the reſiſtance will be as the denſity of the fluid.

3. On the ſame hypotheſis the reſiſtance will be as the ſquare of the velocity. For the reſiſtance muſt vary as the number of particles which ſtrike the plane in a given time, multiplied into the force of each againſt the plane; but both the number and

the force is as the velocity, and conſequently the reſiſtance is as the ſquare of the velocity.

This proof ſuppoſes that after the body ſtrikes a particle the action of that particle entirely ceaſes: whereas the particles, after they are ſtruck, muſt neceſſarily diverge, and act upon the particles behind them; thus cauſing ſome difference between theory and experiment. This hypotheſis, however, on account of its ſimplicity, is generally retained, and corrected afterwards by deductions from actual experiments.

This ratio of the ſquare of the velocity may be otherwiſe derived, thus.

It is evident that the reſiſtance to a plane, moving perpendicularly through an infinite fluid, at reſt, is equal to the preſſure or force of the fluid on the plane at reſt, and the fluid moving with the ſame velocity, and in the contrary direction, to that of the plane in the former caſe. But the force of the fluid in motion muſt be equal to the weight or preſſure which generates that motion; and which, it is known, is equal to the weight or preſſure of a column of the fluid, whoſe baſe is equal to the plane, and its altitude equal to the height through which a body muſt fall, by the force of gravity, to acquire the velocity of the fluid: and that altitude is, for the ſake of brevity, called the altitude due to the velocity. So that, if a denote the area of the plane, v the velocity, and n the ſpecific gravity of the fluid; then, the altitude due to the velocity v being $\frac{v^2}{2g}$, the whole reſiſtance, or motive force m, will be $a \times n \times \frac{v^2}{2g} = \frac{anv^2}{2g}$; g being $32\frac{1}{6}$ feet. And hence, *cæteris paribus*, the reſiſtance is as the ſquare of the velocity.

4. If the direction of the motion, inſtead of being perpendicular to the plane, as above ſuppoſed, be inclined to it in any angle, then the reſiſtance to the plane, in the direction of the motion as aſſigned above, will be diminiſhed in the triplicate ratio of radius to the ſine of the angle of inclination, or in the ratio of 1 to s^3, where s is the ſine of inclination.

For AB (fig. 7. pl. XVIII.) being the direction of the plane, and BD that of the motion, ABD the angle whoſe ſine is s; the number of particles or quantity of the fluid which ſtrikes the plane will be diminiſhed in the ratio of 1 to s; and the force of each particle will likewiſe be diminiſhed in the ſame ratio: ſo that, on both theſe accounts, the whole reſiſtance will be diminiſhed in the ratio of 1 to s^2; that is, in the duplicate ratio of radius to the ſine of ABD. But further, it muſt be conſidered that this whole reſiſtance is exerted in the direction BE perpendicular to the plane; and any force in direction BE is to its effect in direction AE parallel to BD as AE to BE, or as 1 to s. Conſequently, on all theſe accounts, the reſiſtance in the direction of the motion is diminiſhed in the ratio of 1 to s^3. And if this be compared with the reſult of the preceding ſtep, we ſhall have for the whole reſiſtance, or the motive force on the plane, $m = \frac{anv^2s^3}{2g}$.

5. If w repreſent the weight of the body whoſe plane face a is reſiſted by the abſolute force m, then the retarding force $f = \frac{m}{w} = \frac{anv^2s^3}{2gw}$.

6. And, if the body be a cylinder whoſe face or end is a, and diameter d, or radius r, moving in the direction of its axis; then, becauſe $s = 1$, and $a = \pi r^2 = \frac{1}{4}\pi d^2$, where $\pi = 3{\cdot}141593$; the reſiſting force m will be $= \frac{\pi nd^2v^2}{8g} = \frac{\pi nr^2v^2}{2g}$; and the retarding force $f = \frac{\pi nd^2v^2}{8gw} = \frac{\pi nr^2v^2}{2gw}$.

7. This is the value of the reſiſtance when the end of the cylinder is a plane perpendicular to its axis, or to the direction of motion. But were its face a conical ſurface, or an elliptic ſection, or any other figure every-where equally inclined to the axis, the ſine of inclination being s; then, the number of particles of the fluid ſtriking the face being ſtill the ſame, but the force of each, oppoſed to the direction of motion, diminiſhed in the duplicate ratio of radius to the ſine of inclination, the reſiſting force m would be $\frac{\pi nd^2v^2s^2}{8g} = \frac{\pi nr^2v^2s^2}{2g}$.

But if the body were terminated by an end or face of any other form, as a ſpherical one, or ſuch like, where every part of it has a different inclination to the axis; then a further inveſtigation becomes neceſſary, as in the following propoſition.

542. Prop. *To determine the reſiſtance of a fluid to any body moving in it, having a curved end, as a ſphere, a cylinder with a hemiſpherical end, &c.*

1. Let BEAD be a ſection through the axis CA of the ſolid, moving in the direction of that axis. To any point of the curve draw the tangent EG, meeting the axis produced in G: alſo draw the perpendicular ordinates EF, ef indefinitely near to each other; and draw ae parallel to CG. Fig. 11. pl. XVIII.

Putting $CF = x$, $EF = y$, $BE = z$, $s =$ ſine ∠ G to radius 1; then 2 y is the circumference whoſe radius is EF, or the circumference deſcribed by the point E, in revolving about the axis CA; and $2\pi y \times \mathrm{E}e$, or $2\pi y\dot{z}$, is the fluxion of the ſurface, or it is the ſurface deſcribed by $\mathrm{E}e$, in the ſaid revolution about CA; which is the quantity repreſented by *a* in art. 4 of the laſt problem: hence $\frac{n v^2 s^3}{2g} \times 2\pi y\dot{z}$, or $\frac{\pi n v^2 s^3}{g} \times y\dot{z}$ is the reſiſtance on that ring, or the fluxion of the reſiſtance to the body, whatever the figure of it may be: the fluent of which will be the reſiſtance required.

2. In the caſe of a ſpherical ſhape; putting the radius CA or CB $= r$, we have $y = \sqrt{r^2 - x^2}$, $s = \frac{\mathrm{EF}}{\mathrm{EG}} = \frac{\mathrm{CF}}{\mathrm{CE}} = \frac{x}{r}$, and $y\dot{z}$ or $\mathrm{EF} \times \mathrm{E}e = \mathrm{CE} \times ae = r\dot{x}$; therefore the general fluxion $\frac{\pi n v^2}{g} \times s^3 y\dot{z}$ becomes $\frac{\pi n v^2}{g} \times \frac{x^3}{r^3} \times r\dot{x} = \frac{\pi n v^2}{g r^2} \times x^3\dot{x}$; the fluent of which, or $\frac{\pi n v^2}{4 g r^2} x^4$, is the reſiſtance to the ſpherical ſurface generated by BE. And when x or CF is $= r$ or CA, it becomes $\frac{\pi n v^2 r^2}{4g}$ for the reſiſtance on the whole hemiſphere; which is alſo equal to $\frac{\pi n v^2 d^2}{16g}$, where $d = 2r$ the diameter.

3. But the perpendicular reſiſtance to the circle of the ſame diameter d or BD, by art. 6. of the preceding problem, is $\frac{\pi n v^2 d^2}{8g}$; which, being double the former, ſhews that *the reſiſtance to the ſphere is juſt equal to half the direct reſiſtance to a great circle of it, or to a cylinder of the ſame diameter.*

4. Since $\frac{1}{6}\pi d^3$ is the magnitude of the globe; if N denote its denſity or ſpecific gravity, its weight w will be $= \frac{1}{6}\pi d^3 \mathrm{N}$, and therefore the retardive force f or $\frac{m}{w} = \frac{\pi n v^2 d^2}{16g} \times \frac{6}{\pi \mathrm{N} d^3} = \frac{3 n v^2}{8 g \mathrm{N} d}$; which is alſo $= \frac{v^2}{2g\mathrm{S}}$ by art. 243. Hence then $\frac{3n}{4\mathrm{N}d} = \frac{1}{\mathrm{S}}$, and $\mathrm{S} = \frac{\mathrm{N}}{n} \times \frac{4}{3}d$; which is the ſpace that would be deſcribed by the globe, while its whole motion is generated or deſtroyed by a conſtant force which is equal to the force of reſiſtance, if no other force acted on the globe to continue its motion. And if the denſity of the fluid were equal to that of the globe, the reſiſting force is ſuch, as, acting conſtantly on

the globe without any other force, would generate or deſtroy its motion in deſcribing the ſpace $\frac{4}{3}d$, or $\frac{4}{3}$ of its diameter, by that accelerating or retarding force.

5. Hence the greateſt velocity that a globe will acquire by deſcending in a fluid, by means of its relative weight in the fluid, will be found by making the reſiſting force equal to that weight. For, after the velocity is arrived at ſuch a degree that the reſiſting force is equal to the weight that urges it, it will increaſe no longer, and the globe will afterwards continue to deſcend with that velocity uniformly. Now, N and n being the ſeparate ſpecific gravities of the globe and fluid, $N-n$ will be the relative gravity of the globe in the fluid, and therefore $w = \frac{1}{6}\pi d^3 (N-n)$ is the weight by which it is urged; alſo, $m = \frac{\pi n v^2 d^2}{16 g}$ is the reſiſtance; conſequently $\frac{\pi n v^2 d^2}{16 g} = \frac{1}{6}\pi d^3$ $(N-n)$ when the velocity becomes uniform: from which equation is found $v = \sqrt{2g \cdot \frac{4}{3}d \cdot \frac{N-n}{n}}$, for the ſaid uniform or greateſt velocity.

By comparing this value of v with thoſe in arts. 216 and 243, it will appear that the greateſt velocity is equal to the velocity generated by the accelerating force $\frac{N-n}{n}$ in deſcribing the ſpace $\frac{4}{3}d$, or equal to the velocity generated by gravity in freely deſcribing the ſpace $\frac{N-n}{n} \times \frac{4}{3}d$.—If $N = 2n$, or the ſpecific gravity of the globe be double that of the fluid, then $\frac{N-n}{n} = 1 =$ the natural force of gravity; and then the globe will attain its greateſt velocity in deſcribing $\frac{4}{3}d$, or $\frac{4}{3}$ of its diameter.—It is further evident, that if the body be very ſmall it will ſoon acquire its greateſt velocity, whatever its denſity may be.

Exam. If a leaden ball, of 1 inch diameter, deſcend in water, and in air of the ſame denſity as at the earth's ſurface, the three ſpecific gravities being as $11\frac{1}{3}$, and 1, and $\frac{3}{2500}$. Then $v = \sqrt{2 \cdot 32\frac{1}{6} \cdot \frac{4}{36} \cdot 10\frac{1}{3}} = \frac{1}{9}\sqrt{31 \cdot 193} = 8{\cdot}5944$ feet, is the greateſt velocity per ſecond the ball can acquire by deſcending in water. And $v = \sqrt{4 \cdot \frac{193}{12} \cdot \frac{4}{36} \cdot \frac{34}{3} \cdot \frac{2500}{3}}$ nearly $= \frac{50}{9}\sqrt{\frac{34 \cdot 193}{3}} = 259{\cdot}82$, is the greateſt velocity it can acquire in air.

But if the globe were only $\frac{1}{100}$ of an inch diameter, the greateſt velocities it could acquire would be only $\frac{1}{10}$ of theſe, namely $\frac{86}{100}$ of a foot in water, and 26 feet nearly in air: and if the ball were ſtill further diminiſhed the greateſt velocity would alſo be diminiſhed, and that in the ſubduplicate ratio of the diameter of the ball.

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543. It appears from the third ſtep of the preceding article, that the reſiſtance to the motion of a cylinder moving in the direction of its axis is double to that of a globe of equal diameter: and, in experiments, when the bodies move ſlow, this will nearly hold in water, but more accurately in air; becauſe its particles move more freely than thoſe of water, and leſs diſturb each other's motions: but when the motion is more rapid, conſiderable aberrations will ariſe; both from the mutual diſturbance of the particles, and from the fluid not flowing in ſo faſt behind as the body moves forward: in the air, alſo, a new cauſe of aberration will ariſe, from the condenſation of the fluid before the body. Sir Iſaac Newton ſuppoſes, that in a continuous non-elaſtic fluid, infinitely compreſſed, the reſiſtances of a ſphere and cylinder of equal diameters are equal: but this appears to be an error in theory as well as in fact; for the lemma (Lemma 5. book ii. § 7. *Princip.*) on which he founds his inference, has been juſtly called in queſtion. When the motion is ſlow in water, the fluid may be conceived to be nearly of that nature which Newton ſuppoſes; yet the reſiſtances are almoſt as coincident with theory as when the motion is in air: thus M. Borda found the reſiſtance of a ſphere moving in water to be to that of its greateſt circle as 1 to 2·508, and in air the reſiſtances were as 1 to 2·45. The experiments of Dr. Hutton in air give the reſiſtances as 1 to $2\frac{1}{3}$, at a mean.

The reaſon that experiments give the ratio of the reſiſtances greater than that of 2 to 1 ſeems to be this: in theory it is ſuppoſed that the action of every particle of the fluid ceaſes the inſtant it makes its impact on the ſolid; but this is not actually the caſe, as we have before obſerved (art. 541.): and ſince the particles, after impact on the ſphere, ſlide along the curved ſurface, and hence eſcape with more facility than along the face of the cylinder, the error will be greater in the cylinder; that is, the greater reſiſtance will exceed theory more than the leſs. It is alſo to be obſerved, that the difference between the reſiſtances of the globe and cylinder, in water, is greater than in air; which is directly contrary to what might be inferred from Newton's reaſoning, which ſuppoſes them equal in a continuous fluid, but in the ratio of 1 to 2 in a rare fluid.

544. Lemma. If a given angle be divided into two parts, the product, or ſolid, contained under the ſquare of the ſine of the one part and the ſine of the other will be a *maximum* when the tangent of the former is double the tangent of the latter; or, when the ſine of the difference of the parts is one-third of the ſine of the whole given angle.

This is a particular caſe of Prob. v. p. 502, *Simpſon's Fluxions;* and is well known to mathematicians.

545. Prop. *Suppoſe that a plane* ABC (fig. 12. pl. XVIII.) *moving with a velocity and direction repreſented by* bB, *is acted upon by a fluid whoſe particles move with a velocity repreſented by* DB, *and in directions parallel to that line; it is propoſed to determine the angle of inclination* ABD *ſo that the effect of the fluid may be the greateſt poſſible.*

Since a particle impinging on the plane at B moves through the ſpace DB in the time that the plane itſelf would paſs from the poſition abc to ABC, it is manifeſt that the diſtance De of the ſaid particle from the plane (produced) at the beginning of that time will be the meaſure of the relative celerity with which the particles of the fluid approach the plane in a direction perpendicular to it; and conſequently (art. 469.) that the force of the ſtream in that direction will $\propto De^2$: whence, by the reſolution of forces, the efficacy in the propoſed direction BH will $\propto De^2 \times$ ſine ABH $\propto De^2 \times$ ſine abH.

Now the angle bBD being given, as well as the ſides Bb, BD, containing that angle, the remaining angle BbD will be known, as well as the ſide Db: of conſequence, De being the ſine of the angle Dbe to the given radius Db, the effect $De^2 \times$ ſine abH will be a maximum, when ſine $^2 Dbe \times$ ſine abH is a maximum; that is (by the Lemma), when ſine ($Dba \backsim abH$) $= \frac{1}{3}$ ſine BbD: whence the difference being given, the angles themſelves will be known. The geometrical conſtruction is very ſimple: thus, having from the centre b with any radius deſcribed the arc mr, on rb (produced if neceſſary) let fall the perpendicular mp; take $pq = \frac{1}{3} mp$, and draw qs parallel to pr, cutting the circle in s: then biſect the arc ms by the line bae, and the thing required is done. For the ſine sv of sr, that is, of the difference of the angles Dba, abH, is $\frac{1}{3}$ of mp, the ſine of the whole given angle BbD, as it ought to be.

546. To obtain a general theorem expreſſed algebraically, let the velocity bB of the plane be put $= v$, and that of the fluid $= V$; alſo let the angle DBb be called B: and having drawn BFL perpendicular to the plane, or to bFe, put $bF = x$, and $BF = y$. Then, becauſe FB and FL are tangents of the angles FbB, FbL, to the common radius bF, it follows (from the lemma) that $FL = 2\,BF = 2y$; whence, if LR and DQ be

perpendicular to HQ, we have, by ſimilar triangles, Bb : BF :: BL : BR; that is, $v : y :: 3y : \frac{3y^2}{v} =$ BR; and conſequently bR = BR − Bb $\frac{3y^2-v^2}{v}$. Likewiſe, Bb : bF :: BL : LR; that is, $v : x :: 3y : \frac{3xy}{v} =$ LR. But DQ = V . ſine B, and bQ = V. cos. B − v, we have again, by ſimilar triangles, DQ : bQ :: LR : bR, or V. ſine B : V. cos. B − v :: $\frac{3xy}{v} : \frac{3y^2-v^2}{v}$. Multiplying the means and extremes of this analogy, we obtain

$$3y^2 - v^2 = \frac{\text{V.cos. B} - v}{\text{V . sine B}} \times 3xy.$$

Subſtituting in this equation for v^2 its equal x^2+y^2, completing the ſquare and reducing, we at length find

$$\frac{x}{y} = \sqrt{2 + \tfrac{9}{4} \cdot \left(\frac{\text{V . cos. B} - v}{\text{V sine B}}\right)^2} - \frac{3\,(\text{V cos. B} - v)}{2\ \text{V. sine B}}.$$

And this equation manifeſtly expreſſes the natural tangent of the angle bBF, or the cotangent of the required angle FbH.

547. COR. 1. If the given angle DBb be a right angle (as is the caſe when the wind ſtrikes againſt the ſails of a windmill), then is ſine B = 1, and coſine B = 0, the expreſſion for the tangent of bBF (which is here equal to the angle of inclination ABD) will become $\sqrt{2 + \frac{9v^2}{4V^2}} + \frac{3v}{2V}$. This, if v be taken = 0, or the plane be ſuppoſed at reſt, will be barely = √ 2, anſwering to an angle of 54° 44′. But if the velocity of the plane be ſuppoſed $\frac{1}{8}$, $\frac{1}{4}$, $\frac{1}{3}$, $\frac{1}{2}$, or $\frac{1}{1}$, of that of the medium or ſtream, then the angle ABD, found from this theorem, will be equal to 58° 14′, 61° 27′, 63° 26′, 66° 58′ or 74° 19′ reſpectively: ſo that the greater the velocity of the plane, the greater alſo will be the angle of inclination.

Hence it appears, that the ſails of a windmill, in order that the effect may be the greateſt, ought to be more turned towards the wind in the extreme parts where the motion is ſwifteſt than in the parts nearer to the axis of motion; in ſuch a manner that the tangent of the axle formed by the direction of the wind and the ſail may every-where be equal to the expreſſion $\sqrt{2 + \frac{9v^2}{4V^2}} + \frac{3v}{2V}$; the velocity v being proportional to the diſtance from the axis of motion, and increaſing till, at the extremity of the ſail, it is ſometimes equal to V, or even exceeds it.

548. Cor. 2. If the angle DBA, which the direction of the ſtream makes with the plane, be given, inſtead of the angle DBH or DBb; it will then appear that the effect will in that caſe be a maximum when ſine ABH (the angle made by the plane and the direction of its motion) : ſine DBA :: $\frac{1}{3}$ BD : Bb.

For the force in the direction FB varying as De^2, its effect in the direction BH will $\propto De^2 \times \frac{BF}{Bb} \propto \frac{De^2 \times Ee}{Bb}$. Now DB, B$b$, and the angle DBE, being given, DE is thence given. And it is well known (ſee Simpſon's Geom. theor. 17. of max. and min.) that the ſolid of the ſquare of one part of a line into the other part is a maximum, when the former part is the double of the latter. Conſequently De muſt be $=2$ Ee; ſo that Ee, or its equal BF, will be $\frac{1}{3}$ DE.

But, ſine BbF : radius :: BF ($=\frac{1}{3}$DE) : Bb,
and radius : ſine DBA :: BD : DE,
whence, *componendo*, ſine BbF : ſine DBA :: $\frac{1}{3}$ BD : Bb.

549. Cor. 3. The proportion in the preceding corollary can only obtain when Bb is equal to or greater than $\frac{1}{3}$ DE. For, when Bb is leſs than $\frac{1}{3}$ DE, Ee (which is always leſs than Bb) cannot be equal to $\frac{1}{3}$ DE; but will approach the neareſt to it when BF coincides with Bb, that is, when the angle FbH or ABH is of 90°; and in this caſe the effect will be a maximum when the direction of the motion is perpendicular to the plane. If the given angle DBA be a right angle (which appears to be the moſt advantageous, becauſe then DE$=$DB), it follows that ſine ABH will be to radius as $\frac{1}{3}$ of the velocity of the ſtream to the velocity of the plane or ſail. Hence, *if the force of the wind be capable of producing a degree of velocity in a ſhip greater than $\frac{1}{3}$ of its own velocity, it is evident that the ſhip may run ſwifter upon an oblique courſe than when ſhe ſails directly before the wind.* If the velocity be to that of the wind as 1 to 3, and the courſe be 109° 28′, the force of the wind upon the veſſel to promote its motion will be greater than the force in a direct courſe of 180°, in the ratio of $\sqrt[3]{32}$ to $\sqrt[3]{27}$, or of 3·1748 to 3. See *Maclaurin's Fluxions*, art. 919.

550. Prop. *To determine the relations of velocity, ſpace, and time, of a ball moving in a fluid, in which it is projected with a given velocity.*

1. Let $a=$ the firſt velocity of projection, x the ſpace deſcribed in any time t, and v the velocity then. Now, by ſtep 4, art. 542. the accelerative force $f=\frac{3nv^2}{8gNd}$; where N is the denſity of the ball, n that of the fluid, and d the diameter. There-

fore the general equation $v\dot{v} = gf\dot{s}$ becomes $v\dot{v} = \frac{-3nv^2}{8Nd}\dot{x}$; and hence $\frac{\dot{v}}{v} = \frac{-3n}{8Nd}\dot{x} = -b\dot{x}$, putting b for $\frac{3n}{8Nd}$. The correct fluent of this is log. a − log. v, or log. $\frac{a}{v} = bx$. Or, putting $c =$ 2·718281828, the number whoſe hyp. log. is 1, then is $\frac{a}{v} = c^{bx}$, and the velocity $v = \frac{a}{c^{bx}} = ac^{-bx}$.

2. The velocity v at any time being the c^{-bx} part of the firſt velocity, therefore the velocity loſt in any time will be the $1 - c^{-bx}$ part, or the $\frac{c^{bx} - 1}{c^{bx}}$ part of the firſt velocity.

EXAM. 1. If a globe be projected with any velocity in a medium of the ſame denſity with itſelf, and it deſcribe a ſpace equal to $3d$ or 3 of its diameters. Then $x = 3d$, and $b = \frac{3n}{8Nd} = \frac{3}{8d}$; therefore $bx = \frac{9}{8}$, and the velocity loſt is $\frac{c^{bx} - 1}{c^{bx}} = \frac{2{\cdot}08}{3{\cdot}08}$, or nearly $\frac{2}{3}$ of the projectile velocity.

EXAMP. 2. If an iron ball of 2 inches diameter were projected with a velocity of 1200 feet per ſecond; to find the velocity loſt after moving through any ſpace, as ſuppoſe 500 feet of air: we ſhould have $d = \frac{2}{12} = \frac{1}{6}$, $a = 1200$, $x = 500$, $N = 7\frac{1}{3}$, $n = {\cdot}0012$; and therefore $bx = \frac{3nx}{8Nd} = \frac{3.12.500.3.6}{8.22.10000} = \frac{81}{440}$, and $v = \frac{1200}{c^{\frac{81}{440}}} = 998$ feet per ſecond: having loſt 202 feet, or nearly $\frac{1}{6}$ of its firſt velocity.

EXAM. 3. If the earth revolved about the ſun, in a medium as denſe as the atmoſphere near the earth's ſurface; and it were required to find the quantity of motion loſt in a year. Then, ſince the earth's mean denſity is about $4\frac{1}{2}$, and its diſtance from the ſun 12000 of its diameters, we have 24000 × 3·1416 = 75398 diameters $= x$, and $bx = \frac{3 \cdot 75398 . 12 . 2}{8 . 10000 . 9} =$ 7·5398; hence $\frac{c^{bx} - 1}{c^{bx}} = \frac{1880}{1881}$ parts are loſt of the firſt motion in the ſpace of a year, and only the $\frac{1}{1881}$ part remains.

3. To find the time t; we have $\dot{t} = \frac{\dot{s}}{v} = \frac{\dot{x}}{v} = \frac{c^{bx}\dot{x}}{a}$. Now,

to find the fluent of this, put $z = c^{bx}$; then is $bx = \log. z$, and $b\dot{x} = \frac{\dot{z}}{z}$, or $\dot{x} = \frac{\dot{z}}{bz}$; conſequently $\dot{t}$ or $\frac{c^{bx}\dot{x}}{a} = \frac{z\dot{x}}{a} = \frac{\dot{z}}{ab}$; and hence $t = \frac{z}{ab} = \frac{c^{bx}}{ab}$. But as t and x vaniſh together, and when $x = 0$, the quantity $\frac{c^{bx}}{ab}$ is $= \frac{1}{ab}$; therefore, by correction, $t = \frac{c^{bx} - 1}{ab} = \frac{1}{bv} - \frac{1}{ba} = \frac{1}{b}\left(\frac{1}{v} - \frac{1}{a}\right)$ the time ſought; where $b = \frac{3n}{8Nd}$, and $v = \frac{a}{c^{bx}}$ the velocity.

551. Prop. *To determine the relations of ſpace, time, and velocity, when a globe deſcends, by its own weight, in an infinite fluid.*

The foregoing notation remaining, viz. d = diameter, N and n the denſity of the ball and fluid, and v, s, t, the velocity, ſpace, and time, in motion; we have $\frac{1}{6}\pi d^3$ = the magnitude of the ball, and $\frac{1}{6}\pi d^3 (N - n)$ = its weight in the fluid, alſo $m = \frac{\pi n d^2 v^2}{16g}$ = its reſiſtance from the fluid; conſequently $\frac{1}{6}\pi d^3 (N - n) - \frac{\pi n d^2 v^2}{16g}$ is the motive force by which the ball is urged; which being divided by $\frac{1}{6}\pi N d^3$, the quantity of matter moved, gives $f = 1 - \frac{n}{N} - \frac{3nv^2}{8gNd}$ for the accelerative force.

2. Hence $v\dot{v} = gf\dot{s}$, and $\dot{s} = \frac{v\dot{v}}{gf} = \frac{Nv\dot{v}}{g(N-n) - \frac{3n}{8d}v^2} = \frac{1}{b} \times \frac{v\dot{v}}{a - v^2}$, putting $b = \frac{3n}{8Nd}$, and $\frac{1}{a} = \frac{3n}{g \cdot 8d(N-n)}$, or $ab = g$ nearly; the fluent of which is $s = \frac{1}{2b} \times \log.$ of $\frac{a}{a - v^2}$, an expreſſion for the ſpace s, in terms of the velocity v, when s and v begin together.

3. To determine v in terms of s, put $c = 2{\cdot}718281828$; then, ſince the log. of $\frac{a}{a - v^2} = 2bs$, therefore $\frac{a}{a - v^2} = c^{2bs}$, or $\frac{a - v^2}{a} = c^{-2bs}$; and hence $v = \sqrt{a - ac^{-2bs}}$, the velocity ſought.

4. The greateſt velocity is to be found, as in ſtep. 5 of art. 542. by making f or $1 - \frac{n}{N} - \frac{3nv^2}{8gNd} = 0$, which gives $v =$

$\sqrt{g \cdot 8d \cdot \frac{N-n}{3n}} = \sqrt{a}$. The ſame value is alſo obtained by making the fluxion of v^2, or of $a - ac^{-2bs}$, $= 0$. And the ſame value of v is obtained by making s infinite, for then $c^{-2bs} = 0$. But this velocity $\sqrt{a}$ cannot be attained in any finite time, and it only denotes the velocity to which the general value of v or $\sqrt{a - ac^{-2bs}}$ continually approaches. It is evident, however, that it will approximate towards it the faſter, the greater b is, or the leſs d is; and that, the diameters being very ſmall, the bodies deſcend by nearly uniform velocities, which are directly in the ſubduplicate ratio of the diameters.

5. To find the time t; we have $\dot{t} = \frac{\dot{s}}{v} = \sqrt{\frac{1}{a}} \times \frac{\dot{s}}{\sqrt{1 - c^{-2bs}}}$. Then, to find the fluent of this fluxion, put $z = \sqrt{1 - c^{-2bs}} = \frac{v}{\sqrt{a}}$, or $z^2 = 1 - c^{-2bs}$; hence $z\dot{z} = b\dot{s}c^{-2bs}$, and $\dot{s} = \frac{z\dot{z}}{bc^{-2bs}} = \frac{1}{b} \cdot \frac{z\dot{z}}{1 - z^2}$; conſequently $\dot{t} = \frac{1}{b\sqrt{a}} \cdot \frac{\dot{z}}{1 - z^2}$, and therefore the fluent is $t = \frac{1}{2b\sqrt{a}} \times \log. \frac{1+z}{1-z} = \frac{1}{2b\sqrt{a}} \times \log. \frac{1 + \sqrt{1 - c^{-2bs}}}{1 - \sqrt{1 - c^{-2bs}}} = \frac{1}{2b\sqrt{a}} \times \log. \frac{\sqrt{a} + v}{\sqrt{a} - v}$, which is the general expreſſion for the time.

Note. If the globe be ſo light as to aſcend in the fluid, it is only neceſſary to change the ſigns of the firſt two terms in the value of f, or the accelerating force, by which it becomes $f = \frac{n}{N} - 1 - \frac{3nv^2}{8gNd}$; and then to proceed in all reſpects as before.

For more examples ſee Dr. *Hutton's Select Exerciſes*; from whence ſeveral articles in this chapter were taken.

* * *

The theory of the reſiſtance of fluids when conſidered in its utmoſt extent, is very intricate and perplexing. Beſides the propoſitions on this ſubject in *Newton's Principia*, lib. 2, the reader may be referred to the theory of *John Bernoulli* in his Diſſertation on the Communication of Motion, and that of M. *D'Alembert* in his *Hydrodynamica*. The latter theory is genuine and unexceptionable, but extremely intricate, requiring a minute acquaintance with the moſt abſtruſe analyſis; and it furniſhes only one *new* propoſition that is of any practical utility. The theory of M. Georges Juan in his *Examen Maritime*, and that of M. *La Grange*, a ſketch of which may be ſeen in *Prony's Architecture Hydraulique*, include all the conſiderations that ought to be compriſed in ſuch diſcuſſions: but they preſent numerous difficulties inſeparable from the analyſis itſelf, and furniſh but few practical reſults.

CHAPTER VI.

Experiments on the Resistance of Fluids.

552. In the preceding chapter we have given ſome of the moſt uſeful propoſitions in the ordinary theory of the reſiſtance of fluids, and have referred to the works of other authors in which more rigorous theories are exhibited. But in all theories reſpecting the motions and actions of fluids, there are aſſumptions which depend on the judgment of the inveſtigator; ſo that it is always proper to put every ſuch theory to the teſt of experiments. If it be highly deſirable that theſe branches of ſcience be reduced to rules capable of being employed in practice with certainty, and if, as is our opinion, this can never be completely attained by theory alone, it becomes a matter of great importance to ſubject theſe points to an extenſive and varied ſeries of experiments; to diſcuſs theſe experiments with care, and to compare them with the theory adopted, in order to diſcover where it is deficient, and to ſupply the deficiencies. A multiplicity of facts attentively analyſed, and reduced as far as poſſible to general laws, may correct the reſults of theory, or compoſe of themſelves the outlines of a kind of theory which may be readily adapted to the uſual occaſions of practice. Under theſe impreſſions we have already ſtated the reſults of experiments on the effluence of fluids, and the motion of water wheels (chaps. 2 and 4, book IV.); and ſhall now exhibit the moſt curious and important reſults which have been deduced from experiments on the reſiſtance of fluids.

553. Experiments on this ſubject are by no means numerous; at leaſt ſuch as can be depended on for the foundation of any practical application. The firſt that have this character are thoſe publiſhed by Mr. Robins in 1742, in his treatiſe on gunnery. They were repeated with ſome additions by the Chevalier Borda, and ſome account of them publiſhed in the Memoirs of the Academy of Sciences in 1763. In the Philoſophical Tranſactions of the Royal Society of London, vol. lxxiii. there are ſome experiments of the ſame kind on a larger ſcale by Mr. Edgeworth.

In all theſe experiments the reſiſtances were found to be very nearly in the proportion of the ſquares of the velocities; but they were found conſiderably greater than the weight of the column of air whoſe height would produce the velocity in a falling body. Mr. Robins's experiments on a ſquare of 16 inches, deſcribing 25·2 feet per ſecond, indicate the reſiſtance to be to this weight nearly as 4 to 3. Borda's experiments upon the ſame ſurface, ſtate the diſproportion to be ſtill greater.

The reſiſtances are found not to be in the proportion of the ſurfaces, but increaſe conſiderably faſter. Surfaces of 9, 16, 36, and 81 inches, moving with one velocity, had reſiſtances in the proportion of 9, 17½, 42¾, and 104¾.

Now as this deviation from the proportion of the ſurfaces increaſes with great regularity, it is moſt probable that it continues to increaſe in ſurfaces of ſtill greater extent; and theſe are the moſt generally to be met with in practice in the action of wind on ſhips and mills.

Borda's experiments on 81 inches ſhew that the impulſe of wind moving one foot per ſecond, is about $\frac{1}{500}$ of a pound on a ſquare foot. Therefore, to find the impulſe on a foot correſponding to any velocity, divide the ſquare of the velocity by 500, and we obtain the impulſe in pounds. Mr. Rouſe of Leiceſterſhire made many experiments, which are mentioned with great approbation by Mr. Smeaton. His great ſagacity and experience in the erection of windmills oblige us to pay a conſiderable deference to his judgment. Theſe experiments confirm our opinion, that the impulſes increaſe faſter than the ſurfaces. The following table was calculated from Mr. Rouſe's obſervations, and may be conſidered as pretty near the truth.

Velocity in Ft.	Impulse on a Ft. in Pounds.
0	0·000
10	0·129
20	0·915
30	2·059
40	3·660
50	5·718
60	8·234
70	11·207
80	14·638
90	18·526
100	22·872
110	27·675
120	32·926
130	38·654
140	44·830
150	51·462

If we multiply the ſquare of the velocity in feet by 16, the product will be the impulſe or reſiſtance on a ſquare foot in grains, according to Mr. Rouſe's numbers.

554. The greateſt deviation from the theory occurs in the oblique impulſes. Mr. Robins compared the reſiſtance of a wedge, whoſe angle was 90°, with the reſiſtance of its baſe; and inſtead of finding it leſs in the proportion of $\sqrt{2}$ to 1, as determined by the theory, he found it greater in the proportion of 55 to 68 nearly; and when he formed the body into a pyramid, of which the ſides had the ſame ſurface and the ſame inclination as the ſides of the wedge, the reſiſtance of the baſe and face were now as 55 to 39 nearly: ſo that here the ſame ſurface with the ſame inclination had its reſiſtance reduced from 68 to 39 by being put into this form. Similar deviations occur in the experiments of the Chevalier Borda; and it may be collected from both, that the reſiſtances diminiſh more nearly in the proportion of the ſines of incidence than in the proportion of the ſquares of thoſe ſines.

The irregularity in the reſiſtance of curved ſurfaces is as great as in plane ſurfaces. In general, the theory gives the oblique impulſes on plane ſurfaces much too ſmall, and the impulſes on curved ſurfaces too great. The reſiſtance of a ſphere does not exceed the fourth part of the reſiſtance of its great circle, inſtead of being its half; but the anomaly is ſuch as to leave hardly any room for calculation. It would be very deſirable to have the experiments on this ſubject repeated in a greater variety of caſes, and on larger ſurfaces, ſo that the errors of the experiments may be of leſs conſequence.

Mr. Robins having proved that in very great changes of velocity the reſiſtance does not accurately follow the duplicate ratio of the velocity, lays down two poſitions, which he ſuppoſes may be of ſome ſervice, particularly in the practice of artillery, till a more complete and accurate theory of reſiſtance, and the changes of its augmentation, may be diſcovered. The firſt of theſe is, that till the velocity of the projectile ſurpaſs 1100 or 1200 feet per ſecond, the reſiſtance may be eſteemed in the duplicate ratio of the velocity. The ſecond is, that when the velocity exceeds 1100 or 1200 feet in a ſecond, then the abſolute quantity of the reſiſtance will be nearly 3 times as great as it ſhould be from a compariſon with the ſmaller velocities. We ſhall ſoon ſee, however, that there is no *abrupt* change in the law of reſiſtance; but that it is ſlow and continual, from the ſmalleſt to the greateſt velocities.

555. Dr. *Hutton*, of the Royal Academy Woolwich, has likewiſe proſecuted the ſubject of the reſiſtance of the air experimentally, and that to a conſiderable extent. His experiments were made not only with the whirling machine, invented by Mr. Robins, but with cannon balls of different weights, from 1 lb. to 6 lb.; alſo with figures of various ſhapes, and with planes ſet at a variety of angles of inclination to the path of motion. From theſe curious experiments the doctor has aſcertained the reſiſtance of bodies to all velocities, from 1 to 2000 feet per ſecond; the bodies being different, and their faces at different angles of inclination. Some of his general tables and concluſions deduced from thoſe experiments are as follow.

Table I. *Reſiſtances of different bodies.*

Veloc. per sec.	Small hemis. flat side.	Large hemis.		Cone.		Cylinder.	Whole globe.	Resis. as the power of the veloc.
		flat side.	round side.	vertex.	base.			
feet.	oz.	oz.	oz.	oz.	oz.	oz.	oz.	
3	·028	·051	·020	·028	·064	·050	·027	
4	·048	·096	·039	·048	·109	·090	·047	
5	·072	·148	·063	·071	·162	·143	·068	
6	·103	·211	·092	·098	·225	·205	·094	
7	·141	·284	·123	·129	·298	·278	·125	
8	·184	·368	·160	·168	·382	·360	·162	
9	·233	·464	·199	·211	·478	·456	·205	
10	·287	·573	·242	·260	·587	·565	·255	
11	·349	·698	·292	·315	·712	·688	·310	2·052
12	·418	·836	·347	·376	·850	·826	·370	2·042
13	·492	·988	·409	·440	1·000	·979	·435	2·036
14	·573	1·154	·478	·512	1·166	1·145	·505	2·031
15	·661	1·336	·552	·589	1·346	1·327	·581	2·031
16	·754	1·538	·634	·673	1·546	1·526	·663	2·033
17	·853	1·757	·722	·762	1·763	1·745	·752	2·038
18	·959	1·928	·818	·858	2·002	1·986	·848	2·044
19	1·073	2·998	·922	·959	2·260	2·246	·949	2·047
20	1·196	2·542	1·033	1·069	2·540	2·528	1·057	2·051
Mean propor. Nos.	140	288	119	126	291	285	124	2·040
1	2	3	4	5	6	7	8	9

In this table are contained the reſiſtances to ſeveral forms of bodies, when moved with ſeveral degrees of velocity, from 3 feet per ſecond to 20. The names of the bodies are at the tops of the columns, as alſo which end went foremoſt through the air; the different velocities are in the firſt column, and the reſiſtances on the ſame line, in their ſeveral columns, in avoirdupois ounces and decimal parts. So on the firſt line are contained the reſiſtances when the bodies move with a velocity of 3 feet in a ſecond, viz. in the 2d column for the ſmall hemiſphere, of $4\frac{3}{4}$ inches diameter, its reſiſtance ·028 oz. when the flat ſide went foremoſt; in the 3d and 4th columns the reſiſtances to a larger hemiſphere, firſt with the flat ſide, and next the round ſide foremoſt, the diameter of this, as well as all the following figures being $6\frac{5}{8}$ inches, and therefore the area of the great circle = 32 ſq. inches, or $\frac{2}{9}$ of a ſq. foot; then in the 5th and 6th columns are the reſiſtances to a cone, firſt its vertex and then its baſe foremoſt, the altitude of the cone being $6\frac{5}{8}$ inches, the ſame as the diameter of its baſe; in the 7th column the reſiſtance to the end of the cylinder, and in the 8th that againſt the whole globe or ſphere. All the numbers ſhew the real weights which are equal to the reſiſtances; and at the bottoms of the columns are placed proportional numbers, which ſhew the mean proportions of the reſiſtances of all the figures to one another with any velocity. Laſtly, in the 9th column are placed the exponents of the power of the velocity which the reſiſtances in the 8th column bear to each other, viz. which that of the 10 feet velocity bears to each of the following ones, the medium of all of them being as the 2·04 power of the velocity; that is, very little above the ſquare or ſecond power of the velocity, ſo far as the velocities in this table extend.

556. From this table the following inferences are eaſily deduced.

I. That the reſiſtance is nearly in the ſame proportion as the ſurfaces; a ſmall increaſe only taking place in the greater ſurfaces, and for the greater velocities. Thus, by comparing together the numbers in the 2d and 3d columns for the baſes of the two hemiſpheres, the areas of which baſes are in the proportion of $17\frac{3}{4}$ to 32, or 5 to 9 very nearly, it appears that the numbers in thoſe two columns expreſſing the reſiſtances, are nearly as 1 to 2, or 5 to 10, as far as the velocity of 12 feet; but after that, the reſiſtances on the greater ſurface increaſe gradually more and more above that proportion.

II. The reſiſtance to the ſame ſurface with different velocities, is, in theſe ſlow motions, nearly as the ſquare of the velocity; but gradually increaſes more and more above that proportion as the velocity increaſes. This is manifeſt from all the columns; and the index of the power of the velocity is ſet down in the 9th column, for the reſiſtances in the 8th, the medium being 2·04; by which it appears that the reſiſtance to the ſame body is, in theſe ſlow motions, as the 2·04 power of the velocity, or nearly as the ſquare of it.

III. The round ends, and ſharp ends, of ſolids, ſuffer leſs reſiſtance than the flat or plane ends, of the ſame diameter; but the ſharper end has not always the leſs reſiſtance. Thus, the cylinder and the flat ends of the hemiſphere and cone have more reſiſtance than the round or ſharp ends of the ſame; but the round ſide of the hemiſphere has leſs reſiſtance than the ſharper end of the cone.

IV. The reſiſtance on the baſe of the hemiſphere, is to that on the round, or whole ſphere, as $2\frac{1}{3}$ to 1, inſtead of 2 to 1, as the theory gives that relation. Alſo the experimented reſiſtance on each of theſe, is nearly $\frac{1}{4}$ more than the quantity aſſigned by the theory.

V. The reſiſtance on the baſe of the cone is to that on the vertex nearly as $2\frac{3}{10}$ to 1; and in the ſame ratio is radius to the ſine of the angle of inclination of the ſide of the cone to its path or axis. So that, in this inſtance, the reſiſtance is directly as the ſine of the angle of incidence, the tranſverſe ſection being the ſame.

VI. When the hinder parts of different bodies are of different forms, the reſiſtances are different, though the fore parts be exactly alike and equal; owing probably to the different preſſures of the air on the hinder parts. Thus, the reſiſtance to the fore part of the cylinder is leſs than on the equal flat ſurface of the cone, or of the hemiſphere; becauſe the hinder part of the cylinder is more preſſed or puſhed by the following air than thoſe of the other two figures; alſo, for the ſame reaſon, the baſe of the hemiſphere ſuffers a leſs reſiſtance than that of the cone, and the round ſide of the hemiſphere leſs than the whole ſphere.

557. Table II. *Reſiſtances both by experiment and theory, to a globe of* 1·965 *inches diameter.*

Veloc. per sec. in feet.	Resist. by exper. oz.	Resist. by theory. oz.	Ratio of exper. to theory.	Resist. as the power of the veloc.
5	0·006	0·005	1·20	
10	0·024½	0·020	1·23	
15	0·055	0·044	1·25	
20	0·100	0·079	1·27	
25	0·157	0·123	1·28	2·022
30	0·23	0·177	1·30	2·055
40	0·42	0·314	1·33	2·068
50	0·67	0·491	1·36	2·075
100	2·72	1·964	1·38	2·059
200	11	7·9	1·40	2·041
300	25	18·7	1·41	2·039
400	45	31·4	1·43	2·039
500	72	49	1·47	2·044
600	107	71	1·51	2·051
700	151	96	1·57	2·059
800	205	126	1·63	2·067
900	271	159	1·70	2·077
1000	350	196	1·78	2·086
1100	442	238	1·86	2·095
1200	546	283	1·90	2·102
1300	661	332	1·99	2·107
1400	785	385	2·04	2·111
1500	916	442	2·07	2·113
1600	1051	503	2·09	2·113
1700	1186	568	2·08	2·111
1800	1319	636	2·07	2·108
1900	1447	709	2·04	2·104
2000	1569	786	2·00	2·098

In the firſt column of this table are contained the ſeveral velocities, gradually from 0 up to the great velocity of 2000 feet per ſecond, with which a ball or globe moved. In the 2d column are the experimented reſiſtances in avoirdupois ounces. In the 3d column are the correſpondent reſiſtances, as computed by the theory. In the 4th column are the ratios of theſe two reſiſtances, or the quotients of the former divided by the latter. And in the 5th or laſt, the indexes of the power of the velocity which is proportional to the experimented reſiſtance; which are found by comparing the reſiſtance of 20 feet velocity with each of the following ones.

From the 2d, 3d, and 4th columns it appears, that at the beginning of the motion the experimented reſiſtance is nearly equal to that computed by theory; but that, as the velocity increaſes, the experimented reſiſtance gradually exceeds the other more and more, till at the velocity of 1300 feet the former becomes juſt double the latter; after which the difference increaſes a little further, till about the velocity of 1600 or 1700, where that exceſs is the greateſt, and is rather leſs than $2\frac{1}{10}$; after this, the difference decreaſes gradually as the velocity increaſes, and at the velocity of 2000 the former reſiſtance again becomes juſt double the latter.

From the laſt column it appears that, near the beginning, or in ſlow motions, the reſiſtances are nearly as the ſquare of the velocities; but that the ratio gradually increaſes, with ſome ſmall variation, till at the velocity of 1500 or 1600 feet it becomes as the $2\frac{1}{9}$ power of the velocity nearly, which is its higheſt aſcent; and after that it gradually decreaſes again, as the velocity goes higher. And ſimilar concluſions have alſo been derived from experiments with larger balls or globes.

558. Table III. *Reſiſtance to a plane, ſet at various angles of inclination to its path.*

Angle with the path.	Experim. reſistances. oz.	Resist. by this formula. $\cdot 84s^{1\cdot 842c}$	Sines of the angles to radius ·840.
0°	·000	·000	·000
5	·015	·009	·073
10	·044	·035	·146
15	·082	·076	·217
20	·133	·131	·287
25	·200	·199	·355
30	·278	·278	·420
35	·362	·363	·482
40	·448	·450	·540
45	·534	·535	·594
50	·619	·613	·643
55	·684	·680	·688
60	·729	·736	·727
65	·770	·778	·761
70	·803	·808	·789
75	·823	·826	·811
80	·835	·836	·827
85	·839	·839	·838
90	·840	·840	·840

In the 2d column of this table are contained the actual experimented reſiſtences in ounces, to a plane of 32 ſquare inches, or $\frac{2}{9}$ of a ſquare foot, moved through the air with a velocity of exactly 12 feet per ſecond, when the plane was ſet ſo as to make, with the direction of its path, the correſponding angles in the firſt column.

And from theſe Dr. Hutton deduced this formula, viz. $\cdot 84s^{1\cdot 84c}$, which brings out very nearly the ſame numbers, and is a general theorem for every angle, for the ſame plane of $\frac{2}{9}$ of a foot, and moved with the ſame velocity of 12 feet in a ſecond of time; where s is the ſine, and c the coſine of the angles of inclination in the firſt column.

If a theorem be deſired for any other velocity v, and any other plane whoſe area is a, it will be thus: $\frac{1}{38} a v^2 s^{1\cdot 842c}$, or more nearly $\frac{1}{42} a v^{2\cdot 04} s^{1\cdot 842c}$; which exhibits the reſiſtance very nearly to any plane ſurface whoſe area is a, moved through the air with the velocity v, in a direction making with that plane an angle, whoſe ſine is s and coſine c.

If the fluid be water, or any other whoſe denſity is different from that of air, the formula for the reſiſtance muſt be increaſed in proportion to the denſity.

By this theorem were computed the numbers in the 3d column, which, it is evident, agree very nearly with the experimental reſiſtances in the 2d column; excepting in two or three of the ſmall numbers near the beginning, which are of the leaſt conſequence. In all other caſes, the theorem gives nearly the true reſiſtance. In the 4th or laſt column are entered the ſines of the angles of the firſt column, to the radius ·84, in order to compare them with the reſiſtances in the other columns. From whence it appears that thoſe reſiſtances bear no ſort of analogy to the ſines of the angles, nor yet to the ſquares, or any other power of the ſines. In the beginning of the columns, the ſines much exceed the reſiſtances all the way till the angle is between 55° and 60°; after which the ſines are leſs than the reſiſtances all the way to the end, or, till the angle becomes a right angle. See *Hutton's Dictionary*, art. RESISTANCE *of Fluids*.

See alſo a paper by Mr. Vince on this ſubject in the Phil. Tranſ. for 1798, or Nicholſon's Journal, vol. iii. p. 506.

559. Meſſrs. *D'Alembert*, *Condorcet*, and *Boſſut*, purſuant to the directions of M. *Turgot*, comptroller-general of finances, made, in the year 1775, a ſeries of experiments, in order to perfect internal navigation; and the reſiſtance of fluids was the

principal object of their reſearches. Their experiments, however, were confined to very ſmall velocities, and to *direct* reſiſtances; ſo that their reſults, as might be expected, did not differ ſenſibly from thoſe furniſhed by the ordinary theory. But, in 1778, Boſſut and Condorcet directed their attention to oblique reſiſtances, and undertook "A new ſeries of experiments, "chiefly intended to diſcover the law according to which the "reſiſtance experienced by an angular prow diminiſhes in pro- "portion as the angle of that prow becomes more acute; the "fluid being indefinite, ſuch as the ſea."

Theſe experiments were made in a great reſervoir ſituated on the north ſide of the ancient Boulevards of Paris. The veſſels were priſmatic (excepting the prow, which was an iſoſceles wedge), and their motion was produced by the deſcent of a weight, which, by means of a cord, and different pullies, cauſed the veſſels to advance. During the firſt inſtants of the deſcent of the weight, the motion was accelerated; but the veſſel ſoon acquired a uniform velocity, and the reſiſtance of the water, relative to that velocity, makes the equilibrium with the moving force of the weight due to gravity, &c.

The following table compriſes 69 experiments of Boſſut on ſimple angular prows. The firſt five were made with a boat in form of a rectangular parallelopiped, the length of which was 4 feet, breadth 2 feet, depth in the fluid 2 feet, and height of the part jutting out of the fluid about 7 feet.

The other 64 experiments were made with 14 veſſels in form of a priſm, the prows being iſoſceles wedges, having their vertices forwards: theſe vertical angles increaſed gradually from 12°, 24°, 36°, &c. to 180°; the length of each veſſel was 4 feet, breadth 2 feet, depth in the water 2 feet, height out of the water about 7 feet: all in Paris meaſure. The titles of the columns require no explanation.

Angle of the prow.	Weight of the mover in pounds.	Time of describing 96 feet uniformly.	Angle of the prow.	Weight of the mover in pounds.	Time of describing 96 feet uniformly.
180°	61·8	78·08″	96°	60·5	63·00″
	112·5	57·51		110·5	46·45
	162·5	47·44		162·5	38·05
	212·5	41·49		212·5	32·66
	262·6	37·32		262·5	29·27
168°	62·5	77·50	84°	60·5	60·55
	112·5	56·95		110·5	44·56
	162·5	47·22		162·5	35·78
	212·5	41·26		212·5	31·25
	262·5	37·12		262·5	27·51
156°	62·5	75·09	72°	62·5	57·50
	112·5	56·15		112·5	42·75
	162·5	46·44		162·5	34·85
	212·5	41·03		212·5	29·65
	262·5	36·52		262·5	25·86
144°	62·5	73·38	60°	60·5	55·45
	112·5	54·75		112·5	40·04
	162·5	45·35		162·5	33·05
	212·5	39·58		212·5	28·25
	262·5	37·57		262·5	24·77
132°	60·5	72·08	48°	62·5	52·51
	111·5	53·25		112·5	38·05
	162·5	43·75		162·5	31·61
	212·5	38·26		212·5	27·56
	262·5	34·30		262·5	24·30
120°	62·0	68·32	36°	62·5	51·15
	112·5	50·84		112·5	36·96
	162·5	41·84		162·5	30·53
	212·5	36·62	24°	61·5	49·48
	262·5	32·77		112·5	35·76
108°	61·5	65·85		162·5	30·23
	112·5	48·75	12°	62·5	49·38
	162·5	39·50		112·5	35·33
	212·5	34·46		162·5	30·01
	262·5	31·05			

The direct reſiſtance on the end of the veſſel without an iſoſceles prow may be repreſented by any arbitrary number, as 10000 for inſtance.

And if the angle of the prow, or that formed at the fore-part of the veſſel or ſummit of the triangular ſection, be x, the reſiſtance will be expreſſed by the formula 10000 coſ.^2x. To correct this expreſſion by means of the foregoing table, we may obſerve that, when the angle x undergoes a variation of 12°, each of the angles at the baſe of the iſoſceles prow will vary 6°: calling this latter variation q, M. Boſſut deduced from the table the additive correction $3{\cdot}153\left(\frac{x}{q}\right)^{3{\cdot}25}$, x and q being expreſſed in decimal parts of the radius unity. Thus the whole reſiſtance againſt the prow, eſtimated in the oppoſite direction to that of the motion of the boat, is equal to $10000 \text{ coſ.}^2x + 3{\cdot}153\left(\frac{x}{q}\right)^{3{\cdot}25}$. And this formula will be ſufficiently exact when the extreme angle of the prow is greater than 12°: but when that angle is leſs than 12° the theorem will err in exceſs.

560. Such are the concluſions and rules which have been deduced from ſome of the moſt judicious experiments on the reſiſtance of fluids to the motion of bodies paſſing through them. Conſidering the number and importance of theſe inferences with reſpect to the magnitude and extent of the experiments from which they were drawn, we cannot but earneſtly wiſh that thoſe who have leiſure and opportunity to purſue ſuch enquiries, would ſtill further extend and vary their reſearches in this department of experimental philoſophy. Much valuable knowledge may be expected to reſult from labours of this kind: for it is, probably, only by reiterated experiments and obſervations that ſufficient data can be obtained to perfect the theory of two momentous branches of mechanical ſcience, viz. thoſe relating to the motion of military projectiles, and to the manœuvres of veſſels at ſea.

END OF VOLUME I.